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RELATIVELY HEREDITARY RADICAL CLASSES OF RINGS

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Dedicated to the memory of Barry Jones.

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Abstract. Radical classes of rings are studied which, while not hereditary, are closed with respect to ideals of some kind: maximal, prime and finite index ideals among others. In some, but not all cases, the ideal property is characterized by the corresponding class of factor rings; for instance maximal ideals are characterized by the simple rings. Such characterizations sometimes make it possible to prove results for several types of ideal simultaneously. Several results concerning hereditary radicals are generalized to various types of relatively hereditary ones, e.g. if $\mathcal{R}$ is hereditary then for $I \triangleleft A$ we have $\mathcal{R}(I) = I \cap \mathcal{R}(A)$ and hereditary classes define hereditary lower radical classes. In the construction of some examples, use is made of A-radical classes and this leads to some consideration of radical classes of abelian groups.

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1. INTRODUCTION

Although stronger versions of the hereditary property for radical (and other) classes of rings, e.g. *left hereditary* (hereditary for left ideals) and *strongly hereditary* (hereditary for subrings) have been extensively studied, almost no attention has been given to *weak* versions of hereditariness for rings or other structures. Thus nothing seems to be known about radical classes which are hereditary for maximal ideals, or for prime ideals for example.

In fact to our knowledge the only weak hereditary property for radical classes of rings that has been studied previously is that considered by Sands [13]: he defined a radical class $\mathcal{R}$ to be *radically hereditary* if for every $A \in \mathcal{R}$ we have $\mathcal{U}(A) \in \mathcal{R}$ for every radical class $\mathcal{U}$.

For other structures the story is similar. In a paper by the second author written over half a century ago [4] the radical classes of abelian groups which are hereditary for *pure* subgroups were described and there were some results for other types of subgroups (generalized purity) in [6]. A different characterization of the pure-hereditary radical classes – as kernels of tensor product functors – was later given in [15]. On the other hand, nothing seems to be known in the case of groups. Radical classes of groups which are hereditary for *characteristic* or *fully invariant* subgroups might be worth looking at.

For a radical class $\mathcal{R}$ with semi-simple class $\mathcal{S}$, the hereditariness of $\mathcal{R}$ is equivalent to each of the following.

(1) $\mathcal{R}(I) = I \cap \mathcal{R}(A)$ for every ideal I of every ring A.
(2) $\mathcal{S}$ is closed under essential extensions.
(3) $\mathcal{R}$ is hereditary for essential ideals.

Also the hereditary property is preserved under the lower radical construction in the sense that if $\mathcal{M}$ is a hereditary class, then its lower radical class $L(\mathcal{M})$ is also hereditary.

Among other things we shall seek generalizations of (1)-(3) and the lower radical result to classes with weaker properties. For example the analogue of (1) holds for radical classes which are hereditary for maximal, prime or finite index ideals, while that of (3) holds in the maximal and finite index cases but the prime case remain open. If a class is hereditary for maximal or finite index ideals then its lower radical class has the same property and again the prime case is open. On the other hand, if a class is hereditary for persistent ideals or radically hereditary, its lower radical class need not enjoy the same property. Some of our examples of relatively hereditary radical classes are obtained as intersections of A-radical classes and hereditary radical classes. Using some results from the literature and such examples we are then able to deduce a couple of "sporadic" results: normal radical classes and lower radical classes defined by zerorings are hereditary for ideals of finite index.

Some types of relative hereditariness can be treated simultaneously by means of the following concept.

For a non-empty class $\mathcal{C}$ of non-zero rings which is hereditary for non-zero ideals an ideal I of a ring A is called a $\mathcal{C}^{\leftarrow}$ *ideal* if $A/I \in \mathcal{C}$.

When $\mathcal{C}$ is the class of simple (respectively prime, respectively non-zero finite) rings, the $\mathcal{C}^{\leftarrow}$ ideals the maximal(respectively prime, respectively finite index) ideals (with the last not including the whole ring).

We prefer not to use the term $\mathcal{C}$ *ideal* (by analogy with prime ideals and prime rings) because of conflict with the usage "$\mathcal{R}$-ideal" meaning an ideal in a radical class $\mathcal{R}$, and to some extent with similar terminology elsewhere. On occasion we impose an extra condition on the class $\mathcal{C}$ – that it be closed under non-zero homomorphic images – in order to prove something, but it remains unknown whether this extra condition is necessary or not.

Recall that an ideal I of a ring A is *essential* if $I \cap J \neq 0$ for every non-zero ideal J of A.

We call an ideal I of a ring A *persistent* if it satisfies the condition

$$A \triangleleft B \Rightarrow I \triangleleft B.$$

Since we shall only consider associative rings, persistent ideals include radicals of rings (by (ADS)), idempotent ideals and semiprime ideals (by the Andrunakievich Lemma). Every ideal of a ring with identity is persistent as a consequence of a ring with identity being a direct summand whenever it's an ideal.

We shall use the following notation:

- $I \triangleleft A$: I is an ideal of A;
- $I \triangleleft' A$: I is a prime ideal of A;
- $I \triangleleft_m A$: I is a maximal ideal of A;
- $I \triangleleft_{fi} A$: I is an ideal of A with finite index;
- $I \triangleleft^{\bullet} A$: I is an essential ideal of A;
- $I \triangleleft_p A$: I is a persistent ideal of A.

Expressions such as "$\mathcal{R}$ is hereditary for maximal ideals" mean, of course, that if A is in $\mathcal{R}$ then so is any maximal ideal of A etc.

A radical class $\mathcal{R}$ is said to be *radically hereditary* (Sands [13]) if for every $A \in \mathcal{R}$ we have $\mathcal{U}(A) \in \mathcal{R}$ for every radical class $\mathcal{U}$.

The term "radical class" is always used in the Kurosh-Amitsur sense of a homomorphically closed class $\mathcal{R}$ of rings such that each ring A has a largest ideal $\mathcal{R}(A) \in \mathcal{R}$ and $\mathcal{R}(A/\mathcal{R}(A)) = 0$ for each A. Equivalently, a radical class is a non-empty class which is homomorphically closed, closed under extensions and closed under unions of ascending chains of ideals,

We also deal with radical classes of abelian groups, which can be defined the same way *mutatis mutandis*, but more conveniently as non-empty classes closed under homomorphic images, extensions and direct sums. For unexplained terms and results pertaining to radical classes of rings, see [9], for abelian groups see [3] and radical classes of abelian groups are discussed in [8].

The following notation will be used at several places in the paper. The (radical) class of torsion abelian groups will be called $\mathcal{T}$ and that of abelian p-groups (p prime) $\mathcal{T}_p$. If a ring A is an algebra over a commutative ring K with identity, $A * K$ will denote the unital extension of A given by the adjunction of the identity of K, i.e. the ring on $A \times K$ with componentwise addition and multiplication given by

$$(a,k)(b,\ell) = (ab + kb + \ell a, k\ell).$$

In several places we need to consider additive groups of rings and ring structures on abelian groups. There we shall use the following notation.

- $(\cdot)^+$: additive group of a ring;
- $(\cdot)^0$: zeroring (all products zero) on the additive group of a ring and more generally, on an abelian group.

All rings are associative and "ring" does not mean "ring with identity".

2. Results and examples

Proposition 1. *A radical class of rings is hereditary for principal ideals if and only if it is hereditary.*

Proof. If $\mathcal{R}$ is hereditary for principal ideals then for $I \triangleleft A \in \mathcal{R}$, $[i]$ (= principal ideal of A generated by i) is in $\mathcal{R}$ for every $i \in i$ and then $I = \sum_{i \in I}[i] \in \mathcal{R}$. □

The only other case we are aware of where some sort of relative hereditariness implies hereditariness is contained in the next result.

Theorem 1. *The following conditions are equivalent for a radical class $\mathcal{R}$ of rings.*

(i) $\mathcal{R}$ *is hereditary.*
(ii) $\mathcal{R}$ *is hereditary for essential ideals.*
(iii) $I \triangleleft A \Rightarrow \mathcal{R}(I) = I \cap \mathcal{R}(A)$.
(iv) $I \triangleleft^{\bullet} A \Rightarrow \mathcal{R}(I) = I \cap \mathcal{R}(A)$.

Proof.

$(i) \Rightarrow (ii)$**:** Clearly.
$(ii) \Rightarrow (i)$**:** If $\mathcal{R}$ is hereditary for essential ideals and $I \triangleleft A \in \mathcal{R}$, then as $0 \cap I = 0$, Zorn's Lemma ensures that A has an ideal M which is maximal with respect to having zero intersection with I. Then

$$I \cong I/(I \cap M) \cong (I+M)/M \triangleleft^{\bullet} A/M \in \mathcal{R}.$$

$(i) \Leftrightarrow (iii)$**:** See [9], p.46, Corollary 3.2.4.
$(iii) \Rightarrow (iv)$**:** This is clear.
$(iv) \Rightarrow (ii)$**:** If $I \triangleleft^{\bullet} A \in \mathcal{R}$, then $\mathcal{R}(I) = I \cap \mathcal{R}(A) = I \cap A = I$, i.e. $I \in \mathcal{R}$.

□

If $\mathcal{R}$ is a radical class with semi-simple class $\mathcal{S}$, then $\mathcal{R}$ is hereditary if and only if $\mathcal{S}$ is closed under essential extensions. This result is due to Armendariz [1] and Ryabukhin [12]; see also [9], p.47, Proposition 3.2.6. Hence we have

Corollary 1. *For a radical class $\mathcal{R}$ with semi-simple class $\mathcal{S}$, the following conditions are equivalent.*

(i) $I \triangleleft^{\bullet} A$ *and* $A \in \mathcal{R} \Rightarrow I \in \mathcal{R}$.
(ii) $I \triangleleft^{\bullet} A$ *and* $I \in \mathcal{S} \Rightarrow A \in \mathcal{S}$.

It would be interesting to know if there are any other kinds of ideal that satisfy the analogous "duality". Of course the conditions of Corollary 1 are equivalent to those of Theorem 1. Certainly arbitrary ideals don't satisfy the duality condition. For if $\mathcal{R}$ is a hereditary radical class which is neither $\{0\}$ nor the class of all rings, $\mathcal{S}$ its semi-simple class, let A, B be non-zero rings in $\mathcal{R}$, $\mathcal{S}$ respectively. Then $B \in \mathcal{S}$ and

$B \triangleleft A \oplus B$ but $A \oplus B \notin \mathcal{S}$ (though $\mathcal{R}$ is hereditary). In the same way, any type of ideal which includes direct summands fails to satisfy the condition.

Before considering some examples, we introduce an idea which makes it possible to treat several relatively hereditary properties simultaneously. For a non-empty class $\mathcal{C}$ of non-zero rings which is hereditary for non-zero ideals, an ideal I of a ring A will be called a $\mathcal{C}^{\leftarrow}$ *ideal* if $A/I \in \mathcal{C}$.

For example, if $\mathcal{C}$ is the class of simple (respectively prime, respectively non-zero finite) rings then the $\mathcal{C}^{\leftarrow}$ ideals are the maximal (respectively prime, respectively finite index) ideals. We exclude the ring 0 from $\mathcal{C}$ so that the ring itself is not a $\mathcal{C}^{\leftarrow}$ ideal, consistent with the standard usage for maximal and prime ideals.

We now obtain some generalizations of (1) of Section 1.

Theorem 2. *Let $\mathcal{C}$ be a non-empty class of non-zero rings which is hereditary for non-zero ideals. Then a radical class $\mathcal{R}$ is hereditary for $\mathcal{C}^{\leftarrow}$ ideals if and only if $\mathcal{R}(I) = I \cap \mathcal{R}(A)$ for every $\mathcal{C}^{\leftarrow}$ ideal I of every ring A.*

Proof. "If" is proved by the usual simple argument. If $A \in \mathcal{R}$ and I is a $\mathcal{C}^{\leftarrow}$ ideal of A, then $\mathcal{R}(I) = I \cap \mathcal{R}(A) = I \cap A = I$.

"Only if". Suppose now that $\mathcal{R}$ is hereditary for $\mathcal{C}^{\leftarrow}$ ideals. If I is a $\mathcal{C}^{\leftarrow}$ ideal of a ring A, then

$$\mathcal{R}(A)/(I \cap \mathcal{R}(A)) \cong (\mathcal{R}(A)+I)/I \triangleleft A/I \in \mathcal{C}.$$

Hence $\mathcal{R}(A)/I \cap \mathcal{R}(A) = 0$ or $\mathcal{R}(A)/I \cap \mathcal{R}(A) \in \mathcal{C}$. In the first case, $\mathcal{R}(A) = I \cap \mathcal{R}(A) \triangleleft I$ so $\mathcal{R}(A) \subseteq \mathcal{R}(I)$ and hence

$$\mathcal{R}(A) = \mathcal{R}(I) = I \cap \mathcal{R}(I) = I \cap \mathcal{R}(A).$$

In the second case, $I \cap \mathcal{R}(A)$ is a $\mathcal{C}^{\leftarrow}$ ideal of $\mathcal{R}(A) \in \mathcal{R}$, so by assumption, $I \cap \mathcal{R}(A)$ is in $\mathcal{R}$. Also $I \cap \mathcal{R}(A) \triangleleft I$ so

$$I \cap \mathcal{R}(A) \subseteq \mathcal{R}(I) \subseteq I \cap \mathcal{R}(A).$$

Hence $\mathcal{R}(I) = I \cap \mathcal{R}(A)$. □

From our theorem above we can now deduce

Corollary 2. *If a radical class R is hereditary for maximal (respectively prime, respectively finite index) ideals, then $\mathcal{R}(I) = I \cap \mathcal{R}(A)$ for every maximal (respectively prime, respectively finite index) ideal I of every ring A.*

We have a generalization of (2) of Section 1 using classes $\mathcal{C}$, but we may need an extra condition on $\mathcal{C}$.

Theorem 3. *If $\mathcal{C}$ is a class of non-zero rings which is hereditary for non-zero ideals* **and closed under non-zero homomorphic images**, *then a radical class $\mathcal{R}$ is hereditary for $\mathcal{C}^{\leftarrow}$ ideals if and only if it is hereditary for essential $\mathcal{C}^{\leftarrow}$ ideals.*

Proof. If $\mathcal{R}$ is hereditary for essential $\mathcal{C}^{\leftarrow}$ ideals and I is a $\mathcal{C}^{\leftarrow}$ ideal of $A \in \mathcal{R}$, then as $0 \cap I = 0$, Zorn's Lemma ensures that A has an ideal M which is maximal with respect to having zero intersection with I. Then

$$I \cong I/(I \cap M) \cong (I+M)/M \triangleleft^{\bullet} A/M \in \mathcal{R}. \qquad (*)$$

and $(A/M)/((I+M)/M \in \mathcal{C}$ or $A/M = (I+M)/M$, and hence $A = I+M = I \oplus M$. In the former case $I \in \mathcal{R}$ by $(*)$ and in the latter, I as a direct summand of A, is again in $\mathcal{R}$. □

Problem 1. *Do we need the extra condition in Theorem 3?*

Note that special classes of prime rings satisfy the original conditions imposed on $\mathcal{C}$ in Theorem 2 and non-empty classes of simple rings satisfy the conditions of $\mathcal{C}$ in Theorem 3. On the other hand, the approach based on classes $\mathcal{C}$ has its limitations, as some types of ideals can't be defined in terms of classes of rings.

Proposition 2. *There is no class $\mathcal{C}$ of rings defining either*

(i) the essential ideals or

(ii) the persistent ideals

as the $\mathcal{C}^{\leftarrow}$ ideals.

Proof.

(i)**:** Suppose the essential ideals are the $\mathcal{C}^{\leftarrow}$ ideals for some $\mathcal{C}$. Then for any prime p we have $p\mathbb{Z} \triangleleft^{\bullet} \mathbb{Z}$ so $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} \in \mathcal{C}$. But for any ring R, we have $R \oplus \mathbb{Z}_p/R \cong \mathbb{Z}_p$, while R is a non-essential ideal.

(ii)**:** Suppose there is a class $\mathcal{C}$ such the persistent ideals are the $\mathcal{C}^{\leftarrow}$ ideals. Let $A = \mathbb{Z} \oplus (\mathbb{Q}/\mathbb{Z})^0$. Then $\mathbb{Z}$ is an idempotent ideal of A and therefore a persistent ideal, so $(\mathbb{Q}/\mathbb{Z})^0 \in \mathcal{C}$.

Now in $\mathbb{Q}[X]$ let

$$I = \{a_1X + a_2X^2 + \ldots : a_1 \in \mathbb{Z},\ a_2, \dots \in \mathbb{Q}\}.$$

Then I is an ideal of (X), the ideal of $\mathbb{Q}[X]$ generated by X. Define

$$f : (X) \to \mathbb{Q}^+/\mathbb{Z}^+; \qquad b_1X + b_2X^2 + \cdots \mapsto b_1 + \mathbb{Z}.$$

This a surjective abelian group homomorphism with kernel I. If

$$\gamma = c_1X + c_2X^2 + \ldots, \qquad \delta = d_1X + d_2X^2 + \cdots \in (X),$$

then

$$\gamma\delta = c_1d_1X^2 + (c_1d_2 + c_2d_1)X^3 + \ldots,$$

so $f(\gamma\delta) = 0$. If we now view $\mathbb{Q}/\mathbb{Z}$ as $(\mathbb{Q}/\mathbb{Z})^0$ we have $f(\gamma\delta) = 0 = f(\gamma)f(\delta)$ for all $\gamma, \delta \in (X)$ so f becomes a ring homomorphism with kernel I and $(X)/I \cong (\mathbb{Q}/\mathbb{Z})^0$, so I is a $\mathcal{C}^{\leftarrow}$ ideal of (X). But $(X) \triangleleft \mathbb{Q}[X]$ and, e.g. $3X \in I$ but $\frac{1}{2}3X = \frac{3}{2}X \notin I$ (though $\frac{1}{2} \in \mathbb{Q}[X]$). Hence I is not an ideal of $\mathbb{Q}[X]$ and consequently not a persistent ideal of (X).

□

By Theorem 1 radical classes which are hereditary for essential ideals satisfy the conclusion of Theorem 3, so by Proposition 2 the existence of a class $\mathcal{C}$ as in Theorem 3 is not *necessary* for the conclusion of that result.

Problem 2. *Do the persistent ideals satisfy the conclusion of Theorem 3, i.e. if a radical class is hereditary for essential persistent ideals, must it be hereditary for all persidtent ideals?*

Proposition 3. *If a radical class $\mathcal{R}$ contains $\mathbb{Z}$ and is hereditary for persistent ideals, then it is hereditary.*

Proof. If $A \in \mathcal{R}$ then $A * \mathbb{Z} \in \mathcal{R}$ and if $I \triangleleft A$ then I is a persistent ideal of $A * \mathbb{Z}$, so I is in $\mathcal{R}$. □

Proposition 4. *Let $\mathcal{R}$ be a radical class consisting of idempotent rings such that each ring in $\mathcal{R}$ is an ideal of a ring with identity in $\mathcal{R}$. If $\mathcal{R}$ is hereditary for persistent ideals then it is hereditary.*

Proof. If $A \in \mathcal{R}$, let $A \triangleleft B$ where B has an identity and is in $\mathcal{R}$. For an ideal I of A, let I^* be the ideal of B generated by I. Since B has an identity, I^* is a persistent ideal and hence is in $\mathcal{R}$. But then I^* is idempotent, while by the Andrunakievich Lemma $(I^*)^3 \subseteq I$. Hence $I = I^* \in \mathcal{R}$. □

This is well and good, but we have not yet given any examples of relatively hereditary radical classes which are not hereditary. To produce some examples we begin with A- *radical classes*.

For every radical class $\mathcal{U}$ of abelian groups we get a radical class $\mathcal{U}^*$ of rings by defining

$$\mathcal{U}^* = \{A : A^+ \in \mathcal{U}\}.$$

A radical class $\mathcal{R}$ of rings is called an A-*radical class* it satisfies the condition

$$A \in \mathcal{R} \text{ and } A^+ \cong B^+ \Rightarrow B \in \mathcal{R}.$$

The *A*-radical classes are precisely the classes $\mathcal{U}^*$ defined above. For all this see [9], pp.165-166.

Let S be a set of primes, S^* the multiplicative semigroup generated by S. An abelian group G is *S-divisible* if $nG = G$ for all $n \in S^*$, i.e. if it satisfies the condition

$$((\forall g \in G)(\forall n \in S^*))(\exists g_n \in G)(ng_n = g).$$

Clearly G is S-divisible if and only if $pG = G$ for every $p \in S$.

A subgroup H of an abelian group G is *S-pure* if $nH = H \cap nG$ for all $n \in S^*$. This is *not* equivalent to the condition $pH = H \cap pG$ for all $p \in S$; cf. *neat* subgroups [3]. We call an abelian group *S-torsion* if each of its elements has order in S^* and *S-torsion-free* if none of its non-zero elements has order in S^*. When S is the set of all primes we get the standard notions *divisible group, pure subgroup, torsion and torsion-free group*.

Example 1. The class $\mathcal{D}_S$ of S-divisible abelian groups is a radical class of abelian groups. We shall consider the A-radical class $\mathcal{D}_S^*$.

If $I \triangleleft_m R \in \mathcal{D}_S^*$, then R/I is a simple ring in $\mathcal{D}_S^*$. Only simple rings of characteristic 0 or a prime not in S are additively S-divisible, and these are also additively S-torsion-free. But then I^+ is an S-pure subgroup of R^+ and therefore S-divisible (since R^+ is). Hence $I \in \mathcal{D}_S^*$, so that $\mathcal{D}_S^*$ is hereditary for maximal ideals.

But unless $S = \varnothing$, $\mathcal{D}_S^*$ is not hereditary; let $\mathbb{Z}^0, \mathbb{Q}^0$ be the zerorings on $\mathbb{Z}^+, \mathbb{Q}^+$ respectively. Then $\mathbb{Z}^0 \triangleleft \mathbb{Q}^0 \in \mathcal{D}_S^*$, but $\mathbb{Z}^0 \notin \mathcal{D}_S^*$.

Clearly the intersection of two radical classes which are hereditary for maximal ideals is itself hereditary for maximal ideals. This gives us a family of examples.

If $\mathcal{R}$ is a hereditary radical class of rings then $\mathcal{R} \cap \mathcal{D}_S^*$ is hereditary for maximal ideals.

We need to be a little cautious with this example. If $\mathcal{R}$ contains all nilpotent rings, or equivalently, if $\mathbb{Z}^0 \in \mathcal{R}$, then we can use the $\mathbb{Z}^0, \mathbb{Q}^0$ example to see that $\mathcal{R} \cap \mathcal{D}_S^*$ is not hereditary. On the other hand, for the radical class $\mathcal{V}$ of *regular* rings, $\mathcal{V} \cap \mathcal{D}_S^*$ is also the class of S-torsion-free regular rings and this is hereditary. (See, e.g. [3] for the additive structure of regular rings.)

Example 2. The radical class $\mathcal{D}_S^*$ is hereditary for prime ideals. For if $I \triangleleft' A \in \mathcal{D}_S^*$, then A/I is an S-divisible divisible prime ring. But its S-torsion ideal is then contained in the annihilator and hence is zero. Thus A/I is additively S-torsion-free and we can argue as in the previous example.

Example 3. The class $\mathcal{D}_S^*$ is hereditary for ideals of finite index. For if $I \triangleleft A \in \mathcal{D}_S^*$ and A/I is finite, then as $A/I \in \mathcal{D}_S^*$, it must be $\{p\colon\ p \notin S\}$-torsion and hence S-torsion-free.

As with Example 2 we see that for every hereditary radical class $\mathcal{R}$, the class $\mathcal{R} \cap \mathcal{D}_S^*$ is hereditary for prime ideals and for ideals of finite index.

Problem 3. *Is there a radical class which is hereditary for prime ideals but not maximal ideals or the other way around?*

Using a method like that used for the divisible radical and its relatives, we can get a family of radical classes which are hereditary for ideals of finite index. We shall make use of the following two assertions concerning abelian groups.

(α) *If B is a subgroup of a torsion-free abelian group C and C/B is a torsion group, then every radical class $\mathcal{U}$ of abelian groups which contains B also contains C.* (Corollary 1.2 of [5])

(β) *If a radical class of abelian groups contains a group L, it also contains its torsion subgroup* . (Theorem 5.2 of [2])

The proof of the following result is based on arguments in [5] used to prove that radical classes of abelian groups are closed under quasi-isomorphisms.

Theorem 4. *Every radical class $\mathcal{U}$ of abelian groups is hereditary for subgroups of finite index.*

Proof. Let $\mathcal{U}$ be a radical class of abelian groups, G a group in $\mathcal{U}$ and H a subgroup with $|G/H| = n$ (finite). Then $\mathcal{T}(G) \in \mathcal{U}$ by (β) and $\mathcal{T}(H) = H \cap \mathcal{T}(G) \subseteq \mathcal{T}(G)$. Also

$$\mathcal{T}(G)/\mathcal{T}(H) = \mathcal{T}(G)/\mathcal{T}(G) \cap H \cong (\mathcal{T}(G)+H)/H \subseteq G/H,$$

so $|\mathcal{T}(G)/\mathcal{T}(H)| \leq |G/H| = n$, and thus $n\mathcal{T}(G) \subseteq \mathcal{T}(H)$. From this it is clear that $n\mathcal{T}_p(G) \subseteq \mathcal{T}_p(H)$ for every prime p, where $\mathcal{T}_p(G)$ is the largest p-subgroup of G and so on. If $\mathcal{U}$ contains no non-zero p-groups, then by (β), $\mathcal{T}_p(G) = 0$ and thus $\mathcal{T}_p(H) = 0 \in \mathcal{U}$. If $\mathcal{U}$ contains only the divisible p-groups, then

$$\mathcal{T}_p(G) = n\mathcal{T}_p(G) \subseteq \mathcal{T}_p(H) \subseteq \mathcal{T}_p(G),$$

so $\mathcal{T}_p(H) = \mathcal{T}_p(G) \in \mathcal{U}$ (by(β) again). The remaining possibility is that $\mathcal{U}$ contains all p-groups ([2], Theorem 2.6) and then $\mathcal{T}_p(H) \in \mathcal{U}$ anyway. Hence $\mathcal{T}(H) \in \mathcal{U}$.

Now $H/\mathcal{T}(H) = H/(H \cap \mathcal{T}(G)) \cong (H+\mathcal{T}(G))/\mathcal{T}(G)$ so we have an exact sequence

$$0 \to H/\mathcal{T}(H) \to G/\mathcal{T}(G) \to G/(H+\mathcal{T}(G)) \to 0.$$

Since G is in $\mathcal{U}$, so also is $G/(H+\mathcal{T}(G))$. Also $|G/(H+\mathcal{T}(G))| \leq |G/H| = n$.

Let f denote the monomorphism from our exact sequence. Then $|(G/\mathcal{T}(G)/f(H/\mathcal{T}(H))| \leq n$. Hence

$$G/\mathcal{T}(G) \cong n(G/\mathcal{T}(G)) \subseteq f(H/\mathcal{T}(H),$$

the isomorphism being given by "multiplication by n". (This is an isomorphism since $G/\mathcal{T}(G)$ is torsion-free.) Clearly $nf(H/\mathcal{T}(H)) \subseteq n(G/\mathcal{T}(G))$, so in the exact sequence

$$0 \to n(G/\mathcal{T}(G)) \to f(H/\mathcal{T}(H) \to f(H/\mathcal{T}(H))/n(G/\mathcal{T}(G)) \to 0$$

the first two terms are torsion-free and the third is a torsion group. Moreover $n(G/\mathcal{T}(G)) \cong G/\mathcal{T}(G) \in \mathcal{U}$, so by (α) we have $H/\mathcal{T}(H) \cong f(H/\mathcal{T}(H)) \in \mathcal{U}$. Since, as shown above, $\mathcal{T}(H) \in \mathcal{U}$, we have $H \in \mathcal{U}$ and the theorem is proved. □

Corollary 3. *Every A-radical class $\mathcal{U}^*$ is hereditary for ideals of finite index.*

As with S-divisible rings and maximal ideals, we can now give further examples of radical classes which are hereditary for ideals of finite index.

Example 4. For every hereditary radical class $\mathcal{R}$ of rings and every A-radical class $\mathcal{U}^*$, $\mathcal{R} \cap \mathcal{U}^*$ is a radical class which is hereditary for ideals of finite index.

A radical class $\mathcal{R}$ is *normal* if for every Morita context (A,V,W,B) we have $V\mathcal{R}(B)W \subseteq \mathcal{R}(A)$. An N-radical class is a radical class which contains all nilpotent rings and is left and right hereditary and left and right strong. For a discussion of these concepts see pp. 149-158 of [9]. Note that N-radical classes are hereditary.

Jaegermann and Sands have shown that every normal radical class is an intersection of an *N-radical class* and an A-radical class. ([11], Theorem 10). Thus as a consequence we have

Theorem 5. *Every normal radical class is hereditary for ideals of finite index.*

It was shown in [7] that the lower radical class defined by any class of zerorings is the intersection of an A-radical class with the prime (= Baer) radical class. This gives us another result.

Theorem 6. *Every class of zerorings defines a lower radical class that is hereditary for ideals of finite index.*

3. The lower radical construction

It was first shown by Hoffman and Leavitt [10] that if a class $\mathcal{M}$ of rings is hereditary, then so is its lower radical class $L(\mathcal{M})$. Along with $(1)-(3)$ of Section 1, the possibility of generalizing *this* result for relatively hereditary classes is a significant question.

Here we show that a class which is hereditary for $\mathcal{C}^{\leftarrow}$ ideals determines a lower radical class with the same property, but at the expense of an extra assumption about $\mathcal{C}$. In the final section of the paper we shall see examples in which relative hereditary properties are not preserved under the lower radical construction.

Recall the Tangeman-Kreiling lower radical construction ([14]; see also [9], pp.28-29).

Let $\mathcal{M}$ be a class of rings. We inductively define, for each ordinal γ, a class $\mathcal{M}_\gamma$ as follows.

$\mathcal{M}_1$ is the homomorphic closure of $\mathcal{M}$.

$\mathcal{M}_\mu$ being defined for all $\mu < \lambda$,

(i) if λ is a successor $\nu+1$, let $\mathcal{M}_\lambda$ be the class of rings B with an ideal $I \in \mathcal{M}_\nu$ such that also $B/I \in \mathcal{M}_\nu$, and

(ii) if λ is a limit, let $\mathcal{M}_\lambda$ consist of all rings which are unions of ascending chains of ideals from various $\mathcal{M}_\mu$ with $\mu < \lambda$.

Then the lower radical class $L(\mathcal{M})$ is the union of all the $\mathcal{M}_\gamma$.

Theorem 7. *Let $\mathcal{C}$ be a non-empty class of non-zero rings which is hereditary for non-zero ideals and* **closed under non-zero homomorphic images***. If a class $\mathcal{M}$ of rings is hereditary for $\mathcal{C}^{\leftarrow}$ ideals, then so is its lower radical class $L(\mathcal{M})$.*

Proof. If $A/I \in \mathcal{M}_1$ ($A \in \mathcal{M}$) and J/I is a $\mathcal{C}^{\leftarrow}$ ideal of A/I, then $A/J \cong (A/I)/(J/I)$ $\in \mathcal{C}$ so J is a $\mathcal{C}^{\leftarrow}$ ideal of A, whence $J \in \mathcal{M}$, $J/I \in \mathcal{M}_1$ and finally $\mathcal{M}_1$ is hereditary for $\mathcal{C}^{\leftarrow}$ ideals.

If $\mathcal{M}_\nu$ is hereditary for $\mathcal{C}^{\leftarrow}$ ideals and $B \in \mathcal{M}_{\nu+1}$, let A be an ideal of B such that both A and B/A are in $\mathcal{M}_\nu$. If I is a $\mathcal{C}^{\leftarrow}$ ideal of B, then

$$A/A \cap I \cong (A+I)/I \triangleleft B/I \in \mathcal{C},$$

so $A/A \cap I \in \mathcal{C}$ or $A/A \cap I = 0$. In the former case $A \cap I$ is a $\mathcal{C}^{\leftarrow}$ ideal of $A \in \mathcal{M}_\nu$, so $A \cap I \in \mathcal{M}_\nu$. In the latter, $A \cap I = A \in \mathcal{M}_\nu$ so in any case we have $A \cap I \in \mathcal{M}_\nu$.

Now

$$B/I \in \mathcal{C}, \text{ so } B/(I+A) \in \mathcal{C} \text{ or } B/(I+A) = 0. \qquad (\bullet)$$

If it is in $\mathcal{C}$, then $(I+A)/A$ is a $\mathcal{C}^{\leftarrow}$ ideal of $B/A \in \mathcal{M}_\nu$ whence

$$I/A \cap I \cong (I+A)/A \in \mathcal{M}_\nu.$$

But then since $A \cap I \in \mathcal{M}_\nu$, it follows that $I \in \mathcal{M}_{\nu+1}$. If, on the other hand, $B/(I+A) = 0$, then

$$I/A \cap I \cong (I+A)/A = B/A \in \mathcal{M}_\nu,$$

so again I is in $\mathcal{M}_{\nu+1}$. This class is therefore hereditary for $\mathcal{C}^{\leftarrow}$ ideals.

Now let λ be a limit ordinal such that $\mathcal{M}_\mu$ is hereditary for $\mathcal{C}^{\leftarrow}$ ideals for all $\mu < \lambda$. Let B be in $\mathcal{M}_\lambda$. Then $B = \bigcup I_x$ where the I_x form a chain of ideals of B, and for each x, $I_x \in \mathcal{M}_{\mu_x}$ for some $\mu_x < \lambda$. If J is a $\mathcal{C}^{\leftarrow}$ ideal of B, then $J = \bigcup J \cap I_x$ where each $J \cap I_x \triangleleft J$. Now

$$I_x/I_x \cap J \cong (I_x + J)/J \triangleleft B/J \in \mathcal{C},$$

so $I_x/J \cap I_x \in \mathcal{C}$ or $I_x/J \cap I_x = 0$. If the former, then $J \cap I_x$ is a $\mathcal{C}^{\leftarrow}$ ideal of $I_x \in \mathcal{M}_{\mu_x}$ and so $J \cap I_x \in \mathcal{M}_{\mu_x}$. On the other hand, if $I_x/J \cap I_x = 0$, then

$$J \cap I_x = I_x \in \mathcal{M}_{\mu_x}.$$

Since $J \cap I_x \in \mathcal{M}_{\mu_x}$ for all x, J is in $\mathcal{M}_\lambda$.

We have now shown that each $\mathcal{M}_\gamma$ is hereditary for $\mathcal{C}^{\leftarrow}$ ideals, and hence the same is true of the lower radical class $L(\mathcal{M})$.

□

Again it's unclear whether we really need the extra hypothesis of non-zero homomorphic closure; (•) indicates the only point of the proof at which we used it. Meanwhile, as with some other questions, the case of prime ideals remains open. In the final section we shall show that some relative properties are not preserved by the lower radical construction.

4. Connections between relative hereditary properties

We saw earlier that a radical class is hereditary for essential ideals (respectively principal ideals) if and only if it is hereditary. If a radical class is hereditary for maximal ideals, must it be hereditary for prime ideals? This, and similar questions will be our concern in this section. Our first result connects maximal and finite index ideals, and its proof relies on several applications of a lemma which we shall soon prove. First, though, we need some terminology.

For a property $(*)$ of ideals, let $\mathcal{C}(*)$ be the class of rings B such that if $\mathcal{R}$ is a radical class which is hereditary for $(*)$-ideals, $I \triangleleft A \in \mathcal{R}$ and $A/I \cong B$, then $I \in \mathcal{R}$. Then by definition, any radical class which is hereditary for $(*)$-ideals is hereditary for $\mathcal{C}(*)$-ideals.

This notation will be retained in the following result and its proof.

Lemma 1. *Let B be a ring with a finite series*

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_n \subseteq J_{n+1} = B$$

of ideals such that each $J_{i+1}/J_i \in \mathcal{C}(*)$. Then $B \in \mathcal{C}(*)$.

Proof. Supposing the assertion is true for $n = k$, consider a chain

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_k \subseteq L_{k+1} \subseteq L_{k+2} = A$$

of ideals of a ring A such that each $L_{i+1}/L_i \in \mathcal{C}(*)$. Then

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_k \subseteq L_{k+1}$$

is a chain of ideals of L_{k+1} and all the relevant factors are in $\mathcal{C}(*)$, so $L_{k+1} \in \mathcal{C}(*)$. Also $A/L_{k+1} = L_{k+2}/L_{k+1} \in \mathcal{C}(*)$.

Let $\mathcal{R}$ be a radical class which is hereditary for $(*)$-ideals. If $I \triangleleft R \in \mathcal{R}$ and $R/I \cong A$, then R has an ideal S such that $S/I \cong L_{k+1}$ and $(R/I)/(S/I) \cong A/L_{k+1}$. Hence $R/S \cong A/L_{k+1} \in \mathcal{C}(*)$, so $S \in \mathcal{R}$. But then as $S/I \cong L_{k+1} \in \mathcal{C}(*)$ it follows that $I \in \mathcal{R}$. This means that $A \in \mathcal{C}(*)$. Since the case $n = 0$ is clear, we have the result by induction. □

Corollary 4. *If $B_1, B_2, \ldots, B_n \in \mathcal{C}(*)$ (n finite), then $B_1 \oplus B_2 \oplus \cdots \oplus B_n \in \mathcal{C}(*)$.*

Theorem 8. *Every radical class which is hereditary for maximal ideals is also hereditary for ideals of finite index.*

Proof. In the notation of the previous two results, let $(*)$ be the property of maximality. Then we have to show that all finite rings are in $\mathcal{C}(*)$. We do this by combining a structure theorem for finite rings with multiple uses of the lemma. Recall that a finite ring A has a largest nilpotent ideal N and A/N is isomorphic to a finite direct sum of matrix rings over finite fields.

(i)**:** *For each prime p, the zeroring on the cyclic group of order p^n is in $\mathcal{C}(*)$ for $n = 1, 2, 3, \ldots$.*

If a is a group element of order p^n then we have a series

$$0 =< p^n a >\subseteq< p^{n-1}a >\subseteq \cdots \subseteq< pa >\subseteq< a >$$

of cyclic groups with each $< p^{i-1}a > / < p^i a >$ cyclic of order p, so the corresponding series

$$0 =< p^n a >^0\subseteq< p^{n-1}a >^0\subseteq \cdots \subseteq< pa >^0\subseteq< a >^0$$

of ideals of the zeroring $< a >^0$ has simple factors which are in $\mathcal{C}(*)$ whence also $< a >^0\in \mathcal{C}(*)$.

(ii)**:** *Every finite p-zeroring is in* $\mathcal{C}(*)$, as each such ring is a direct sum of finitely many zerorings on cyclic p-groups.

If R is a nilpotent p-ring with $R^n = 0$, we have a series

$$0 = R^n \subseteq R^{n-1} \subseteq R^2 \subseteq R$$

of ideals of R for which all factors are zerorings, so

(iii)**:** *all finite nilpotent p-rings are in* $\mathcal{C}(*)$ *for every prime p.*

Every finite nilpotent ring is a direct sum of nilpotent p-rings for finitely many primes p. Hence

(iv)**:** *every finite nilpotent ring is in* $\mathcal{C}(*)$.

As matrix rings over fields are simple,

(v)**:** *all finite direct sums of matrix rings over finite fields are in* $\mathcal{C}(*)$.

The result now follows from *(iv), (v)* and the lemma.

□

We do not know whether radical classes which are hereditary for maximal ideals must be hereditary for prime ideals or *vice versa*. Adding ideals of finite index to the mix gives us the following result.

Theorem 9. *If a radical class* $\mathcal{R}$ *is hereditary for both prime ideals and ideals of finite index, then* $\mathcal{R}$ *is hereditary for maximal ideals.*

Proof. If $I \triangleleft_m A \in \mathcal{R}$ then A/I simple so is prime or a finite zeroring and either implies that $I \in \mathcal{R}$. □

Theorem 10. *A radical class which is hereditary for ideals of finite index need not be hereditary for prime ideals.*

Proof. Let $\mathcal{R}$ be the upper radical class defined by $\{\mathbb{Z}^0\}$. Since $\mathbb{Z}^0$ is isomorphic to all its non-zero ideals, $\mathcal{R}$ is the class of rings which don't have $\mathbb{Z}^0$ as a homomorphic image or, alternatively, the class of rings with no non-zero homomorphisms to $\mathbb{Z}^0$.

Let $I \triangleleft A \in \mathcal{R}$ with $n(A/I) = 0$ for some integer $n \neq 0$ and suppose $I \notin \mathcal{R}$. Then (using (ADS)!) we have $I/\mathcal{T}^*(I) \triangleleft A/\mathcal{T}^*(I) \in \mathcal{R}$ and every non-zero homomorphism from I to $\mathbb{Z}^0$ must factor through $I/\mathcal{T}^*(I)$ which accordingly is not in $\mathcal{R}$. Consequently we lose no generality by assuming that I is additively torsion-free.

Now $I \cap \mathcal{T}^*(A) = \mathcal{T}^*(I) = 0$. It follows that A has an ideal M which contains $\mathcal{T}^*(A)$ and is maximal with respect to having zero intersection with I. Then

$$I \cong I/I \cap M \cong (I+M)/M \triangleleft^{\bullet} A/M \in \mathcal{R}$$

and

$$|(A/M)/((I+M)/M)| = |A/(I+M)| \leq |A/I| = n.$$

Moreover, as I is in the semi-simple class corresponding to the hereditary radical class $\mathcal{T}^*$, its essential extension A/M is also additively torsion-free. Thus we may assume that A is torsion-free also.

Now $nA \subseteq I$, so $nA \triangleleft I$. Hence there is a surjective homomorphism $f : nA \to \mathbb{Z}^0$ (since only the zero map from I to $\mathbb{Z}^0$ can take nI to 0, $\mathbb{Z}^0$ being torsion-free, and $nI = nA$ as I^+ is a pure subgroup of A^+). Define $g \colon A \to \mathbb{Z}^0$ by setting $g(a) = f(na)$ for all $a \in A$. For all $a,b \in A$ we have $g(a+b) = f(n(a+b)) = f(na+nb) = f(na)+f(nb) = g(a)+g(b)$. As well, $ng(ab) = nf(nab) = f(n^2ab) = f(na \cdot nb) = f(na)f(nb) = 0$ ($\mathbb{Z}^0$ is a zeroring!). But $\mathbb{Z}^0$ is torsion-free, so $g(ab) = 0 = g(a)g(b)$. Thus g is a homomorphism (surjective in fact) from $A \in \mathcal{R}$ to $\mathbb{Z}^0$ and with this contradiction we see that $\mathcal{R}$ is indeed hereditary for ideals of finite index.

Consider now the ring $\mathbb{Z}^0 * \mathbb{Z}$ obtained by the adjunction of the identity of $\mathbb{Z}$ to $\mathbb{Z}^0$. As a non-zero ring with identity, this is in $\mathcal{R}$, but it has $\mathbb{Z}^0$ as a prime ideal. Hence $\mathcal{R}$ is not hereditary for prime ideals. □

The classes $\mathcal{D}_S^*$ with $S \neq \varnothing$ have been useful to us and enjoy many relatively hereditary properties. We end this section by showing that $\mathcal{D}_S^*$ is not hereditary for persistent ideals, though it is radically hereditary.

Proposition 5. *(i) If $A \in \mathcal{D}_S^*$ and I is an idempotent ideal of A, then $I \in \mathcal{D}_S^*$.*
(ii) $\mathcal{D}_S^*$ is radically hereditary.
(iii) $\mathcal{D}_S^*$ is hereditary for semiprime ideals.
(iv) $\mathcal{D}_S^*$ is not hereditary for persistent ideals.

Proof.

(i)**:** If

$$I^2 = I \triangleleft A \in \mathcal{D}_S^*$$

and $x \in I$, Then $x = i_1 j_1 + i_2 j_2 + \cdots + i_n j_n$ for some $i_1, j_1, \ldots, i_n, j_n \in I$. For $m \in S^*$, by the S-divisibility of A there exist $a_1, a_2, \ldots, a_n \in A$ such that $j_1 = ma_1, j_2 = ma_2, \ldots, j_n = ma_n$ and then

$$x = m(i_1 a_1 + i_2 a_2 + \cdots + i_n a_n),$$

where the bracketed term is in I. Thus I is S-divisible.

(ii)**:** This follows from Lemma 16 of [11].

(iii)**:** Like the proof in Example 2.

(iv)**:** The zeroring $\mathbb{Z}(p^\infty)^0$ is in $\mathcal{D}_S^*$, but for $p \in S$, $\mathbb{Z}(p)^0 \cong \{x \in \mathbb{Z}(p^\infty)^0 \colon px = 0\}$ is not, though it is a persistent ideal.

□

5. A USEFUL GROUP

Let V be a two dimensional $\mathbb{Q}$-vector space with basis $\{x,y\}$. We define our group G as a subgroup of V. Let

$$G =< p^{-\infty}x,\ q^{-\infty}y,\ t^{-\infty}(x+y) >,$$

where $< . >$ means "group generated by", p,q and t are distinct primes and $p^{-\infty}$ is shorthand for a list of all negative powers of p and so on.

For a set S of primes, let $\mathbb{Q}(S) = \{\frac{m}{n} : m \in \mathbb{Z}, n \in S^*\}$. When S has a single element p we write $\mathbb{Q}(p)$ rather than $\mathbb{Q}(\{p\})$ and when S is the set of all primes, we write $\mathbb{Q}$ as usual. We shall first examine some radical classes of abelian groups and then pass to radical classes of rings *via* A-radicals.

Clearly $< p^{-\infty}x >$ is p-divisible. On the other hand, routine calculations show that y and $x+y$ have zero p-height. Thus G is not p-divisible, so as it has rank 2 we see that

$$\mathbb{Q}(p) \cong < p^{-\infty}x >= \mathcal{D}_p(G).$$

Moreover, $\mathcal{D}_p(G)$ is a pure subgroup (by direct calculation or reference to Proposition 1.1 of [5]), so $G/ < p^{-\infty}x >=< q^{-\infty}\bar{y},\ t^{-\infty}\bar{y} >$ where $\bar{y}$ is the coset of y. This is torsion-free of rank 1 and some more routine calculation shows that it is isomorphic to $\mathbb{Q}(\{q,t\})$. Summarizing, then, we have an exact sequence

$$0 \to \mathbb{Q}(p) \to G \to \mathbb{Q}(\{q,t\}) \to 0.$$

Let $\mathcal{U}$ denote the lower radical class defined by $\{\mathbb{Q}(p), \mathbb{Q}(\{q,t\})$. Then $G \in \mathcal{U}$, but by arguments like those above for p, we have $\mathbb{Q}(q) \cong \mathcal{D}_q(G)$. Comparing the types of the rational groups concerned, we see that $\mathcal{D}_q(G) \notin \mathcal{U}$.

By considering A-radicals, we can use the foregoing to get an informative example.

Example 5. For $\mathcal{U}$ etc. as above we have

$$G^0 \in \mathcal{U} \text{ but } \mathcal{D}_q^*(G^0) \cong \mathbb{Q}(q) \notin \mathcal{U}^*,$$

so $\mathcal{U}^*$ is not radically hereditary. Furthermore, for primes $s \neq p$ (including q and t) $\mathcal{U}$ contains non-zero cyclic s-groups and therefore all s-groups. Similarly it contains all s-groups for $s \neq q,t$, including p. It follows that $\mathcal{T} \subseteq \mathcal{U}$, where $\mathcal{T}$ is the class of torsion groups. Thus for abelian group radicals, $\mathcal{U}$ is the lower radical class defined by the homomorphically closed class$\{\mathbb{Q}(p), \mathbb{Q}(\{q,t\}\} \cup \mathcal{T}$.

Proposition 6. *For a non-empty set S of primes, every non-zero persistent ideal of* $\mathbb{Q}(S)^0$ *is isomorphic to* $\mathbb{Q}(S)^0$.

Proof. Let $\mathbb{Q}(S)^*$ denote the subring of the rationals whose additive group is $\mathbb{Q}(S)$. If $H^0 \neq 0$ is a persistent ideal of $\mathbb{Q}(S)^0$ then $H^0 \triangleleft \mathbb{Q}(S)^0 * (Q)(S)^*$. But then, for $h \in H$,

$p \in S$ and $n \in \mathbb{Z}^+$ we have (writing elements of $\mathbb{Q}(S)^0 * \mathbb{Q}(S)^*$ as ordered pairs for clarity),

$$\frac{h}{p^n} = (h,0)(0,p^{-n}) \in H^0,$$

so H is S-divisible and therefore has type at least as great of that of $\mathbb{Q}(S)$. (For *types*, see [3].) The types must therefore be equal so that $H \cong \mathbb{Q}(S)$. □

As $\mathcal{T}$ is hereditary, the class $\{\mathbb{Q}(p), \mathbb{Q}(\{q,t\})\} \cup \mathcal{T}$ is hereditary for persistent ideals, but its lower radical class $\mathcal{U}$ is not radically hereditary (and hence not hereditary for persistent ideals). Turning to ring radicals we see that the lower radical class $L(\{H^0 : H \in \mathcal{U}\})$ is not radically hereditary though it is determined by a subclass which is hereditary for persistent ideals. This gives us a two-part theorem.

Theorem 11. *A class of rings which is radically hereditary (respectively hereditary for persistent ideals) may define a lower radical class which is not.*

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