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A NUMERICAL APPROACH FOR SOLVING VOLTERRA DELAY INTEGRO-DIFFERENTIAL EQUATION

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Abstract. In this paper, we propose a finite difference technique for approximating solutions of first-order Volterra delay integro-differential equation. The presented numerical method acquires a second-order convergence in discrete maximum norm. The derived results are numerically validated in test problems to support the theoretical analysis.

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1. Introduction

After Volterra's establishment, the Volterra integro-differential Equations (VIDEs) appeared [13]. It then arise in a variety of physical applications such as diffusion process, heat transfer, glassforming process, neutron diffusion, nanohydrodynamics [24]. Detailed information about VIDEs in literature can be found in biology, engineering and physics applications books [23].

In this work the following Volterra delay integro-differential equation (VDIDE) is being analyzed:

$$u'(t)+a(t)u(t)+b(t)u(t-r)+\int_{t-r}^{t}K(t,s)u(s)ds=f(t), \quad t\in I, \tag{1.1}$$

$$u(t)=\varphi(t), \quad -r\le t\le 0. \tag{1.2}$$

$I=(0,T]=\cup_{p=1}^{m}I_p, \quad I_p=\{t:r_{p-1}<t\le r_p\}, 1\le p\le m$ and $r_s=sr$, for $0\le s\le m, \bar{I}=[0,T]$, $I_0=[-r,0]$. $a(t)\ge 0, f(t), b(t)(t\in\bar{I}), \varphi(t)(t\in I_0)$ and $K(t,s)((t,s)\in \bar{I}\times\bar{I})$ given functions, r is a constant delay. Moreover, we will assume that $a,b,f\in C(\bar{I}), \varphi\in C^2(I_0)$ and $\frac{\partial^2 K}{\partial s^2}\in C(\bar{I}^2)(s=0,1,2)$.

Integral equations (IEs) play an essential role in applied mathematics, since they arise from various biological, physical and engineering problems. They can be used to model various processes occurring in viscoelasticity, elasticity, diffraction problems, hydrodynamics, stochastic processes, epidemic studies [18], scattering in quantum mechanics, dynamical systems, scattering of time-harmonic acoustic waves [20], etc. Many initial value problems (IVPs) and boundary value problems involving ordinary differential equations and partial differential equations can also be rewritten as IEs. Many efforts have been made over time to study the solvability and characteristics of these equations [12], as well as approaches for approximating their solutions [22].

There has been a rise in interest in integro-differential equations (IDEs) in recent years, with various studies focusing on developing more advanced and efficient approaches [21]. The IDEs be an important branch of modern mathematics. Engineering, electrostatics, mechanics, elasticity theory, potential and mathematical physics are just a few of the fields where it commonly occurs.

Researchers have gradually developed an interest in the field of Volterra/Fredholm IEs during the last fifty years and significant efforts have been made to numerically solve them. The numerical solution to IEs can be found using [7]. There were also numerous notable research in the field of IDEs. [11], for example, investigated the numerical solution of non-linear IDEs using the meshless approach.

The Fredholm integro-differential equations (FIDEs) are solved using a different kind of analytical and computational approaches in the literature [8].

Many physical processes of interest are described by delay differential equation (DDE) in biology [9], medicine, chemistry, physics, engineering and economics, among other fields [14]. The initial-boundary value problem for a linear pseudo-parabolic equation (PPE) with delay is addressed in [4]. The stability bounds for the investigated problem are obtained using the method of energy estimates.The one dimensional initial-boundary problem for a PPE with time delay in the second spatial derivative is considered in [1]. In this work constructed a higher order difference approach to solve the problem numerically and obtained an error estimate for its solution. [6] discusses a singularly perturbed IVP for a quasilinear second-order DDE. An exponentially fitted difference scheme constructed in an equidistant mesh with first-order uniform convergence in the discrete maximum norm. The singularly perturbed IVP for a linear first-order VIDE with delay is addressed in [15]. The authors develop and analyze a numerical method with uniform convergence in the small parameter. The differential part of the problem is discretized using implicit difference rules, and the integral part is discretized using composite numerical quadrature rules. Established layer-adapted mesh error estimations for the approximate solution.

In recent years, the numerical solution of VDIDEs has gotten a lot of attention [2]. In [3] the researchers not only examined the stability inequalities of VDIDEs with a given order of derivatives, but also considered the high-order form of the VDIDEs.

For approximating solutions of second-order Fredholm-Volterra IEs, the author of [16] recommends a straightforward numerical approach. The method is based on Picard iteration and uses an appropriate quadrature formula. The author shows the existence and uniqueness of the solution under certain conditions and provides error estimates for the approximation. In terms of Taylor polynomials around any point, [17] presents a Taylor approach for finding the approximate solution to high-order linear Volterra-Fredholm IEs under mixed conditions. [5] give an overview of approximating techniques for VDIDEs and other similar problems.

Singularly perturbed Volterra/Fredholm IDEs appear in many scientific applications. [19] reviewed the literature on Volterra integral and IDEs that are singularly perturbed. The IVP for a quasilinear VDIDE with small parameter is studied in [25]. The method of integral identities is used to design and analyze a fitted difference scheme using exponential basis functions and interpolating quadrature rules with the weight and remainder term in integral form. The authors shown that the method displays first-order uniform convergence in perturbation parameter. [10] provide a comprehensive discussion of several strategies for the Volterra/Fredholm integral or IDEs.

The current article is divided into four sections and deals with a numerical approximation of the first-order VDIDE. In the Introduction, works in the area of IDEs were compared and newer studies were presented. Further, the authors construct the absolutely stable finite difference scheme in an equidistant mesh. The composite middle rectangle rule was used for the integral term of (1.1). The approximate problem's stability bounds are addressed in Section 3. In the discrete maximum norm, the authors shown that the method has a second-order accuracy. The paper concludes with numerical examples.

2. The Construction of Difference Scheme

We introduce the uniform mesh ω_{N_0} on

$$\bar{I} : \omega_{N_0} = \{t_i = ih, i = 1, 2, ..., N_0; h = T/N_0 = r/N\}$$

(for simplicity we suppose that T/r is integer; i.e., $T = mr$), which contains N mesh points at each subinterval $I_p (1 \leq p \leq m)$:

$$\omega_{N,p} = \{t_i : (p-1)N + 1 \leq i \leq pN\}, \quad 1 \leq p \leq m,$$

and consequently

$$\omega_{N_0} = \bigcup_{p=1}^{m} \omega_{N,p}.$$

To simplify the notation, we get $g_i = g(t_i)$ for any function $g(t)$ and y_i represent an approximation of $u(t)$ at t_i. Also, for any mesh function g_i defined on ω_{N_0} we use

$$g_{\bar{t},i} = \frac{g_i - g_{i-1}}{h}, \quad g_i^{(0.5)} = \frac{1}{2}(g_i + g_{i-1}), \quad g_{i-\frac{1}{2}} = g\left(t_{i-\frac{1}{2}}\right) = g\left(t_i - \frac{h}{2}\right)$$

and discrete maximum norms

$$\|g\|_{\infty,p} \equiv \|g\|_{\infty,\omega_{N,p}} = \max_{\bar{\omega}_{N,p}} |g(t)|, \quad \|g\|_{\infty,N_0} \equiv \|g\|_{\infty,\omega_{N_0}}.$$

Setting $t = t_{i-\frac{1}{2}}$ in (1.1) we have

$$u'\left(t_{i-\frac{1}{2}}\right) + a\left(t_{i-\frac{1}{2}}\right) u\left(t_{i-\frac{1}{2}}\right) + b\left(t_{i-\frac{1}{2}}\right) u\left(t_{i-\frac{1}{2}} - r\right)$$
$$+ \int_{t_{i-\frac{1}{2}}-r}^{t_{i-\frac{1}{2}}} K\left(t_{i-\frac{1}{2}}, s\right) u(s)ds = f\left(t_{i-\frac{1}{2}}\right), \qquad (1 \le i \le N_0). \tag{2.1}$$

Next, we use the relations

$$u'\left(t_{i-\frac{1}{2}}\right) = u_{\bar{t},i} + R_i^{(1)}, \qquad R_i^{(1)} = -\frac{h^2}{24} u'''(\xi_i^{(1)}), \quad \xi_i^{(1)} \in (t_{i-1}, t_i), \tag{2.2}$$

$$u\left(t_{i-\frac{1}{2}}\right) = u_i^{(0,5)} + R_i^{(2)}, \qquad R_i^{(2)} = -\frac{h^2}{8} u''(\xi_i^{(2)}), \quad \xi_i^{(2)} \in (t_{i-1}, t_i), \tag{2.3}$$

$$u\left(t_{i-\frac{1}{2}} - r\right) = u_{i-N}^{(0,5)} + R_i^{(3)}, \qquad R_i^{(3)} = -\frac{h^2}{8} u''(\xi_i^{(3)}), \quad \xi_i^{(3)} \in (t_{i-1-N}, t_{i-N}). \tag{2.4}$$

For the integral term of (2.1), after applying composite rectangle rule, we have

$$\int_{t_{i-\frac{1}{2}}-r}^{t_{i-\frac{1}{2}}} K\left(t_{i-\frac{1}{2}}, s\right) u(s)ds = h \sum_{j=i-N+1}^{i-1} K\left(t_{i-\frac{1}{2}}, t_i\right) u_i + R_i^{(4)},$$

where

$$R_i^{(4)} = \frac{h^2 r}{24} \frac{d^2}{ds^2} \left(K\left(t_{i-\frac{1}{2}}, (\xi_i^{(4)}\right) u\left(\xi_i^{(4)}\right)\right), \quad t_{i-\frac{1}{2}} - r < \xi_i < t_{i-\frac{1}{2}}. \tag{2.5}$$

Consequently we get the exact relation for $u(t_i)$

$$u_{\bar{t},i} + a_{i-\frac{1}{2}} u_i^{(0.5)} + b_{i-\frac{1}{2}} u_{i-N}^{(0.5)} + h \sum_{j=i-N+1}^{i-1} K_{i-\frac{1}{2},j} u_j = f_{i-\frac{1}{2}} - R_i, \tag{2.6}$$

with remainder term

$$R_i = R_i^{(1)} + a_{i-\frac{1}{2}} R_i^{(2)} + b_{i-\frac{1}{2}} R_i^{(3)} + R_i^{(4)}, \quad 1 \le i \le N_0,$$

where $R_i^{(k)} (k = 1,2,3,4)$ are defined by (2.2), (2.3), (2.4) and (2.5), respectively. Based on (2.6), we propose the following difference scheme for approximating (1.1)-(1.2):

$$y_{\bar{t},i} + a_{i-\frac{1}{2}} y_i^{(0.5)} + b_{i-\frac{1}{2}} y_{i-N}^{(0.5)} + h \sum_{j=i-N+1}^{i-1} K_{i-\frac{1}{2},j} y_j = f_{i-\frac{1}{2}}, \quad 1 \le i \le N_0, \tag{2.7}$$

$$y_i = \varphi_i, \quad -N \le i \le 0. \tag{2.8}$$

3. Stability Bound and Error Estimate

Lemma 1. *Consider the following difference problem:*

$$v_{\bar{t},i} + A_i v_i^{(0.5)} = F_i, \quad 1 \le i \le N, \tag{3.1}$$

$$v_0 = \mu \tag{3.2}$$

with $t_i = x_0 + ih, \quad h = (X - x_0)/N, A_i \ge 0$. *Let* $| F_i | \le \mathcal{F}_i$ *and* $\mathcal{F}_i$ *a nondecreasing function. Then the solution of* (3.1)-(3.2) *satisfies*

$$|v_i| \le |\mu| + (X - x_0)\,\mathcal{F}_i, \quad 1 \le i \le N. \tag{3.3}$$

Proof. The difference equation can be written as

$$v_i = \frac{2 - hA_i}{2 + hA_i} v_{i-1} + \frac{2h}{2 + hA_i} F_i.$$

From here, it follows that

$$|v_i| \le v_{i-1} + h|F_i|.$$

Therefore,

$$|v_i| \le |v_0| + h \sum_{j=1}^{i} |F_j| \le |v_0| + ih\mathcal{F}_i,$$

which implies the validity of (3.3). □

Let $z_i = y_i - u_i$. Then the error of approximate solution z_i by (2.7) and (2.8) will be

$$z_{\bar{t},i} + a_{i-\frac{1}{2}} z_i^{(0.5)} + b_{i-\frac{1}{2}} z_{i-N}^{(0.5)} + h \sum_{j=i-N+1}^{i-1} K_{i-\frac{1}{2},j} z_j = R_i, \quad 1 \le i \le N_0, \tag{3.4}$$

$$z_i = 0, \quad -N \le i \le 0. \tag{3.5}$$

Lemma 2. *The error function* z_i *will satisfy*

$$\|z\|_{\infty,N_0} \le \frac{\left[1 + r\left(\|b\|_{\infty,\bar{I}} + T\bar{K}\right) e^{r^2 T\bar{K}}\right]^m - 1}{\|b\|_{\infty,\bar{I}} + T\bar{K}} \|R\|_{\infty,N_0}, \tag{3.6}$$

where $\bar{K} = \max\limits_{\bar{I}\times\bar{I}} |K(t,s)|$.

Proof. For $t_i \in \omega_{N,p}$, we have

$$|R_i - b_{i-\frac{1}{2}} z_{i-N}^{(0.5)} - h \sum_{j=i-N+1}^{i-1} K_{i-\frac{1}{2},j} z_j| \le \|R\|_{\infty,N,p} + \|b\|_{\infty,\bar{I}} \|z\|_{\infty,p-1} + h\bar{K} \sum_{j=i-N+2}^{i} |z_{j-1}|$$

$$\le \|R\|_{\infty,p} + \|b\|_{\infty,\bar{I}} \|z\|_{\infty,p-1} + T\bar{K}\|z\|_{\infty,p-1} + h\bar{K} \sum_{j=(p-1)N+1}^{i} |z_{j-1}|$$

$$\leq \|R\|_{\infty,\omega_{N_0}} + (\|b\|_\infty + T\bar{K})\,\|z\|_{\infty,p-1} + h\bar{K}\sum_{j=(p-1)N+1}^{i}|z_{j-1}|.$$

Then after applying Lemma 1 to (3.4) on $\omega_{N,p}$ we get

$$|z_i| \leq |z_{(p-1)N}| + r\|R\|_{\infty,N_0} + r\left(\|b\|_{\infty,\bar{I}+T\bar{K}}\right)\|z\|_{\infty,p-1} + rh\bar{K}\sum_{j=(p-1)N+1}^{i}|z_{j-1}|,$$

so

$$|z_i| \leq r\|R\|_{\infty,N_0} + \left(1 + r\|b\|_{\infty,\bar{I}+T\bar{K}}\right)\|z\|_{\infty,p-1} + rh\bar{K}\sum_{j=(p-1)N+1}^{i}|z_{j-1}|.$$

From here, by using the difference analogue of Gronwall's inequality, we arrive at

$$|z_i| \leq re^{\bar{K}r^2t_i}\|R\|_{\infty,N_0} + \left[1 + r\left(\|b\|_{\infty,\bar{I}} + T\bar{K}\right)\right]e^{\bar{K}r^2t_i}\|z\|_{\infty,p-1}.$$

Therefore we have the first-order difference inequality of the form

$$\|z\|_{\infty,p} \leq A\|z\|_{\infty,p-1} + B \tag{3.7}$$

with

$$\begin{aligned} A &= 1 + r\left(\|b\|_{\infty,\bar{I}} + T\bar{K}\right)e^{r^2T\bar{K}}, \\ B &= re^{r^2T\bar{K}}\|R\|_{\infty,N_0}. \end{aligned} \tag{3.8}$$

Resolving (3.7), by taking into consideration (3.8) we arrive at

$$\|z\|_{\infty,p} \leq \frac{A^p - 1}{A - 1}B, \quad 1 \leq p \leq m,$$

which proves (3.6). □

Lemma 3. *Let be fulfilled the assumptions above and $u \in C^3[0,T]$. Then for the truncation error R_i, the following estimate holds:*

$$\|R\|_{\infty,\omega_{N_0}} \leq Ch^2.$$

Proof. From explicit expressions of $R_i^{(k)}$ $(k = 1,2,3,4)$, it is not hard to observe that $\|R^{(k)}\|_{\infty,\omega_{N_0}} \leq Ch^2$. This, along with (2.2), (2.3), (2.4) and (2.5) give the desired result. □

Theorem 1. *Let u the solution of* (1.1)-(1.2) *and y the solution of* (2.7)-(2.8)*. Under the assuming conditions the solution of difference problem* (2.7)-(2.8) *second-order uniformly convergent to the solution of* (1.1)-(1.2)*:*

$$\|y - u\|_{\infty,\bar{\omega}_{N_0}} \leq Ch^2.$$

Proof. It is evidently from Lemma 2 and Lemma 3. □

4. Numerical Results

In this section, we present a numerical results which illustrate the presented method.

Example 1. Consider the test problem:

$$u'(t)+\frac{2}{3}u(t)-\frac{1}{3}\int_{t-1}^{t}u(s)ds=-\frac{1}{3}+\frac{1}{3}t-\frac{1}{2}(1-e^{-\frac{2t}{3}}),\quad t\in(0,2],$$
$$u(t)=1,\quad -1\le t\le 0.$$

The exact solution is

$$u(t)=\begin{cases} e^{-\frac{2}{3}t}, & 0\le t\le 1,\\ \frac{e^{\frac{1}{3}(t-1)}-e^{1-t}}{2}-1+e^{-\frac{2}{3}t}+e^{-\frac{2}{3}(1-t)}, & 1\le t\le 2.\end{cases}$$

We define the exact errors

$$e^N=\|y-u\|_{\infty,\bar{\omega}_{N_0}},$$

where y is the approximate solution of u for the various values of N. Experimental rates of convergence are computed by

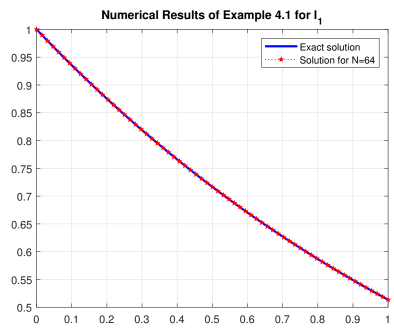
$$p^N=\frac{\ln\left(e^N/e^{2N}\right)}{\ln 2}.$$

From Table 1 we observe that the experimental rate of convergence is monotonically increasing towards 2, so in agreement with the theoretical analysis from Theorem 1.

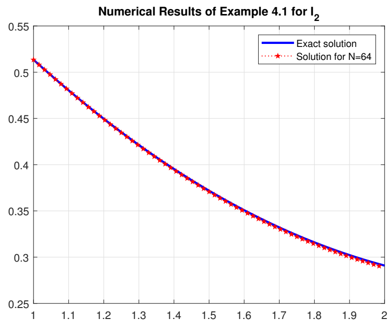
TABLE 1. The numerical results for the test problem.

$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
0.031121	0.008514	0.002250	0.000572	0.000144
1.87	1.92	1.98	1.99	

The graph of the exact and approximation solution of Example 1 in the interval I_1 is given in the graphs below:



The graph of the exact and approximation solution of Example 1 in the interval I_2 is given in the graphs below:



Example 2. Consider the IVP

$$u'(t) + tu(t) + u(t-1) + \int_{t-1}^{t} e^{ts} ds = \sin\frac{\pi t}{4}, \quad t \in (0,2],$$

$$u(t) = \sqrt{1-t}, \quad -1 \le t \le 0,$$

whose exact solution is not available. Using the double mesh method, we evaluate errors in the computed solution and convergence rates. We calculate another approximate solution y^{2N} on a mesh that is obtained by uniformly bisecting the original mesh $\bar{\omega}_{N_0}$. We calculate the errors for various values of N by

$$E^N = \left\| y^N - y^{2N} \right\|_{\infty, \bar{\omega}_{N_0}}.$$

Furthermore, the experimental convergence rates are calculated in the same way as in Example 2.

TABLE 2. The numerical results for the test problem.

$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
0.048089	0.013620	0.003675	0.000958	0.000243
1.82	1.89	1.94	1.98	

5. CONCLUSION

We have considered the numerical approximation of first-order VDID problem. In discrete maximum norm, the method has a second-order accuracy. The obtained theoretical results have been tested on particular problems. The method given here can be used to solve more complex linear and nonlinear VDIDEs.

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