Miskolc Mathematical Notes
Vol. 26 (2025), No. 2, pp. 661–673

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2025.4557

INVERSE SPECTRAL PROBLEM FOR STURM-LIOUVILLE OPERATOR WITH HIGHER ORDER POLYNOMIALS OF SPECTRAL PARAMETER IN THE BOUNDARY CONDITION

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Received 28 January, 2023

Abstract. This paper is concerned with an inverse scattering problem for Sturm-Liouville operator with polynomials of spectral parameter in boundary condition. We provide scattering data and present some spectral properties. In order to study inverse scattering problem, the main equation is derived. As a consequence, we show that the potential can be recovered uniquely from scattering data.

2010 *Mathematics Subject Classification:* 34L25; 34B24; 34B07; 34A55

Keywords: inverse problem, scattering data, Sturm-Liouville operator, nonlinear spectral parameter, main equation

1. INTRODUCTION

Inverse problems in spectral theory often appear in mathematics, mechanics, physics and other branches of natural sciences. The inverse scattering problem deals with the recovery of potential from their spectral characteristics, and besides, problems involving spectral parameter in equation and boundary conditions are an important part of spectral theory. Sturm-Liouville problems arise in studies of heat conduction problems and vibrating string problems when boundary conditions depend on spectral parameter. Cohen [2] presented a method to solve an initial boundary value problem arising in the theory of diffusion and heat flow. Many examples of spectral problems which appear in mechanical engineering and contain eigen parameter in the boundary conditions were presented in the book [5].

In this paper we consider the boundary value problem generated by Sturm-Liouville operator and spectral parameter dependent boundary condition:

$$\ell u(x) := -u''(x) + \left\{q(x) - \lambda^2\right\} u(x) = 0, \qquad (0 \le x < \infty), \tag{1.1}$$

$$V(u) := \sum_{k=0}^{n} \left(\alpha_k u'(0) - \beta_k u(0)\right)(i\lambda)^k = 0, \tag{1.2}$$

where λ is a spectral parameter, $q(x)$ is a real valued function with the condition

$$\int_0^\infty (1+x)\,|q(x)|\,dx < \infty \tag{1.3}$$

and for $\alpha_i, \beta_j \in \mathbb{R}$ $(i, j = 1, \ldots, n)$

$$\begin{vmatrix} \alpha_i & \alpha_{i+2} \\ \beta_i & \beta_{i+2} \end{vmatrix} \geq 0, \quad \begin{vmatrix} \alpha_i & \alpha_{i+k} \\ \beta_i & \beta_{i+k} \end{vmatrix} = 0, \quad k \neq 2. \tag{1.4}$$

Marchenko [18] and Levitan [11] studied inverse scattering problem of the Sturm-Liouville operator on the half line with a boundary condition at the origin when there is no spectral parameter. Inverse problem for second order differential operator pencil on the axis was studied in [6–8, 20]. Problems with spectral parameter dependent boundary condition were examined in finite interval by several authors in [1,9,10,16, 17,21,22] and on the half line in [3,4,12–15,19].

The aim of this paper is to present inverse scattering problem for Sturm-Liouville operator with polynomials of spectral parameter in the boundary condition. In this view, we will extend the Marchenko method to the boundary value problem (1.1)-(1.3), which also introduces a process that can be applied to various other inverse problems where boundary condition contains polynomials of spectral parameter.

The remaining paper is organized as follows: In section (2), the scattering data corresponding to the boundary value problem (1.1)-(1.3) is provided and its properties are investigated. In Section (3), the main equation required for recovering potential uniquely is derived. Finally, we give the uniqueness of the solution of the main equation and the reconstruction of the potential of equation (1.1) in Section (4).

2. Scattering Data

The direct scattering problem concerns the determination of spectral characteristics for the boundary value problem when $q(x)$ is known. Therefore, we shall provide scattering data for (1.1)-(1.3) and analyze its properties.

It is well known in [18] that the equation (1.1) has a unique solution $e(\lambda, x)$ which satisfies the asymptotic behavior, for $\operatorname{Im}\lambda \geq 0$,

$$\lim_{x\to+\infty} e^{-i\lambda x} e(\lambda, x) = 1,$$

when the condition (1.3) is satisfied. It is called Jost solution and can be expressed by

$$e(\lambda, x) = e^{i\lambda x} + \int_x^\infty K(x,t)\, e^{i\lambda t}\, dt, \tag{2.1}$$

where the kernel function $K(x,t)$ satisfies the estimate

$$|K(x,t)| \le \frac{1}{2}\Omega\left(\frac{x+t}{2}\right)\exp\left\{\Omega_1(x)-\Omega_1\left(\frac{x+t}{2}\right)\right\}$$

and the functions $\Omega(x)$ and $\Omega_1(x)$ have the following notations:

$$\Omega(x) \equiv \int_x^\infty |q(t)|\,dt, \qquad \Omega_1(x) \equiv \int_x^\infty \Omega(x)\,dt.$$

In addition,

$$K(x,x) = \frac{1}{2}\int_x^\infty q(t)\,dt. \tag{2.2}$$

Moreover, $e(\lambda,x)$ is an analytic function of λ in the upper half plane ($\operatorname{Im}\lambda > 0$) and is continuous on the real line. The following estimates hold on the half plane $\operatorname{Im}\lambda \ge 0$:

$$|e(\lambda,x)| \le \exp\{-\operatorname{Im}\lambda x + \Omega_1(x)\},$$

$$\left|e(\lambda,x) - e^{i\lambda x}\right| \le \left\{\Omega_1(x) - \Omega_1\left(x+\frac{1}{|\lambda|}\right)\right\}\exp\{-\operatorname{Im}\lambda x + \Omega_1(x)\}, \tag{2.3}$$

$$\left|e'(\lambda,x) - i\lambda e^{i\lambda x}\right| \le \Omega(x)\exp\{-\operatorname{Im}\lambda x + \Omega_1(x)\}. \tag{2.4}$$

Let $W[f,g] = f'g - g'f$ denote the Wronskian. For real $\lambda \ne 0$, the functions $e(\lambda,x)$ and $\overline{e(\lambda,x)}$ form a fundamental system of solutions of equation (1.1) and their Wronskian is independent of x and equal to $2i\lambda$.

Let

$$p_1(\lambda) := \sum_{k=0}^{n}\alpha_k(i\lambda)^k, \qquad p_2(\lambda) := \sum_{k=0}^{n}\beta_k(i\lambda)^k$$

and denote by $\phi(\lambda,x)$ the solution of (1.1) satisfying the conditions

$$\phi(\lambda,0) = p_1(\lambda), \qquad \phi'(\lambda,0) = p_2(\lambda).$$

It is evident that the solution $\phi(\lambda,x)$ holds the boundary condition (1.2).

After stating the above preliminaries, we can now present the following lemmas.

Lemma 1. *The following identity is valid:*

$$\frac{2i\lambda\phi(\lambda,x)}{\Theta(\lambda)} = \overline{e(\lambda,x)} - S(\lambda)e(\lambda,x) \tag{2.5}$$

for all real $\lambda \ne 0$ *and*

$$S(\lambda) = \frac{p_1(\lambda)\overline{e'(\lambda,0)} - p_2(\lambda)\overline{e(\lambda,0)}}{p_1(\lambda)e'(\lambda,0) - p_2(\lambda)e(\lambda,0)}, \tag{2.6}$$

$$S(\lambda) = \overline{S(-\lambda)}, \qquad |S(\lambda)| = 1, \tag{2.7}$$

where

$$\Theta(\lambda) = p_1(\lambda) e'(\lambda,0) - p_2(\lambda) e(\lambda,0). \tag{2.8}$$

Proof. Because of that $e(\lambda,x)$ and $\overline{e(\lambda,x)}$ constitute the fundamental set of solutions of (1.1) for the real $\lambda \neq 0$, the function $\phi(\lambda,x)$ can be represented as a linear combination of these functions:

$$\phi(\lambda,x) = c_1(\lambda) e(\lambda,x) + c_2(\lambda) \overline{e(\lambda,x)},$$

where $c_1(\lambda)$ and $c_2(\lambda)$ are provided by evaluating the values of the Wronskians:

$$W[e(\lambda,x), \phi(\lambda,x)] = c_2(\lambda) 2i\lambda = p_1(\lambda) e'(\lambda,0) - p_2(\lambda) e(\lambda,0),$$
$$W\left[\overline{e(\lambda,x)}, \phi(\lambda,x)\right] = -c_1(\lambda) 2i\lambda = p_1(\lambda) \overline{e'(\lambda,0)} - p_2(\lambda) \overline{e(\lambda,0)}.$$

Thus, we have

$$\phi(\lambda,x) = -\frac{p_1(\lambda)\overline{e'(\lambda,0)} - p_2(\lambda)\overline{e(\lambda,0)}}{2i\lambda} e(\lambda,x) + \frac{p_1(\lambda) e'(\lambda,0) - p_2(\lambda) e(\lambda,0)}{2i\lambda} \overline{e(\lambda,x)}. \tag{2.9}$$

Let us define the function $\Theta(\lambda)$ in the form (2.8). Now, it is necessary to show $\Theta(\lambda) \neq 0$ for all real $\lambda \neq 0$. Assume the contrary, then there exists $\lambda_0 \in \mathbb{R}$, $\lambda_0 \neq 0$, such that

$$p_1(\lambda_0) e'(\lambda_0,0) = p_2(\lambda_0) e(\lambda_0,0).$$

Also,

$$W\left[e(\lambda_0,0), \overline{e(\lambda_0,0)}\right] = 2i\lambda_0$$

is satisfied. By using these relations, we obtain

$$\frac{|e(\lambda_0,0)|^2}{|p_1(\lambda_0)|^2} \sum_{j=0}^{n-2} (\alpha_j \beta_{j+2} - \alpha_{j+2}\beta_j) |\lambda_0|^{2j} \left(\overline{\lambda_0}^2 - \lambda_0^2\right) = 2i\lambda_0$$

and so $0 = 2i\lambda_0$, yielding contradiction since $\lambda_0 \neq 0$. Thus, by dividing equality (2.9) by $\frac{1}{2i\lambda}\Theta(\lambda)$, we obtain (2.5) where $S(\lambda)$ is defined with (2.6).

To complete proof, we need to show that $S(\lambda)$ satisfies the conditions in (2.7). Let us use the notation $\Theta_1(\lambda)$ for the numerator of $S(\lambda)$. Since $q(x)$ is real, it follows that $\overline{e(\lambda,0)} = e(-\lambda,0)$ for real $\lambda \neq 0$. Also, $\overline{p_1(\lambda)} = p_1(-\lambda)$ and $\overline{p_2(\lambda)} = p_2(-\lambda)$ hold for real $\lambda \neq 0$. Hence $\overline{\Theta(\lambda)} = \Theta(-\lambda)$ and $\overline{\Theta_1(\lambda)} = \Theta_1(-\lambda)$, which yields $S(\lambda) = \overline{S(-\lambda)}$.

By (2.6), we get for all real $\lambda \neq 0$

$$|S(\lambda)|^2 = S(\lambda) \cdot \overline{S(\lambda)}$$

$$= \frac{|p_1(\lambda)|^2 \cdot |e'(\lambda,0)|^2 + |p_2(\lambda)|^2 \cdot |e(\lambda,0)|^2 - 2\mathrm{Re}\left(p_1(\lambda) \cdot \overline{p_2(\lambda)} \cdot e(\lambda,0) \cdot \overline{e'(\lambda,0)}\right)}{|p_1(\lambda)|^2 \cdot |e'(\lambda,0)|^2 + |p_2(\lambda)|^2 \cdot |e(\lambda,0)|^2 - 2\mathrm{Re}\left(p_1(\lambda) \cdot \overline{p_2(\lambda)} \cdot e'(\lambda,0) \cdot \overline{e(\lambda,0)}\right)}.$$

Since the condition (1.4) holds and λ is real, we obtain

$$\begin{aligned}
&2\mathrm{Re}\left(p_1(\lambda) \cdot \overline{p_2(\lambda)} \cdot e'(\lambda,0) \cdot \overline{e(\lambda,0)}\right) - 2\mathrm{Re}\left(p_1(\lambda) \cdot \overline{p_2(\lambda)} \cdot e(\lambda,0) \cdot \overline{e'(\lambda,0)}\right) \\
&\quad = p_1(\lambda) \cdot \overline{p_2(\lambda)} \cdot e'(\lambda,0) \cdot \overline{e(\lambda,0)} - p_1(\lambda) \cdot \overline{p_2(\lambda)} \cdot e(\lambda,0) \cdot \overline{e'(\lambda,0)} \\
&\quad\quad - \overline{p_1(\lambda)} \cdot p_2(\lambda) \cdot e'(\lambda,0) \cdot \overline{e(\lambda,0)} + \overline{p_1(\lambda)} \cdot p_2(\lambda) \cdot e(\lambda,0) \cdot \overline{e'(\lambda,0)} \\
&\quad = \left[p_1(\lambda) \cdot \overline{p_2(\lambda)} - p_2(\lambda) \cdot \overline{p_1(\lambda)}\right] \cdot \left[e'(\lambda,0) \cdot \overline{e(\lambda,0)} - e(\lambda,0) \cdot \overline{e'(\lambda,0)}\right] \\
&\quad = \sum_{j=0}^{n-2} (\alpha_j \beta_{j+2} - \alpha_{j+2}\beta_j) |\lambda|^{2j} \left(\lambda^2 - \overline{\lambda}^2\right) \cdot W\left[e(\lambda,0), \overline{e(\lambda,0)}\right] \\
&\quad = 0 \cdot 2i\lambda = 0.
\end{aligned}$$

By combining the results, it follows that $|S(\lambda)|^2 = 1$, i.e., $|S(\lambda)| = 1$ for all real $\lambda \neq 0$.
The proof is completed. □

The function $S(\lambda)$ defined by (2.6) is called the *scattering function* of the boundary value problem (1.1)-(1.3).

Lemma 2. *The function $\Theta(\lambda)$ may have only a finite number of zeros on the half plane* ($\mathrm{Im}\,\lambda > 0$). *All the zeros are simple and lie on the imaginary axis.*

Proof. By the proof of Lemma 1, we have $\Theta(\lambda) \neq 0$ for all real $\lambda \neq 0$ and the point $\lambda = 0$ is the possible zero of $\Theta(\lambda)$. Since $e(\lambda,0)$ and $e'(\lambda,0)$ are analytic in the upper plane $\mathrm{Im}\,\lambda > 0$, it follows from the expression $\Theta(\lambda)$ that it has the same property. We now show that the set of zeros is bounded. Assume the contrary, i.e. let the set of zeros be unbounded. Then, there exists the numbers λ_k such that $\Theta(\lambda_k) = 0$ for $\mathrm{Im}\,\lambda_k > 0$ and $|\lambda_k| \to \infty$, and hence we have

$$e'(\lambda_k,0) = \frac{p_2(\lambda_k)}{p_1(\lambda_k)} e(\lambda_k,0).$$

With the help of (2.4)

$$\left|\frac{p_2(\lambda_k)}{p_1(\lambda_k)} e(\lambda_k,0) - i\lambda_k\right| \leq \Omega(0) \exp\{\Omega_1(0)\},$$

we have

$$|\lambda_k| \leq \left|\frac{p_2(\lambda_k)}{p_1(\lambda_k)} e(\lambda_k,0)\right| + \Omega(0) \exp\{\Omega_1(0)\}.$$

Since the equality (2.3) and $\lim_{k\to+\infty} e(\lambda_k,0) = 1$, the right side of the equality has a finite limit. We arrived at the contradiction, which yields that the set $\{\lambda_k\}$ is bounded.

Thus, the set of zeros of $\Theta(\lambda)$ is bounded and form at most countable set having zero the only possible limit point.

To show that zeros of $\Theta(\lambda)$ lie on the imaginary axis, we assume that λ_1 and λ_2 are zeros of $\Theta(\lambda)$, which follows that they satisfy the equation (1.1):

$$-e''(\lambda_1,x)+q(x)e(\lambda_1,x)=\lambda_1{}^2e(\lambda_1,x), \tag{2.10}$$

$$-\overline{e''(\lambda_2,x)}+q(x)\overline{e(\lambda_2,x)}=\overline{\lambda_2{}^2}\overline{e(\lambda_2,x)}. \tag{2.11}$$

Let us multiply (2.10) by $\overline{e(\lambda_2,x)}$ and (2.11) by $e(\lambda_1,x)$, subtract the second from the first and finally integrate this result according to x from 0 to infinity. As a result, we obtain

$$\left(\lambda_1{}^2-\overline{\lambda_2}^2\right)\int_0^\infty e(\lambda_1,x)\overline{e(\lambda_2,x)}\mathrm{d}x-W\left[e(\lambda_1,x),\overline{e(\lambda_2,x)}\right]\Big|_{x=0}=0. \tag{2.12}$$

On the other hand, since λ_j's $(j=1,2)$ are zeros of $\Theta(\lambda)$, we have

$$W\left[e(\lambda_1,x),\overline{e(\lambda_2,x)}\right]\Big|_{x=0}=\left(\frac{p_2(\lambda_1)}{p_1(\lambda_1)}-\frac{\overline{p_2(\lambda_2)}}{\overline{p_1(\lambda_2)}}\right)e(\lambda_1,0)\overline{e(\lambda_2,0)}.$$

When we take $\lambda_1=\lambda_2=\mu$ and substitute in (2.12), the following result is obtained

$$\left(\mu^2-\overline{\mu}^2\right)\left[\frac{|e(\mu,0)|^2}{|p_1(\mu)|^2}\sum_{m=0}^{n-2}\left(\alpha_m\beta_{m+2}-\alpha_{m+2}\beta_m\right)|\mu|^{2m}+\int_0^\infty|e(\mu,x)|^2\,\mathrm{d}x\right]=0.$$

Thanks to the condition (1.4), the expression in the parentheses is positive, which implies $\mu^2=\overline{\mu}^2$. Since μ isn't real, we conclude that μ is pure imaginary, i.e. $\mu=i\lambda$, where $\lambda\geq 0$.

Next, let us prove that there are only finitely many zeros. Let δ denote the infimum of the distances between two neighboring zeros of $\Theta(\lambda)$ and show $\delta>0$. Assume the opposite and let $\{i\lambda_k\}$ and $\left\{i\widehat{\lambda}_k\right\}$ be two sequences of zeros of the function $\Theta(\lambda)$ such that

$$\lim_{k\to\infty}\left(\widehat{\lambda}_k-\lambda_k\right)=0,\quad 0<\lambda_k\leq\widehat{\lambda}_k,\quad \max_k\widehat{\lambda}_k<M.$$

It follows from the estimate (2.3) that, for A large enough, the inequality

$$e(i\lambda,x)>\frac{1}{2}e^{-\lambda x}$$

is satisfied uniformly with respect to $x\in[A,\infty)$ and $\lambda\in[0,\infty)$. Thus, we obtain

$$\int_A^\infty e\left(i\widehat{\lambda}_k,x\right)\overline{e(i\lambda_k,x)}\mathrm{d}x>\frac{1}{4}\frac{e^{-A\left(\widehat{\lambda}_k+\lambda_k\right)}}{\left(\widehat{\lambda}_k+\lambda_k\right)}>\frac{e^{-2AM}}{8M}. \tag{2.13}$$

On the other hand, the equality (2.12) yields

$$0=\int_0^{\infty} e\left(i\widehat{\lambda}_k,x\right)\overline{e\left(i\lambda_k,x\right)}\mathrm{d}x+\frac{e\left(i\widehat{\lambda}_k,0\right)\overline{e\left(i\lambda_k,0\right)}}{\overline{p_1\left(i\lambda_k\right)}p_1\left(i\widehat{\lambda}_k\right)}\sum_{m=0}^{n-2}\left(\alpha_m\beta_{m+2}-\alpha_{m+2}\beta_m\right)\left(\widehat{\lambda}_k\lambda_k\right)^m$$

$$=\int_0^{A} e\left(i\widehat{\lambda}_k,x\right)\left[\overline{e\left(i\lambda_k,x\right)}-\overline{e\left(i\widehat{\lambda}_k,x\right)}\right]\mathrm{d}x+\int_0^{A} e\left(i\widehat{\lambda}_k,x\right)\overline{e\left(i\widehat{\lambda}_k,x\right)}\mathrm{d}x$$

$$+\int_A^{\infty} e\left(i\widehat{\lambda}_k,x\right)\overline{e\left(i\lambda_k,x\right)}\mathrm{d}x+\frac{e\left(i\widehat{\lambda}_k,0\right)\overline{e\left(i\lambda_k,0\right)}}{\overline{p_1\left(i\lambda_k\right)}p_1\left(i\widehat{\lambda}_k\right)}\sum_{m=0}^{n-2}\left(\alpha_m\beta_{m+2}-\alpha_{m+2}\beta_m\right)\left(\widehat{\lambda}_k\lambda_k\right)^m$$

and letting $k\to\infty$, we get

$$\lim_{k\to\infty}\int_A^{\infty} e\left(i\widehat{\lambda}_k,x\right)\overline{e\left(i\lambda_k,x\right)}\mathrm{d}x\le 0 \tag{2.14}$$

since

$$\lim_{k\to\infty}\left[\overline{e\left(i\lambda_k,x\right)}-\overline{e\left(i\widehat{\lambda}_k,x\right)}\right]=0$$

holds uniformly with respect to $x\in[0,A]$. Comparing (2.13) and (2.14), we obtain a contradiction, yielding $\delta>0$ and hence we conclude that the function $\Theta(\lambda)$ has only a finite number of zeros: $i\lambda_k,\ k=1,\dots,N$.

Finally, let us show that all zeros of the function $\Theta(\lambda)$ are simple. By the derivation of the equation

$$-e''(\lambda,x)+q(x)e(\lambda,x)=\lambda^2 e(\lambda,x) \tag{2.15}$$

with respect to λ, we get

$$-\dot{e}''(\lambda,x)+q(x)\dot{e}(\lambda,x)=\lambda^2\dot{e}(\lambda,x)+2\lambda e(\lambda,x), \tag{2.16}$$

where the dot shows differentiation with respect to λ. Multiplying (2.15) by $\dot{e}(\lambda,x)$ and (2.16) by $e(\lambda,x)$ and subtracting the second from the first and integrating this relation with respect to x over $(0,\infty)$, we obtain

$$2\lambda\int_0^{\infty}\left[e(\lambda,x)\right]^2\mathrm{d}x+W\left[e(\lambda,x),\dot{e}(\lambda,x)\right]\Big|_{x=0}=0.$$

Let λ be a zero of the function $\Theta(\lambda)$. With the help of the expression of the function $\Theta(\lambda)$, the result is found that

$$W\left[e(\lambda,x),\dot{e}(\lambda,x)\right]\Big|_{x=0}$$

$$= -\frac{\dot{\Theta}(\lambda)\, e(\lambda,0)}{p_1(\lambda)} - \left(\frac{e(\lambda,0)}{p_1(\lambda)}\right)^2 i \sum_{k=1}^{n} k \sum_{m=0}^{n-k} (\alpha_m \beta_{m+k} - \alpha_{m+k}\beta_m)\,(i\lambda)^{2m+k-1}$$

and hence we conclude that

$$2\lambda \int_0^\infty [e(\lambda,x)]^2\, dx + \left(\frac{e(\lambda,0)}{p_1(\lambda)}\right)^2 2\lambda \sum_{m=0}^{n-2} (\alpha_m \beta_{m+2} - \alpha_{m+2}\beta_m)\,(i\lambda)^{2m}$$
$$= \frac{\dot{\Theta}(\lambda)\, e(\lambda,0)}{p_1(\lambda)}. \tag{2.17}$$

Substituting $\lambda = i\mu_k$, $\mu_k > 0$, in (2.17) and multiplying $-i$, we find that the left side of the equality is positive. Thus $\dot{\Theta}(i\mu_k) \neq 0$, i.e. the zeros of $\Theta(\lambda)$ are simple, which completes the proof. □

The numbers m_k, $k = 1,\ldots,N$ are defined by

$$m_k^{-2} \equiv \int_0^\infty |e(i\lambda_k,x)|^2\, dx + \frac{|e(i\lambda_k,0)|^2}{|p_1(i\lambda_k)|^2} \sum_{j=0}^{n-2} (\alpha_j \beta_{j+2} - \alpha_{j+2}\beta_j)\,\lambda_k^{2j}$$

and called *norming numbers* for the boundary value problem (1.1)-(1.3).

Definition 1. The collection $\left\{S(\lambda), \{i\lambda_k\}_{k=1}^N, \{m_k\}_{k=1}^N\right\}$ is called the scattering data of the boundary value problem (1.1)-(1.3).

Using (2.1) and substituting related expressions into the function $\Theta(\lambda)$, we get

$$\Theta(\lambda) = p_1(\lambda)\left\{i\lambda - K(0,0) + \int_0^\infty K_x(0,t)\, e^{i\lambda t}\, dt\right\} - p_2(\lambda)\left\{1 + \int_0^\infty K(0,t)\, e^{i\lambda t}\, dt\right\}$$
$$= (i\lambda)^{n+1}\left[\alpha_n + O\left(\frac{1}{\lambda}\right)\right]$$

if $\alpha_n \neq 0$ as $|\lambda| \to \infty$. In a similar way, we obtain

$$\Theta_1(\lambda) = p_1(\lambda)\left\{-i\lambda - K(0,0) + \int_0^\infty K_x(0,t)\, e^{-i\lambda t}\, dt\right\}$$
$$- p_2(\lambda)\left\{1 + \int_0^\infty K(0,t)\, e^{-i\lambda t}\, dt\right\} = (i\lambda)^{n+1}\left[-\alpha_n + O\left(\frac{1}{\lambda}\right)\right].$$

Taking these into account, we deduce that

$$-1 - S(\lambda) = O\left(\frac{1}{\lambda}\right), \qquad |\lambda| \to \infty.$$

In the case that $\alpha_n = 0$, we find that

$$\Theta(\lambda) = (i\lambda)^n \left[\alpha_{n-1} - \beta_n + O\left(\frac{1}{\lambda}\right)\right]$$

and

$$\Theta_1(\lambda) = (i\lambda)^n \left[-\alpha_{n-1} - \beta_n + O\left(\frac{1}{\lambda}\right)\right],$$

as $|\lambda| \to \infty$. Therefore, the following result is obtained

$$\frac{\beta_n + \alpha_{n-1}}{\beta_n - \alpha_{n-1}} - S(\lambda) = O\left(\frac{1}{\lambda}\right), \qquad |\lambda| \to \infty.$$

Let us define

$$S(\infty) = \begin{cases} \frac{\beta_n + \alpha_{n-1}}{\beta_n - \alpha_{n-1}}, & \alpha_n = 0, \\ -1, & \alpha_n \neq 0, \end{cases}$$

$S(\infty) - S(\lambda) \in L_2(-\infty, \infty)$ and hence the function

$$F_S(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (S(\infty) - S(\lambda)) e^{i\lambda x} d\lambda,$$

also belongs to the space $L_2(-\infty, \infty)$.

3. The main equation

In this section, we shall derive the main equation which is extremely important for discussions of inverse scattering problem.

Theorem 1. *For every fixed $x \geq 0$, the kernel $K(x,t)$ to the solution (2.1) satisfies the integral equation*

$$F(x+y) + K(x,y) + \int_x^{\infty} K(x,t) F(t+y)\, dt = 0, \qquad y > x, \tag{3.1}$$

where

$$F(x) = \sum_{k=1}^{N} m_k^2 e^{-\lambda_k x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} (S(\infty) - S(\lambda)) e^{i\lambda x} d\lambda. \tag{3.2}$$

The integral equation (3.1) is called the *main equation* for the boundary value problem (1.1)-(1.3).

Proof. In order to obtain the main equation for the kernel $K(x,t)$ of the solution (2.1), we use the equality (2.5) provided in Lemma 1. Rewriting the identity (2.5), we obtain the following result

$$\frac{2i\lambda\phi(\lambda,x)}{\Theta(\lambda)} - e^{-i\lambda x} + S(\infty) e^{i\lambda x}$$

$$= [S(\infty) - S(\lambda)] e^{i\lambda x} + \int_x^\infty K(x,t) e^{-i\lambda t} dt$$
$$- \int_x^\infty S(\infty) K(x,t) e^{i\lambda t} dt + \int_x^\infty [S(\infty) - S(\lambda)] K(x,t) e^{i\lambda t} dt.$$

Multiplying both sides of this result by $\frac{1}{2\pi} e^{i\lambda y}$ and integrating it according to λ over $(-\infty, \infty)$, we find

$$\frac{1}{2\pi} \int_{-\infty}^\infty \left[\frac{2i\lambda \phi(\lambda, x)}{\Theta(\lambda)} - e^{-i\lambda x} + S(\infty) e^{i\lambda x} \right] e^{i\lambda y} d\lambda$$
$$= \frac{1}{2\pi} \int_{-\infty}^\infty [S(\infty) - S(\lambda)] e^{i\lambda(x+y)} d\lambda$$
$$+ \frac{1}{2\pi} \int_{-\infty}^\infty \int_x^\infty K(x,t) e^{-i\lambda(t-y)} dt d\lambda - \frac{1}{2\pi} \int_{-\infty}^\infty \int_x^\infty S(\infty) K(x,t) e^{i\lambda(t+y)} dt d\lambda$$
$$+ \frac{1}{2\pi} \int_{-\infty}^\infty \int_x^\infty [S(\infty) - S(\lambda)] K(x,t) e^{i\lambda(t+y)} dt d\lambda. \tag{3.3}$$

Here

$$\frac{1}{2\pi} \int_{-\infty}^\infty \int_x^\infty K(x,t) e^{-i\lambda(t-y)} dt d\lambda = \int_x^\infty K(x,t) \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\lambda(t-y)} d\lambda dt$$
$$= \int_x^\infty K(x,t) \delta(t-y) dt = K(x,y)$$

and since $K(x,-y) = 0$ for $y > x$, it follows that

$$\frac{1}{2\pi} \int_{-\infty}^\infty \int_x^\infty S(\infty) K(x,t) e^{i\lambda(t+y)} dt d\lambda = \int_x^\infty S(\infty) K(x,t) \frac{1}{2\pi} \int_{-\infty}^\infty e^{i\lambda(t+y)} d\lambda dt$$
$$= S(\infty) \int_x^\infty K(x,t) \delta(-t-y) dt = S(\infty) K(x,-y) = 0.$$

Therefore, on the right of (3.3), we obtain

$$K(x,y) + F_S(x+y) + \int_x^\infty K(x,t) F_S(t+y) dt, \qquad y > x,$$

where

$$F_S(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[S(\infty)-S(\lambda)]e^{i\lambda x}d\lambda.$$

On the other side, using the residue theorem and Jordan's lemma, we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{2i\lambda\phi(\lambda,x)}{\Theta(\lambda)}-e^{-i\lambda x}+S(\infty)e^{i\lambda x}\right]e^{i\lambda y}d\lambda=-\sum_{k=1}^{N}\frac{2i\lambda_k\phi(i\lambda_k,x)}{\dot{\Theta}(i\lambda_k)}e^{-\lambda_k y}.$$

Taking formula (2.17) into account, we can transform this expression to the form

$$\begin{aligned}&-\sum_{k=1}^{N}\frac{2i\lambda_k\phi(i\lambda_k,x)}{\dot{\Theta}(i\lambda_k)}e^{-\lambda_k y}\\&=-\sum_{k=1}^{N}\frac{2i\lambda_k p_1(i\lambda_k)}{e(i\lambda_k,0)\dot{\Theta}(i\lambda_k)}e(i\lambda_k,x)e^{-\lambda_k y}=-\sum_{k=1}^{N}m_k^2e(i\lambda_k,x)e^{-\lambda_k y}\\&=-\sum_{k=1}^{N}m_k^2\left[e^{-\lambda_k(x+y)}+\int_x^{\infty}K(x,t)e^{-\lambda_k(t+y)}dt\right].\end{aligned}$$

If we substitute this value into the left side of (3.3), we provide integral equation (3.1) and $F(x)$ is defined by the formula (3.2). The theorem is proved. □

4. Solvability of the main equation

The inverse scattering problem deals with the recovery of the coefficient $q(x)$ from the scattering data of the boundary value problem (1.1)-(1.3). From the identity (2.2), it is obvious to determine $q(x)$ that it is sufficient to know $K(x,t)$. Therefore, we suppose that the collection $\left\{S(\lambda),\{i\lambda_k\}_{k=1}^{N},\{m_k\}_{k=1}^{N}\right\}$ is given and construct the equation (3.1) on the basis of this collection. In this equation, we can take kernel $K(x,t)$ as unknown and regard it as a Fredholm-type equation for every fixed x.

Theorem 2. *For every fixed $x\geq 0$, the main equation (3.1) has a unique solution $K(x,.)\in L_1[x,\infty)$.*

Proof. Let us take $g(y)=K(x,y)$ in the equation (3.1). In order to prove the unique solvability of the equation, it suffices to verify that corresponding homogeneous equation has only null solution:

$$g(y)+\int_x^{\infty}g(t)F(t+y)dt=0,\qquad g(y)\in L_1[x,\infty).\tag{4.1}$$

The auxiliary function $F(y)$ in the equation (4.1) has the same properties as the auxiliary function of the boundary value problem without spectral parameter in the

boundary conditions. Therefore, the operators which appear in this equation are compact (the proof of this fact is similar to that of Lemma 3.3.1 in [18]) and hence we obtain the proof of the theorem by applying Theorem 3.3.1 in [18] to equation (4.1). □

Theorem 3. *The scattering data of the boundary value problem* (1.1)-(1.3) *determines potential* $q(x)$ *in equation* (1.1) *uniquely.*

Proof. From given scattering data $\left\{S(\lambda), \{i\lambda_k\}_{k=1}^{N}, \{m_k\}_{k=1}^{N}\right\}$, the main equation (3.1) is established and we find a unique solution $K(x,y)$ for every $x \geq 0$ by solving the main equation with the help of Theorem 2. Thus, we have the kernel $K(x,y)$ of the Jost solution (2.1) and obtain the potential with

$$q(x) = -2\frac{d}{dx}K(x,x).$$

□

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