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CONVERGENCE ANALYSIS OF THE LEGENDRE WAVELETS METHOD FOR A CLASS OF FREDHOLM-VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS OF FRACTIONAL-ORDER

MARZIYEH FELAHAT

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Abstract. In this paper, we develop a Legendre wavelets method for the numerical solution of Fredholm-Volterra integro-differential equations of fractional order. The Caputo sense is used to explain the fractional derivative operator. The proposed scheme is verified by presenting examples that their exact solutions are available. Numerical results show that the approximation errors decay exponentially in L^2-norm.

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1. Introduction

Many natural phenomena in science and engineering may be modeled using mathematical tools in the form of fractional calculus. The integro-differential calculus theory of fractional order allows explaining the natural phenomena more precisely [17, 24]. The integro-differential equations of fractional order has many applications in various scientific fields, such as physics,engineering, economics, biology, etc. Hence, solving these equations is a crucially important task. Moreover, most of these problems cannot be thoroughly solved, so finding a proper approximate solution by implementing numerical methods can be very beneficial. In the last few decades, some numerical methods have been implemented to solve differential and fractional integro-differential equations of fractional order; for example, the spectral method [28], reproducing kernel Hilbert space method [5, 14], wavelet method [18, 19, 27, 30], collocation method [6, 25], fractional differential transform method [2,4,10], variational iteration method [22], homotopy perturbation method [1,22,26], and Adomian decomposition method [21].

In this paper, a new computational method based on the Legendre wavelets is introduced to solve a class of Fredholm-Volterra integro-differential equations of fractional order in the following form:

$$\begin{cases} c_1 y(x) + c_2 y'(x) = f(x) + \lambda \int_0^1 p(x,t) D_{0^+}^{\alpha_1} y(t)\mathrm{d}t + \mu \int_0^x q(x,t) D_{0^+}^{\alpha_2} y(t)\mathrm{d}t, & x \in [0,1], \\ y(0) = c_0, \qquad 0 < \alpha_i < 1, \quad i = 1,2, \end{cases} \tag{1.1}$$

where $D_{0^+}^{\alpha}$ is the Caputo's derivative operator of fractional order α, y is an unknown function, f is a known function on the interval $[0,1]$, p, q are known functions over the range $[0,1]\times[0,1]$, considered to be smooth enough, and c_0, c_1, c_2, λ, μ are constants.

In [9], the existence and uniqueness of the solution to equation (1.1) are investigated. In [3], the authors use a backward and central-difference formula for approximating solutions at mesh points. Yusefi et al. in [28] apply a Chebyshev-Legendre spectral method to approximate the solution of equation (1.1).

In this paper, inspired by [11], we develop an approach on the basis of Legendre wavelet method to solve equation (1.1). Furthermore,we present a precise analysis on the convergence of this method and examine the convergence of the approximate solutions with L^2-norm.

The rest of the paper is organized as follows. In Section 2, we review some of the basic definitions and mathematical tools of the fractional calculus theory needed for later use. We devote Section 3 to introduce the basic definitions of wavelets, the Legendre polynomials of order m and functions approximation using Legendre wavelets. Additionally, in this section, we obtain the integral and derivative operational matrices of equation (1.1). We investigate the convergence of the approximate solution of equation (1.1) using the Legendre wavelets method in Section 4. In Section 5, we introduce some numerical examples of Fredholm-Volterra integro-differential equations of fractional order for describing the proposed algorithm, and in the final section, we give a brief summary of the paper.

2. Preliminaries

In this section, we review the definitions and auxiliary results regarding fractional calculus that will be required later in the paper. The properties of the following fractional integral and derivative operators can be found in [12, 13, 15, 16, 20, 24].

2.1. *Riemann-Liouville fractional integral operator*

The Riemann-Liouville fractional integral operator of order α, $\alpha \geq 0$, on the usual Lebesgue space $L[0,b]$ is given as follows:

$$I_{0_+}^{\alpha} u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t)\mathrm{d}t, \qquad \alpha > 0, \qquad x > 0, \tag{2.1}$$

where Γ is the well-known Gamma function, and $I_{0_+}^{0}u(x) = u(x)$ for $x > 0$. Some of the important properties of the Riemann-Liouville fractional integral operator are as follows:

(a1) $I_{0_+}^{\alpha} I_{0_+}^{\beta} u(x) = I_{0_+}^{\alpha+\beta} u(x)$,

(a2) $I_{0_+}^{\alpha} I_{0_+}^{\beta} u(x) = I_{0_+}^{\beta} I_{0_+}^{\alpha} u(x)$,

(a3) $I_{0_+}^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}$.

2.2. *Caputo's fractional derivative operator*

Let $m \in \mathbb{N}$, then the smallest integer greater than or equal to α, that is, the Caputo's fractional derivative operator of order $\alpha > 0$, is obtained as follows:

$$D_{0_+}^{\alpha} u(x) = \begin{cases} I_{0_+}^{m-\alpha} D^m u(x), & m-1 < \alpha < m, \\ \\ D^m u(x), & \alpha = m \end{cases} \tag{2.2}$$

for $x > 0$. A few important features of the Caputo's fractional derivative operator are as follows:

(b1) $I_{0_+}^{\alpha} D_{0_+}^{\alpha} u(x) = u(x) - \sum_{k=0}^{m-1} u^{(k)}(0^+) \frac{x^k}{k!}$,

(b2) $D_{0_+}^{\alpha} I_{0_+}^{\alpha} u(x) = u(x)$,

(b3) $D_{0_+}^{\alpha} x^{\beta} = \begin{cases} 0, & \beta \in \mathbb{N} \cup \{0\} \textit{ and } \beta < m, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, & \beta \in \mathbb{N} \cup \{0\} \textit{ and } \beta \geq m. \end{cases}$

When $\alpha \in \mathbb{N}$, the Caputo's differential operator is the same as the ordinary differential operator. Similar to the ordinary differential operator, Caputo's fractional differentiation operator is also a linear operator, namely,

$$D_{0_+}^{\alpha} \Big(C_1 u_1(x) + C_2 u_2(x) \Big) = C_1 D_{0_+}^{\alpha} u_1(x) + C_2 D_{0_+}^{\alpha} u_2(x),$$

where C_1, C_2 are constants.

3. Implementation

Wavelets comprise a class of functions formed by translation and dilation of a single function called mother wavelet $\psi(x)$ that is given as:

$$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}} \psi(\frac{x-b}{a}), \quad a, b \in \mathbb{R}, \quad a \neq 0, \tag{3.1}$$

where a and b are the dilation and translation parameters, respectively.

Limiting a and b to discrete values as $a = a_0^{-j}$ and $b = k b_0 a_0^{-j}$, where $a_0 > 1$, $b_0 > 1$, $j, k \in \mathbb{N}$, we can find a class of discrete wavelets as follows:

$$\psi_{j,k}(x) = |a_0|^{\frac{j}{2}} \psi(a_0^j x - k b_0). \tag{3.2}$$

This set of wavelets forms an orthogonal base for $L^2(\mathbb{R})$.

In particular, when $a_0 = 2$ and $b_0 = 1, \psi_{j,k}(x)$ constructs an orthonormal basis, that is

$$\langle \psi_{j,k}, \psi_{l,m} \rangle = \delta_{jl}\delta_{km}, \tag{3.3}$$

where δ_{jl} denotes the Kronecker delta.

3.1. *Legendre wavelets*

The Legendre polynomial of order m represented by $p_m(x)$ is defined on the interval $[-1,1]$ by the following recurrence formula:

$$\begin{aligned} p_0(x) &= 1, \\ p_1(x) &= x, \\ p_{m+2}(x) &= \frac{2m+3}{m+2}\, x p_{m+1}(x) - \frac{m+1}{m+2}\, p_m(x) \quad m = 0,1,2,\ldots. \end{aligned}$$

The Legendre wavelets are given over the range $[0,1)$ as:

$$\psi_{nm}(x) = \begin{cases} (m+\frac{1}{2})^{\frac{1}{2}} 2^{\frac{l}{2}} p_m(2^l x - 2n + 1), & \frac{n-1}{2^{l-1}} \le x < \frac{n}{2^{l-1}}, \\ 0, & \text{otherwise}, \end{cases}$$

where $l \in \mathbb{N}$, $n = 1,2,3,\ldots,2^{l-1}$ and $m = 0,1,2,\ldots,M-1$; m denotes the order of the Legendre polynomial, and M is a positive integer.

3.2. *Approximating functions using Legendre wavelets*

Every function y with square integrability over the range $[0,1]$ can be expanded in terms of the Legendre wavelet polynomials as follows:

$$y(x) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} y_{ij}\psi_{ij}(x), \tag{3.4}$$

where the Fourier coefficients y_{ij} are given as follows:

$$y_{ij} = \langle y(x), \psi_{ij}(x) \rangle.$$

If the infinite series in (3.4) is truncated, equation (3.4) can be rewritten as:

$$y(x) \cong y_N(x) = \sum_{i=1}^{2^{l-1}} \sum_{j=0}^{M-1} y_{ij}\psi_{ij}(x) = \Psi^T(x)\mathbf{Y}, \tag{3.5}$$

where $\mathbf{Y}$ and $\Psi(x)$ are two $N \times 1, (N = 2^{l-1} \times M)$ matrices given by

$$\begin{aligned} \mathbf{Y} &= [y_{10}, y_{11}, \ldots, y_{1M-1}, y_{20}, y_{21}, \ldots, y_{2M-1}, \ldots, y_{2^{l-1}0}, \ldots, y_{2^{l-1}M-1}]^T, \\ \Psi(x) &= [\psi_{10}, \psi_{11}, \ldots, \psi_{1M-1}, \psi_{20}, \ldots, \psi_{2M-1}, \ldots, \psi_{2^{l-1}0}, \ldots, \psi_{2^{l-1}M-1}]^T. \end{aligned} \tag{3.6}$$

Correspondingly, to expand functions including $f \in L^2([0,1])$ and $k,\ h \in L^2([0,1] \times [0,1])$ in terms of Legendre wavelet functions, we have:

$$\begin{aligned} f(x) &\cong f_N(x) = \sum_{i=1}^{N} f_i \psi_i(x) = \Psi^T(x)\mathbf{F}, \\ k(x,t) &\cong k_N(x,t) = \sum_{i=1}^{N}\sum_{j=1}^{N} k_{ij}\psi_i(x)\psi_j(t) = \Psi^T(x)\mathbf{K}\Psi(t), \\ h(x,t) &\cong h_N(x,t) = \sum_{i=1}^{N}\sum_{j=1}^{N} h_{ij}\psi_i(x)\psi_j(t) = \Psi^T(x)\mathbf{H}\Psi(t), \end{aligned} \tag{3.7}$$

where $\mathbf{K}$ and $\mathbf{H}$ are $N \times N$ matrices, and $\mathbf{F}$ is $N \times 1$, given as follows:

$$\begin{aligned} &\mathbf{K} = [k_{ij}]_{N\times N}, \qquad k_{ij} = \Big\langle \psi_i(x), \langle k(x,t), \psi_j(t)\rangle \Big\rangle, \qquad i,j = 1,2,\cdots,N, \\ &\mathbf{H} = [h_{ij}]_{N\times N}, \qquad h_{ij} = \Big\langle \psi_i(x), \langle h(x,t), \psi_j(t)\rangle \Big\rangle, \qquad i,j = 1,2,\cdots,N, \\ &\mathbf{F} = [f_{10},\ldots,f_{1M-1},f_{20},\ldots,f_{2M-1},\ldots,f_{2^{l-1}0},\ldots,f_{2^{l-1}M-1}]^T, \\ &f_{ij} = \big\langle f(x), \psi_{ij}(x)\big\rangle, \quad i,j = 1,2,\cdots,N. \end{aligned}$$

In the following, we will apply our method to Fredholm-Volterra integro-differential equations of fractional order (1.1). First, using the Caputo's fractional derivative operator (2.2), equation (1.1) is obtained as follows:

$$\begin{aligned} c_1 y(x) + c_2 y'(x) = f(x) &+ \frac{\lambda}{\Gamma(1-\alpha_1)} \int_0^1 p(x,t) \int_0^t \frac{y'(s)}{(t-s)^{\alpha_1}} \mathrm{d}s\ \mathrm{d}t \\ &+ \frac{\mu}{\Gamma(1-\alpha_2)} \int_0^x q(x,t) \int_0^t \frac{y'(s)}{(t-s)^{\alpha_2}} \mathrm{d}s\ \mathrm{d}t. \end{aligned} \tag{3.8}$$

Then, by changing the order of integration in equation (3.8), we have:

$$\begin{aligned} c_1 y(x) + c_2 y'(x) = f(x) &+ \frac{\lambda}{\Gamma(1-\alpha_1)} \int_0^1 y'(s) \int_s^1 \frac{p(x,t)}{(t-s)^{\alpha_1}} \mathrm{d}t\ \mathrm{d}s \\ &+ \frac{\mu}{\Gamma(1-\alpha_2)} \int_0^x y'(s) \int_s^x \frac{q(x,t)}{(t-s)^{\alpha_2}} \mathrm{d}t\ \mathrm{d}s. \end{aligned} \tag{3.9}$$

However, we rewrite equation (3.9) as follows:

$$c_1 y(x) + c_2 y'(x) = f(x) + \lambda \int_0^1 k(x,s) y'(s) \mathrm{d}s + \mu \int_0^x h(x,s) y'(s) \mathrm{d}s, \tag{3.10}$$

where $k(x,s) = \frac{1}{\Gamma(1-\alpha_1)} \int_s^1 \frac{p(x,t)}{(t-s)^{\alpha_1}} \mathrm{d}t$ and $h(x,s) = \frac{1}{\Gamma(1-\alpha_2)} \int_s^x \frac{q(x,t)}{(t-s)^{\alpha_2}} \mathrm{d}t$.

Now, by substituting equations (3.5) and (3.7) into equation (3.10) we obtain

$$c_1\Psi^T(x)\mathbf{Y}+c_2\Psi^T(x)\mathbf{DY}\cong\Psi^T(x)\mathbf{F}+\lambda\int_0^1\Psi^T(x)\mathbf{K}\Psi(s)\Psi^T(s)\mathbf{DY}\,\mathrm{d}s \\ +\mu\int_0^x\Psi^T(x)\mathbf{H}\Psi(s)\Psi^T(s)\mathbf{DY}\,\mathrm{d}s. \tag{3.11}$$

Since $\Psi(s)\Psi^T(s)\Big(\mathbf{DY}\Big)=\overline{\mathbf{DY}}\Psi(s)$, where $\overline{\mathbf{DY}}$ is an$N\times N$ matrix, we have

$$c_1\Psi^T(x)\mathbf{Y}+c_2\Psi^T(x)\mathbf{DY}\cong\Psi^T(x)\mathbf{F}+\lambda\Psi^T(x)\mathbf{K}\int_0^1\overline{\mathbf{DY}}\Psi(s)\,\mathrm{d}s \\ +\mu\Psi^T(x)\mathbf{H}\int_0^x\overline{\mathbf{DY}}\Psi(s)\,\mathrm{d}s. \tag{3.12}$$

Moreover, $\int_0^1\Psi(s)\,\mathrm{d}s=\mathbf{w}$ and $\int_0^x\Psi(s)\,\mathrm{d}s=\mathbf{P}\Psi(x)$, where $\mathbf{w}$ and $\mathbf{P}$ are $N\times 1$ and $N\times N$ matrices, respectively. Therefore, we have

$$\Psi^T(x)\Big[c_1\mathbf{Y}+c_2\mathbf{DY}\Big]\cong\Psi^T(x)\mathbf{F}+\lambda\Psi^T(x)\Big(\mathbf{K}\overline{\mathbf{DY}}\mathbf{w}\Big)+\mu\Psi^T(x)\Big(\mathbf{H}\overline{\mathbf{DY}}\mathbf{P}\Big)\Psi(x). \tag{3.13}$$

If we assume $\mathbf{K}\overline{\mathbf{DY}}\mathbf{w}=\mathbf{WY}$ and $\Psi^T(x)\Big(\mathbf{H}\overline{\mathbf{DY}}\mathbf{P}\Big)\Psi(x)=\Psi^T(x)\mathbf{UY}$, where $\mathbf{W}$ and $\mathbf{U}$ are $N\times N$ matrices, we can write

$$\Psi^T(x)\Big[c_1\mathbf{Y}+c_2\mathbf{DY}\Big]\cong\Psi^T(x)\Big[\mathbf{F}+\lambda\mathbf{WY}+\mu\mathbf{UY}\Big]. \tag{3.14}$$

Now, by replacing $\cong$ with $=$ in equation (3.14) and using orthonormality of $\Psi(x)$, we have

$$c_1\mathbf{Y}+c_2\mathbf{DY}=\mathbf{F}+\lambda\mathbf{WY}+\mathbf{UY}. \tag{3.15}$$

From equation (3.15), we obtain the following system:

$$\Big(c_1\mathbf{I}+c_2\mathbf{D}-\lambda\mathbf{W}-\mu\mathbf{U}\Big)\mathbf{Y}=\mathbf{F}. \tag{3.16}$$

Theorem 1 ([11]). *To solve Fredholm-Volterra integral equation, the operational matrix of integration,* $\mathbf{P}$*, is given as:*

$$\mathbf{P}_{N\times N}=\frac{1}{2^l}\begin{bmatrix} T & S & S & \dots & S \\ O & T & S & \dots & S \\ O & O & T & \dots & S \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & T \end{bmatrix},$$

where O, S and T are square matrices of order M given as follows:

$$O_{M\times M}=\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad S_{M\times M}=\begin{bmatrix} 2 & 0 & \dots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$
T_{M\times M} = \begin{bmatrix}
1 & \frac{1}{\sqrt{3}} & 0 & 0 & \dots & 0 & 0 \\
-\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & \dots & 0 & 0 \\
0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0 & \frac{\sqrt{5}}{5\sqrt{7}} & \dots & 0 & 0 \\
0 & 0 & -\frac{\sqrt{7}}{7\sqrt{5}} & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\
0 & 0 & 0 & 0 & \dots & -\frac{\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & 0
\end{bmatrix}.
$$

Theorem 2. *To solve Fredholm-Volterra integro-differential equations of fractional order, the operational matrix of ordinary derivative,* $\mathbf{D}$*, is given as follows:*

$$
\mathbf{D}_{N\times N} = 2^l \begin{bmatrix}
L & O & O & \dots & O \\
O & L & O & \dots & O \\
O & O & L & \dots & O \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
O & O & O & \cdots & L
\end{bmatrix},
$$

where O *and* L *are square matrices of order* M *given by*

$$
O_{M\times M} = \begin{bmatrix}
0 & 0 & \dots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

and

$$
L = \begin{bmatrix}
0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots \\
\sqrt{3\times 1} & 0 & 0 & 0 & \dots & 0 & 0 & \dots \\
0 & \sqrt{5\times 3} & 0 & 0 & \dots & 0 & 0 & \dots \\
\sqrt{7\times 1} & 0 & \sqrt{7\times 5} & 0 & \dots & 0 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots \\
0 & \sqrt{(2i-3)\times 3} & 0 & \sqrt{(2i-3)\times 7} & \dots & \sqrt{(2i-3)(2j-3)} & 0 & \dots \\
\sqrt{(2i-1)\times 1} & 0 & \sqrt{(2i-1)\times 5} & 0 & \dots & 0 & \sqrt{(2i-1)(2j-1)} & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots
\end{bmatrix}.
$$

Proof. First, we find the operational matrix of ordinary derivative for $l = 2$ and $M = 3$. Next, we obtain the general matrix $\mathbf{D}$. The basis functions $\psi_{nm}(x)$, $n = 1,2$, $m = 0,1,2$, and as a result, the matrix

$$
\Psi_{6\times 1}(x) = [\psi_{10}(x) \;\; \psi_{11}(x) \;\; \psi_{12}(x) \;\; \psi_{20}(x) \;\; \psi_{21}(x) \;\; \psi_{22}(x)]^T
$$

for $l = 2$ and $M = 3$, is obtained as follows:

$$\left.\begin{aligned} \psi_{10}(x) &= \sqrt{2}, \\ \psi_{11}(x) &= \sqrt{6}(4x-1), \\ \psi_{12}(x) &= \sqrt{10}\left[\frac{3}{2}(4x-1)^2 - \frac{1}{2}\right], \end{aligned}\right\} \qquad 0 \le x < \frac{1}{2}, \tag{3.17}$$

and

$$\left.\begin{aligned} \psi_{20}(x) &= \sqrt{2}, \\ \psi_{21}(x) &= \sqrt{6}(4x-3), \\ \psi_{22}(x) &= \sqrt{10}\left[\frac{3}{2}(4x-3)^2 - \frac{1}{2}\right], \end{aligned}\right\} \qquad \frac{1}{2} \le x < 1. \tag{3.18}$$

By deriving from equations (3.17) and (3.18), we obtain the following equations:

$$\begin{aligned} \frac{\mathrm{d}\psi_{10}(x)}{\mathrm{d}x} &= 0 \\ &= 0\psi_{10}(x) + 0\psi_{11}(x) + 0\psi_{12}(x) + 0\psi_{20}(x) + 0\psi_{21}(x) + 0\psi_{22}(x) \\ &= 2^2\left[0,0,0,0,0,0\right]\Psi_{6\times 1}(x). \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}\psi_{11}(x)}{\mathrm{d}x} &= 4\sqrt{3} \\ &= 4\sqrt{3}\psi_{10}(x) + 0\psi_{11}(x) + 0\psi_{12}(x) + 0\psi_{20}(x) + 0\psi_{21}(x) + 0\psi_{22}(x) \\ &= 2^2\left[\sqrt{3},0,0,0,0,0\right]\Psi_{6\times 1}(x). \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}\psi_{12}(x)}{\mathrm{d}x} &= 12\sqrt{10}(4x-1) \\ &= 0\psi_{10}(x) + 4\sqrt{15}\psi_{11}(x) + 0\psi_{12}(x) + 0\psi_{20}(x) + 0\psi_{21}(x) + 0\psi_{22}(x) \\ &= 2^2\left[0,\sqrt{5\times 3},0,0,0,0\right]\Psi_{6\times 1}(x). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \frac{\mathrm{d}\psi_{20}(x)}{\mathrm{d}x} &= 0 \\ &= 0\psi_{10}(x) + 0\psi_{11}(x) + 0\psi_{12}(x) + 0\psi_{20}(x) + 0\psi_{21}(x) + 0\psi_{22}(x) \\ &= 2^2\left[0,0,0,0,0,0\right]\Psi_{6\times 1}(x). \end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}\psi_{21}(x)}{\mathrm{d}x} &= 4\sqrt{3}\\ &= 4\sqrt{3}\psi_{10}(x)+0\psi_{11}(x)+0\psi_{12}(x)+0\psi_{20}(x)+0\psi_{21}(x)+0\psi_{22}(x)\\ &= 2^2\left[\sqrt{3\times 1},0,0,0,0,0\right]\Psi_{6\times 1}(x).\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}\psi_{22}(x)}{\mathrm{d}x} &= 12\sqrt{10}(4x-3)\\ &= 0\psi_{10}(x)+4\sqrt{15}\psi_{11}(x)+0\psi_{12}(x)+0\psi_{20}(x)+0\psi_{21}(x)+0\psi_{22}(x)\\ &= 2^2\left[0,\sqrt{5\times 3},0,0,0,0\right]\Psi_{6\times 1}(x).\end{aligned}$$

Therefore, we have

$$\frac{\mathrm{d}\Psi_{6\times 1}(x)}{\mathrm{d}x} = \mathbf{D}_{6\times 6}\Psi_{6\times 1}(x),$$

where $\mathbf{D}_{6\times 6}$ is a square matrix of order 6 that can be structured as a block matrix as follows:

$$\mathbf{D}_{6\times 6} = 2^2\begin{bmatrix} L_{3\times 3} & O_{3\times 3}\\ O_{3\times 3} & L_{3\times 3}\end{bmatrix},$$

where

$$L_{3\times 3} = \begin{bmatrix} 0 & 0 & 0\\ \sqrt{3} & 0 & 0\\ 0 & \sqrt{5\times 3} & 0\end{bmatrix} \quad \text{and} \quad O_{3\times 3} = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

By assuming that the theorem is true for $l-1$ and $M-1$, it can be easily shown that for general l and M, we can write

$$\frac{d\Psi_{6\times 1}(x)}{\mathrm{d}x} = \mathbf{D}_{N\times N}\Psi_{N\times 1}(x), \quad (N = 2^{l-1}\times M), \tag{3.19}$$

where $\mathbf{D}$ is a square matrix of order N that can be structured as a block matrix as follows:

$$\mathbf{D}_{N\times N} = 2^l\begin{bmatrix} L & O & O & \dots & O\\ O & L & O & \dots & O\\ O & O & L & \dots & O\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ O & O & O & \cdots & L\end{bmatrix},$$

where O and L are square matrices of order M as follows:

$$O_{M\times M} = \begin{bmatrix} 0 & 0 & \dots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0\end{bmatrix}$$

and

$$L=\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots \\ \sqrt{3\times 1} & 0 & 0 & 0 & \dots & 0 & 0 & \dots \\ 0 & \sqrt{5\times 3} & 0 & 0 & \dots & 0 & 0 & \dots \\ \sqrt{7\times 1} & 0 & \sqrt{7\times 5} & 0 & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots \\ 0 & \sqrt{(2i-3)\times 3} & 0 & \sqrt{(2i-3)\times 7} & \dots & \sqrt{(2i-3)(2j-3)} & 0 & \dots \\ \sqrt{(2i-1)\times 1} & 0 & \sqrt{(2i-1)\times 5} & 0 & \dots & 0 & \sqrt{(2i-1)(2j-1)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots \end{bmatrix}.$$

□

4. Error estimation

In this section, we prove the convergence of the proposed numerical scheme for fractional Fredholm-Volterra integro-differential equations in L^2-norm. We also review some of the properties and elementary lemmas, required for obtaining the main results.

Lemma 1 ([8]). *Assume that $u, \omega, \beta \in C([0,1])$ with $\beta(x) \geq 0$. If u satisfies the inequality*

$$u(x) \leq \omega(x) + \int_0^x \beta(t)u(t)\mathrm{d}t, \qquad x \in [0,1],$$

then

$$u(x) \leq \omega(x) + \int_0^x \beta(t)\Big(\omega(t)\exp\Big(\int_t^x \beta(s)\mathrm{d}s\Big)\Big)\mathrm{d}t, \qquad x \in [0,1].$$

In other words, if ω is non-decreasing over$[0,1]$, the inequality mentioned above reduces to

$$u(x) \leq \omega(x)\exp\Big(\int_t^x \beta(s)\mathrm{d}s\Big), \qquad x \in [0,1].$$

Lemma 2 ([8]). *Suppose that k is a given kernel function over $[0,1]\times[0,1]$. If $u \in L^p(0,1)$, for $1 \leq p \leq \infty$, the integral*

$$K(u(x)) = \int_0^{x \text{ or } 1} k(x,t)u(t)\mathrm{d}t$$

is well-defined in $L^p(0,1)$, and there exists a γ_0 so that

$$\|Ku\|_p \leq \gamma_0\|u\|_p.$$

Suppose that p_N is the interpolation projection operator from $L^2(0,1)$ to $\mathbb{P}_N(0,1)$, namely, $p_N\colon L^2(0,1) \to \mathbb{P}_N(0,1)$ such that, for each $u \in L^2(0,1)$, we have $p_N(u(x)) = \sum_{j=0}^N u_j\psi_j(x)$, and

$$\int_0^1 \Big(u(x) - p_N(u(x))\Big)v(x)\mathrm{d}x = 0, \quad \forall v \in \mathbb{P}_N(0,1).$$

Moreover, the following inequality for interpolation in Legendre wavelet polynomials and shifted Gauss-Legendre nodal points $r \geq 1$ (or for any fixed $r \leq N$) can be easily written as

$$\begin{aligned} \|u - p_N(u)\|_{H^J(0,1)} &\leq \gamma^* N^{2J-\frac{1}{2}-r} |u|_{r;N}, \\ \|u - p_N(u)\|_{H^J(0,1)} &\leq \delta^* N^{-r} |u|_{r;N}, \end{aligned} \tag{4.1}$$

where $u \in H^I(0,1)$, γ^* and δ^* are constants independent of N and $0 \leq J \leq I$, see [7].

Theorem 3. *For values of N that are large enough, the Legendre wavelet polynomial approximations y_N converge to exact solution in L^2-norm, namely,*

$$\|y - y_N\| \to 0 \quad as \quad N \to \infty.$$

Proof. Suppose that $y_N(x)$ is an approximate solution obtained by using the Legendre wavelet method for equation (3.10). Then, we have

$$c_1 y_N(x) + c_2 y_N'(x) = f_N(x) + \lambda \int_0^1 k_N(x,t) y_N'(t) \mathrm{d}t + \mu \int_0^x h_N(x,t) y_N'(t) \mathrm{d}t. \tag{4.2}$$

Consequently, $f_N(x)$, $k_N(x,t)$ and $h_N(x,t)$ are Legendre interpolation polynomials defined for functions $f(x)$, $k(x,t)$ and $h(x,t)$, respectively. Now, by integrating the sides of equations (3.10) and (4.2), we have

$$\begin{aligned} c_1 \int_0^x y(t) \mathrm{d}t + c_2 y(x) - c_2 y(0) = & \int_0^x f(t) \mathrm{d}t + \lambda \int_0^x \int_0^1 k(t,s) y'(s) \mathrm{d}s \mathrm{d}t \\ & + \mu \int_0^x \int_0^t h(t,s) y'(s) \mathrm{d}s \mathrm{d}t, \end{aligned} \tag{4.3}$$

$$\begin{aligned} c_1 \int_0^x y_N(t) \mathrm{d}t + c_2 y_N(x) - c_2 y(0) = & \int_0^x f_N(t) \mathrm{d}t + \lambda \int_0^x \int_0^1 k_N(t,s) y_N'(s) \mathrm{d}s \mathrm{d}t \\ & + \mu \int_0^x \int_0^t h_N(t,s) y_N'(s) \mathrm{d}s \mathrm{d}t. \end{aligned} \tag{4.4}$$

By subtracting equation (4.4) from equation (4.3), we have

$$\begin{aligned} E_N(y(x)) = & -\frac{c_1}{c_2} \int_0^x E_N(y(t)) \mathrm{d}t + \frac{1}{c_2} \int_0^x E_N(f(t)) \mathrm{d}t \\ & + \frac{\lambda}{c_2} \int_0^x \int_0^1 E_N(k(t,s)) y_N'(s) \mathrm{d}s \mathrm{d}t + \frac{\mu}{c_2} \int_0^x \int_0^t E_N(h(t,s)) y_N'(s) \mathrm{d}s \mathrm{d}t \\ & + \frac{\lambda}{c_2} \int_0^x \int_0^1 k(t,s) D E_N(y(s)) \mathrm{d}s \mathrm{d}t + \frac{\mu}{c_2} \int_0^x \int_0^t h(t,s) D E_N(y(s)) \mathrm{d}s \mathrm{d}t, \end{aligned} \tag{4.5}$$

therefore,

$$
\begin{aligned}
E_N(y(x)) &= y(x) - y_N(x),\\
DE_N(y(s)) &= y'(s) - y'_N(s),\\
E_N(f(t)) &= f(t) - f_N(t),\\
E_N(k(t,s)) &= k(t,s) - k_N(t,s),\\
E_N(r(t,s)) &= h(t,s) - h_N(t,s).
\end{aligned}
$$

From equation (4.5), we can obtain

$$
\begin{aligned}
|E_N(y(x))| &\leq |\frac{c_1}{c_2}| \int_0^x |E_N(y(t))| \mathrm{d}t + |\frac{1}{c_2}| \int_0^x |E_N(f(t))| \mathrm{d}t\\
&+ |\frac{\lambda}{c_2}| \int_0^x | \int_0^1 E_N(k(t,s)) y'_N(s) \mathrm{d}s| \mathrm{d}t + |\frac{\mu}{c_2}| \int_0^x | \int_0^t E_N(h(t,s)) y'_N(s) \mathrm{d}s| \mathrm{d}t\\
&+ |\frac{\lambda}{c_2}| \int_0^x | \int_0^1 k(t,s) DE_N(y(s)) \mathrm{d}s| \mathrm{d}t + |\frac{\mu}{c_2}| \int_0^x | \int_0^t h(t,s) DE_N(y(s)) \mathrm{d}s| \mathrm{d}t\\
&\leq \gamma_1 \int_0^x |E_N(y(t))| \mathrm{d}t + \gamma_2 \int_0^x |E_N(f(t))| \mathrm{d}t\\
&+ \gamma_3 \int_0^x | \int_0^1 E_N(k(t,s)) y'_N(s) \mathrm{d}s| \mathrm{d}t + \gamma_4 \int_0^x | \int_0^t E_N(h(t,s)) y'_N(s) \mathrm{d}s| \mathrm{d}t\\
&+ \gamma_5 \int_0^x | \int_0^1 k(t,s) DE_N(y(s)) \mathrm{d}s| \mathrm{d}t + \gamma_6 \int_0^x | \int_0^t h(t,s) DE_N(y(s)) \mathrm{d}s| \mathrm{d}t.
\end{aligned}
\tag{4.6}
$$

Applying Lemma 1 leads to

$$
\begin{aligned}
|E_N(y(x))| &\leq \exp(\int_0^x \gamma_1 \mathrm{d}t) \{ \gamma_2 \int_0^x |E_N(f(t))| \mathrm{d}t + \gamma_3 \int_0^x | \int_0^1 E_N(k(t,s)) y'_N(s) \mathrm{d}s| \mathrm{d}t\\
&+ \gamma_4 \int_0^x | \int_0^t E_N(h(t,s)) y'_N(s) \mathrm{d}s| \mathrm{d}t + \gamma_5 \int_0^x | \int_0^1 k(t,s) DE_N(y(s)) \mathrm{d}s| \mathrm{d}t\\
&+ \gamma_6 \int_0^x | \int_0^t h(t,s) DE_N(y(s)) \mathrm{d}s| \mathrm{d}t \}\\
&\leq \gamma_7 \int_0^x |E_N(f(t))| \mathrm{d}t + \gamma_8 \int_0^x | \int_0^1 E_N(k(t,s)) y'_N(s) \mathrm{d}s| \mathrm{d}t\\
&+ \gamma_9 \int_0^x | \int_0^t E_N(h(t,s)) y'_N(s) \mathrm{d}s| \mathrm{d}t + \gamma_{10} \int_0^x | \int_0^1 k(t,s) DE_N(y(s)) \mathrm{d}s| \mathrm{d}t\\
&+ \gamma_{11} \int_0^x | \int_0^t h(t,s) DE_N(y(s)) \mathrm{d}s| \mathrm{d}t.
\end{aligned}
\tag{4.7}
$$

Equivalently, by using L^2-norm, we obtain

$$\begin{aligned}\|E_N(y)\|_2 &\le \gamma_7\|\int_0^x |E_N(f(t))|\mathrm{d}t\|_2+\gamma_8\|\int_0^x|\int_0^1 E_N(k(t,s))y_N'(s)\mathrm{d}s|\mathrm{d}t\|_2\\&+\gamma_9\|\int_0^x|\int_0^t E_N(h(t,s))y_N'(s)\mathrm{d}s|\mathrm{d}t\|_2+\gamma_{10}\|\int_0^x|\int_0^1 k(t,s)DE_N(y(s))\mathrm{d}s|\mathrm{d}t\|_2\\&+\gamma_{11}\|\int_0^x|\int_0^t h(t,s)DE_N(y(s))\mathrm{d}s|\mathrm{d}t\|_2.\end{aligned} \tag{4.8}$$

Using Lemma 2, the above inequality reduces to

$$\begin{aligned}\|E_N(y)\|_2 \le \gamma_{12}\|E_N(f)\|_2&+\gamma_{13}\|\int_0^1 E_N(k(.,s))y_N'(s)\mathrm{d}s\|_2\\&+\gamma_{14}\|\int_0^t E_N(h(.,s))y_N'(s)\mathrm{d}s\|_2+\gamma_{15}\|\int_0^1 k(.,s)DE_N(y(s))\mathrm{d}s\|_2\\&+\gamma_{16}\|\int_0^t h(.,s)DE_N(y(s))\mathrm{d}s\|_2.\end{aligned} \tag{4.9}$$

Since the linear derivative operator is continuous and bounded [23], a constant $\delta_0 \ge 0$ exists such that

$$\|y_N'\|_{r;N} \le \delta_0\|y_N\|_{r;N}, \qquad r\in\mathbb{N} \;\; and \;\; r\le N. \tag{4.10}$$

According to equations (4.1) and (4.10) and Lemma 2, we can write

$$\begin{aligned}\|\int_0^1 E_N(k(.,s))y_N'(s)\mathrm{d}s\|_2 &\le \delta^*N^{-r}\|\int_0^1 k(.,s)y_N'(s)\mathrm{d}s\|_{r;N}\\&\le \delta^*\gamma_0N^{-r}\|y_N'\|_{r;N}\\&\le(\delta^*\gamma_0\delta_0)N^{-r}\|y_N\|_{r;N}\\&\le\delta_1N^{-r}(\|y\|_{r;N}+\|E_N(y)\|_{r;N})\\&\le\delta_1N^{-r}(\|y\|_{r;N}+\|E_N(y)\|_{1;N})\\&\le\delta_1N^{-r}(\|y\|_{r;N}+\gamma^*N^{\frac{3}{2}-r}|y|_{r;N}).\end{aligned}$$

Therefore,

$$\|\int_0^1 E_N(k(.,s))y_N'(s)\mathrm{d}s\|_2 \;\le\; \delta_1N^{-r}\|y\|_{r;N}+\delta_2N^{\frac{3}{2}-2r}|y|_{r;N}, \tag{4.11}$$

where $\delta_2=\delta_1\gamma^*$. Similarly,

$$\|\int_0^t E_N(h(.,s))y_N'(s)\mathrm{d}s\|_2 \;\le\; \delta_3N^{-r}\|y\|_{r;N}+\delta_4N^{\frac{3}{2}-2r}|y|_{r;N}. \tag{4.12}$$

In the same way, from Lemma 2 and equation (4.10), we obtain

$$\begin{aligned}\|\int_0^1 k(.,s)DE_N(y(s))\mathrm{d}s\|_2 &\leq \gamma_0\|DE_N(y)\|_2\\ &\leq \gamma_0\delta_0\|E_N(y)\|_2\\ &\leq (\gamma_0\delta_0\delta^*)N^{-r}|y|_{r;N}.\end{aligned}$$

Therefore,

$$\|\int_0^1 k(.,s)DE_N(y(s))\mathrm{d}s\|_2 \leq \delta_5 N^{-r}|y|_{r;N}, \tag{4.13}$$

where $\delta_3 = \gamma_0\delta_0\delta^*$. Similarly,

$$\|\int_0^t h(.,s)DE_N(y(s))\mathrm{d}s\|_2 \leq \delta_6 N^{-r}|y|_{r;N}. \tag{4.14}$$

In a manner similar to equation (4.1), we may write

$$\|E_N(f)\|_2 \leq \gamma^* N^{-r}|f|_{r;N}. \tag{4.15}$$

Ultimately, by substituting (4.11)-(4.15) in (4.9), we get the following result:

$$\begin{aligned}\|E_N(y)\|_2 \leq \gamma_{12}\left(\gamma^* N^{-r}|f|_{r;N}\right) &+ \gamma_{13}\left(\delta_1 N^{-r}\|y\|_{r;N} + \delta_2 N^{\frac{3}{2}-2r}|y|_{r;N}\right)\\ &+\gamma_{14}\left(\delta_3 N^{-r}\|y\|_{r;N} + \delta_4 N^{\frac{3}{2}-2r}|y|_{r;N}\right) + \gamma_{15}\left(\delta_5 N^{-r}|y|_{r;N}\right)\\ &+\gamma_{16}\left(\delta_6 N^{-r}|y|_{r;N}\right).\end{aligned}$$

Hence,

$$\|E_N(y)\|_2 \leq C_1 N^{-r}|f|_{r;N} + C_2 N^{-r}\|y\|_{r;N} + C_3 N^{\frac{3}{2}-2r}|y|_{r;N} + C_4 N^{-r}|y|_{r;N} \tag{4.16}$$

where $C_1 = \gamma^*\gamma_{12}$, $C_2 = \delta_1\gamma_{13} + \delta_3\gamma_{14}$, $C_3 = \delta_2\gamma_{13} + \delta_4\gamma_{14}$ and $C_4 = \delta_5\gamma_{15} + \delta_6\gamma_{16}$. Equation (4.16) proves the convergence of the approximate solution in L^2-norm. The proof is completed here. □

5. Numerical examples

In this section, we use the Legendre wavelet method for solving Fredholm-Volterra integro-differential equations of fractional order. In order to show the efficiency and accuracy of the presented method, four test equations are taken into account as examples. Numerical results in Table 1 as well as Figures 1 and 2 show that approximate solutions converge to exact solutions. All computations were conducted using the Maple software package.

Example 1 ([29]). We take into account the following Fredholm integro-differential equation of fractional order

$$y'(x) = f(x) + \int_0^1 xtD_{0_+}^{\frac{1}{2}}y(t)\,\mathrm{d}t, \qquad 0 < x < 1,$$

TABLE 1. Difference in absolute error of exact and approximate solutions for Examples 1 and 4.

x	Example 1 for $k=2$ and $M=3$	Example 4 for $k=2$ and $M=3$
0.0	$1.33679267695312 \times 10^{-9}$	$4.95687117107094 \times 10^{-10}$
0.1	$1.11257569166759 \times 10^{-9}$	$4.15374405495941 \times 10^{-10}$
0.2	$9.40240333784253 \times 10^{-10}$	$3.31229605283749 \times 10^{-10}$
0.3	$8.19786603303097 \times 10^{-10}$	$2.43252716470520 \times 10^{-10}$
0.4	$7.51214500224126 \times 10^{-10}$	$1.51443739056252 \times 10^{-10}$
0.5	$7.34524024547339 \times 10^{-11}$	$5.58026730409458 \times 10^{-11}$
0.6	$7.69715176272737 \times 10^{-10}$	$4.36704815753983 \times 10^{-10}$
0.7	$8.56787955400318 \times 10^{-10}$	$1.46975724792781 \times 10^{-10}$
0.8	$9.95742361930083 \times 10^{-10}$	$2.54113056611201 \times 10^{-10}$
0.9	$1.18657839586203 \times 10^{-9}$	$3.65082477030659 \times 10^{-10}$
1.0	$1.42929605719617 \times 10^{-9}$	$4.79883986051156 \times 10^{-10}$

where $y(0)=0$ and $y(x)=14x$ are the initial condition and the exact solution, respectively;

$$f(x) = 14\left(1 - \frac{x}{2.5\Gamma(1.5)}\right).$$

Using the presented method, for $k=2$ and $M=3$, the approximate solution is obtained as follows:

$$\begin{aligned} y_{numeric}(x) = &-5.26142047709630 \times 10^{-10} + 13.9999999999075x \\ &- 8.10650176803679 \times 10^{-11}(4x-1)^2 - 8.10650679514575 \times 10^{-11}(4x-3)^2 \\ &\cong 14x. \end{aligned}$$

This is almost the exact solution to the problem. The absolute error between the numerical and exact solution is tabulated in Table 1 for Example 1. The maximum errors with infinite- and L^2-norms are given as follows:

$$\|y_{numeric} - y_{exact}\|_\infty = 1.429296057 \times 10^{-9},$$
$$\|y_{numeric} - y_{exact}\|_{L^2} = 9.705274462 \times 10^{-10}.$$

Example 2 ([29]). We consider the following Fredholm integro-differential equation of fractional order

$$y'(x) = f(x) + \int_0^1 x^2 t^2 D_{0_+}^{\frac{1}{4}} y(t)\,\mathrm{d}t, \qquad 0 < x < 1,$$

with the initial condition $y(0)=0$ and the exact solution $y(x) = 2x^4 - x^{\frac{3}{2}}$. The numerical results for this example with $f(x) = 8x^3 - \frac{3}{2}x^{\frac{1}{2}} - \left(\frac{48}{6.75\Gamma(4.75)} - \frac{\Gamma(2.5)}{4.25\Gamma(2.25)}\right)x^2$ plus values of $k=2$ and $M=5$ are shown in Figure 1.

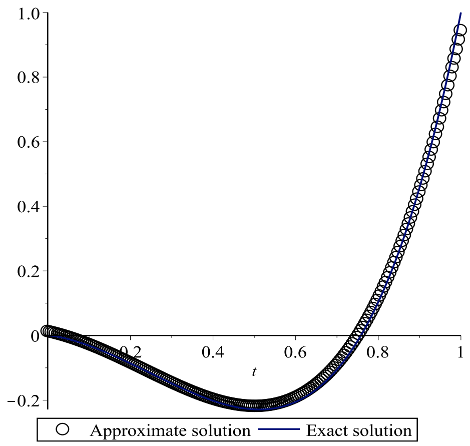

FIGURE 1. Exact and approximate solutions compared for $k=2$ and $M=5$, Example 2.

Example 3. We consider the following Fredholm integro-differential equation of fractional order

$$2y'(x)+y(x)=f(x)+\frac{1}{3}\int_0^x x^3tD_{0_+}^{\frac{1}{2}}y(t)\,\mathrm{d}t, \qquad 0<x<1,$$

with the initial condition $y(0)=1$, the exact solution $y(x)=8x+3x^3$ and $f(x)=16+8x+18x^2+(3-\frac{128}{45\Gamma(0.5)})x^3$. The exact and numerical solutions are shown in Figure 2 for $k=2$ and $M=5$.

Example 4. We consider the following Volterra integro-differential equation of fractional order

$$y'(x)+y(x)=2+x+\frac{x^{\frac{3}{2}}}{\Gamma(2.5)}-\int_0^x D_{0_+}^{\frac{1}{2}}y(t)\,\mathrm{d}t, \qquad 0<x<1,$$

with the initial condition $y(0)=1$ and the exact solution $y(x)=x+1$. For $k=2$ and $M=3$, the approximate solution obtained using our method is given as follows:

$$\begin{aligned} y_{numeric}(x) &= 1.00000000053072+0.999999999074120x-9.09330911332095 \\ &\quad \times 10^{-12}(4x-1)^2-2.88196776492321\times 10^{-12}(4x-3)^2 \cong x+1, \end{aligned}$$

which is equivalent to the exact solution. For Example 4, the absolute error between the numerical and exact solutions is reported in Table 1. The maximum error with infinite- and L^2-norms are as follows:

$$\|y_{numeric}-y_{exact}\|_\infty = 4.956871174\times 10^{-10},$$
$$\|y_{numeric}-y_{exact}\|_{L^2} = 2.847848599\times 10^{-10}.$$

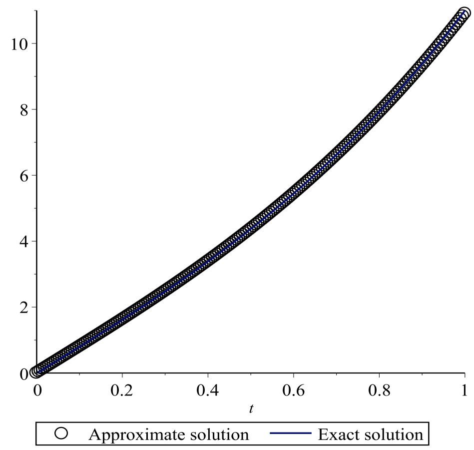

FIGURE 2. Exact and approximate solutions compared for $k = 2$ and $M = 5$, Example 3.

6. CONCLUSIONS

In this paper, we presented a Legendre wavelet approximation of a class of fractional Fredholm-Volterra integro-differential equations. The main purpose of this paper was to show that the approximation errors decay exponentially in L^2-norm. We proved that the method presented in this paper is effective for solving Fredholm-Volterra integro-differential equations of fractional order, and this method has a high convergence rate. The numerical results of the presented method were compared with exact solutions. The acceptable results are in a very good agreement with the exact solutions only for a small number of Legendre wavelet polynomials.

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Author's address

Marziyeh Felahat
Bozorgmehr University Of Qaenat, Department Of Basic Sciences, Qaenat, Iran
E-mail address: m.felahat@yahoo.com