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AN ESTIMATION FOR THE GENERALIZED TERNARY RING HOMOMORPHISM ON NON-ARCHIMEDEAN TERNARY BANACH ALGEBRAS

ISMAIL NIKOUFAR

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Abstract. In this paper, we introduce the notion of the generalized ternary ring homomorphism on non-Archimedean ternary Banach algebras. Using fixed point methods, we prove the superstability and generalized Hyers-Ulam stability of generalized ternary ring homomorphisms on non-Archimedean ternary Banach algebras associated with a Cauchy-Jensen type functional equation.

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1. Introduction

The theory of non-Archimedean spaces has many applications in quantum physics, p-adic strings and superstrings [13]. The methods that are used in non-Archimedean spaces are essentially different from the classical normed space theory [6, 7, 9, 15].

We first review the definition of non-Archimedean spaces [24]. Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $|\cdot|\colon \mathbb{K} \to \mathbb{R}$ such that for all $a,b \in \mathbb{K}$ we have

(1) $|a| \geq 0$ and equality holds if and only if $a = 0$,
(2) $|ab| = |a||b|$,
(3) $|a+b| \leq \max\{|a|,|b|\}$.

Condition (3) is the strict triangle inequality. By condition (2) we have $|1| = |-1| = 1$. By induction on n, one can show that $|n| \leq 1$ for each integer n. Note that for $n = 1$ we have $|1| = 1$. Let $|n| \leq 1$ for each integer n. We prove that $|n+1| \leq 1$. It follows from condition (3) that $|n+1| \leq \max\{|1|,|n|\} = \max\{1,|n|\} = 1$. A non-Archimedean absolute value $|\cdot|$ is non-trivial, i.e., there is an $a_0 \in \mathbb{K}$ such that $|a_0| \notin \{0,1\}$.

Let X be a linear space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial absolute value $|\cdot|$. A function $\|\cdot\|: X \to \mathbb{R}$ is a non-Archimedean norm if it satisfies the following conditions:

(1) $\|x\| = 0$ if and only if $x = 0$,
(2) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$,
(3) $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$.

Then $(X, \|\cdot\|)$ is called a non-Archimedean space. It follows from condition (3) that

$$\|x_m - x_l\| \leq \max\{\|x_{j+1} - x_j\| : l \leq j \leq m-1\}$$

for all $m > l$ and hence a sequence $\{x_m\}$ is Cauchy in X if and only if $\{x_{m+1} - x_m\}$ converges to zero in a non-Archimedean space. A complete non-Archimedean space is a non-Archimedean space that every Cauchy sequence is convergent. A ternary (associative) algebra $(A, [\cdot,\cdot,\cdot])$ is a linear space A over a scalar field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ equipped with a trilinear mapping, the so-called ternary product, $[\cdot,\cdot,\cdot]: A \times A \times A \to A$ such that it is associative in the sense that $[[a,b,c],d,e] = [a,[b,c,d],e] = [a,b,[c,d,e]]$ for all $a,b,c,d,e \in A$. This notion is a natural generalization of the binary case. Indeed, if $(A, \circ)$ is a usual (binary) non-Archimedean algebra, then $[a,b,c] := (a \circ b) \circ c$ induced a ternary product making A into a non-Archimedean ternary algebra which will be called trivial. There are other types of non-Archimedean ternary algebras in which one may consider other versions of associativity. Let A be a non-Archimedean vector space. Then $[a,b,c] = a - b + c$ induced a ternary product making A into a non-Archimedean ternary algebra. A non-Archimedean ternary Banach algebra is a complete non-Archimedean ternary algebra A which the norm satisfies $||[a,b,c]|| \leq ||a|| \cdot ||b|| \cdot ||c||$ for all $a,b,c \in A$.

The first stability problem concerning group homomorphisms was raised by Ulam [25] and affirmatively solved by Hyers [11]. A generalization of Hyers' problem with unbounded Cauchy differences has been considered by Rassias [20], Bourgin [3], and Găvruta [10]. Moreover, Rassias [19] considered the Cauchy difference controlled by a product of different powers of norm. In 1994, Găvruta [10] promoted the stability result into a simple form and reinstated the upper bound by a general control function. This type of stability result accomplished by Găvruta is known as the generalized Hyers-Ulam stability of functional equation. For the history and various aspects of stability theory we refer to [12, 17, 21–23].

Bourgin [2, 3] is the first mathematician dealing with stability of the (ring) homomorphism $f(xy) = f(x)f(y)$. The stability of the approximate homomorphism, the approximate generalized homomorphism $g(xy) = g(x)f(y)$, where f is a (ring) homomorphism, and the derivation on some suitable Banach spaces was studied by a number of mathematicians, see [1, 5–8, 16, 18] and references therein. Let $(A, [\cdot,\cdot,\cdot])$ and $(B, [\cdot,\cdot,\cdot])$ be two non-Archimedean ternary Banach algebras. A mapping $f: A \to B$ is a ternary ring homomorphism or a ternary additive homomorphism if f is additive and satisfies

$$f([x,y,z]) = [f(x), f(y), f(z)]$$

for all $x,y,z \in A$. We introduce the notion of a generalized ternary ring homomorphism between two non-Archimedean ternary Banach algebras A and B as follows. We say that a function $g\colon A \to B$ is a generalized ternary ring homomorphism if g is an additive function and there exists a ternary ring homomorphism $f\colon A \to B$ satisfying

$$g([x,y,z]) = \alpha[g(x), g(y), f(z)] + \beta[g(x), f(y), g(z)] + \gamma[f(x), g(y), g(z)]$$

for all $x,y,z \in A$ and $\alpha,\beta,\gamma \in \mathbb{R}$ with $\alpha+\beta+\gamma = 1$. It is clear that every ternary ring homomorphism g is a generalized ternary ring homomorphism by taking $f = g$ but the converse is not true in general. So, our results recover the stability of the ternary ring homomorphism, see [6, 8].

We now mention the following fixed point theorem [14]. Let (X,d) be a generalized metric space. An operator $F\colon X \to X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \geq 0$ such that $d(Fx,Fy) \leq Ld(x,y)$ for all $x,y \in X$. If the Lipschitz constant L is less than 1, then the operator F is called a strictly contractive operator. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We recall the following theorem by Margolis and Diaz.

Theorem 1 ([14]). *Let (Ω,d) be a complete generalized metric space and $F\colon \Omega \to \Omega$ a strictly contractive mapping with Lipschitz constant L. Then for each $x \in \Omega$, either*

$$d(F^m x, F^{m+1} x) = \infty,$$

for all $m \geq 0$ or there exists a natural number m_0 such that

(1) *$d(F^m x, F^{m+1} x) < \infty$ for all $m \geq m_0$,*
(2) *the sequence $\{F^m x\}$ is convergent to a fixed point y^* of F,*
(3) *y^* is the unique fixed point of F in the set $\Lambda = \{y \in \Omega\colon d(F^{m_0}x, y) < \infty\}$,*
(4) *$d(y,y^*) \leq \frac{1}{1-L} d(y,Fy)$ for all $y \in \Lambda$.*

Throughout this paper we suppose that $(A,[\cdot,\cdot,\cdot])$ and $(B,[\cdot,\cdot,\cdot])$ are two non-Archimedean ternary Banach algebras. For convenience, we use the following abbreviations for two given functions $f_1\colon A \to B$ and $f_2\colon A \to B$,

$$\begin{aligned}
\Delta_m f(x,y) &= mf(\frac{x+y}{m}) - f(x) - f(y),\\
D_{f_1}(x,y,z) &= f_1([x,y,z]) - [f_1(x), f_1(y), f_1(z)],\\
D_{f_1,f_2}(x,y,z) &= f_2([x,y,z]) - \alpha[f_2(x), f_2(y), f_1(z)]\\
&\quad - \beta[f_2(x), f_1(y), f_2(z)] - \gamma[f_1(x), f_2(y), f_2(z)]
\end{aligned}$$

for all $x,y,z \in A$ and $\alpha,\beta,\gamma \in \mathbb{R}$ with $\alpha+\beta+\gamma = 1$, where m is a natural number.

In this paper, using fixed point methods, we investigate the superstability of generalized ternary ring homomorphisms on non-Archimedean ternary Banach algebras.

Moreover, we prove the generalized Hyers-Ulam stability of generalized ternary ring homomorphisms on non-Archimedean ternary Banach algebras associated with the functional equation

$$\Delta_m f(x,y) = 0.$$

2. Superstability of generalized ternary ring homomorphisms on non-Archimedean ternary Banach algebras

In this section we establish the superstability of generalized ternary ring homomorphisms on non-Archimedean ternary Banach algebras. We first need to prove the following lemma.

Lemma 1. *Let $f\colon A \to B$ be a mapping.*

(1) *f is additive if and only if $\Delta_m f(x,y) = 0$ for all $x,y \in A$ and all natural numbers $m \neq 2$,*

(2) *f is additive if and only if $f(0) = 0$ and $\Delta_2 f(x,y) = 0$ for all $x,y \in A$.*

Proof. (1) Since $m \neq 2$, it follows from $\Delta_m f(0,0) = 0$ that $f(0) = 0$. So, $f(x) = \frac{1}{m} f(mx)$ for all $x \in A$. On the other hand, $\Delta_m f(mx,my) = 0$ for all $x,y \in A$. Therefore, f is additive. The converse is obvious.

(2) It follows from $\Delta_2 f(x,0) = 0$ that $f(x) = \frac{1}{2} f(2x)$ for all $x \in A$. Since $\Delta_2 f(2x,2y) = 0$ for all $x,y \in A$, we conclude that f is additive. The converse is obvious. □

In the following two theorems we provide the conditions which give the superstability of generalized ternary ring homomorphisms on on non-Archimedean ternary Banach algebras.

Theorem 2. *Let $f_1\colon A \to B$ and $f_2\colon A \to B$ be two mappings for which there exist some functions $\varphi\colon A \to [0,\infty)$ and $\psi_i\colon A^3 \to [0,\infty)$ such that*

$$\|\Delta_m f_i(x,y)\| \leq \max\{\varphi(x), \varphi(y)\}, \tag{2.1}$$

$$\|D_{f_1}(x,y,z)\| \leq \psi_1(x,y,z), \tag{2.2}$$

$$\|D_{f_1,f_2}(x,y,z)\| \leq \psi_2(x,y,z) \tag{2.3}$$

for all $x,y,z \in A$, $i \in \{1,2\}$, and all natural numbers $m > 2$. If there exists a constant $0 < L < 1$ such that

$$\varphi(mx) \leq |m| L \varphi(x), \tag{2.4}$$

$$\psi_i(mx,my,mz) \leq |m|^3 L \psi_i(x,y,z) \tag{2.5}$$

for all $x,y,z \in A$, then f_2 is a generalized ternary ring homomorphism.

Proof. It follows from (2.4) and (2.5) that

$$\lim_{n\to\infty} \frac{1}{|m|^n} \varphi(m^n x) = 0, \tag{2.6}$$

$$\lim_{n\to\infty}\frac{1}{|m|^{3n}}\psi_i(m^nx,m^ny,m^nz)=0 \tag{2.7}$$

for all $x,y,z\in A$. By (2.6), $\lim_{n\to\infty}\frac{1}{|m|^n}\varphi(0)=0$. Hence, $\varphi(0)=0$. Letting $x=y=0$ in (2.1), we get $\|(m-2)f_i(0)\|\le\varphi(0)=0$. Since $m>2$, $f_i(0)=0$.

Let Ω be the set of all mappings $g\colon A\to B$ and define a generalized metric on Ω as follows

$$d(g,h)=\inf\{t\in(0,\infty)\colon\ \|g(x)-h(x)\|\le t\varphi(x),\ x\in A\}.$$

It is easy to show that (Ω,d) is a generalized complete metric space [4]. Consider the mapping $F\colon\Omega\to\Omega$ defined by $(Fg)(x)=\frac{1}{m}g(mx)$ for all $x\in A$ and all $g\in\Omega$. Let $d(g,h)<t$ for $g,h\in\Omega$ and $t\in(0,\infty)$. Then

$$||g(x)-h(x)||\le t\varphi(x) \tag{2.8}$$

for all $x\in A$. Replace x by mx in (2.8) to find that $||g(mx)-h(mx)||\le t\varphi(mx)$ for all $x\in A$. Using (2.4), we get $||\frac{1}{m}g(mx)-\frac{1}{m}h(mx)||\le Lt\varphi(x)$ for all $x\in A$, which implies $d(Fg,Fh)\le Lt$. Therefore, $d(Fg,Fh)\le Ld(g,h)$ for all $g,h\in\Omega$, that is, F is a strictly contractive mapping of Ω with the Lipschitz constant L. Putting $y=0$ in (2.1), one has

$$\|mf_i(\frac{x}{m})-f_i(x)\|\le\max\{\varphi(x),\varphi(0)\}=\varphi(x) \tag{2.9}$$

for all $x\in A$. Replace x by mx in (2.9) and divide both sides by $|m|$ and then use (2.4) to derive

$$\|f_i(x)-\frac{1}{m}f_i(mx)\|\le\frac{1}{|m|}\varphi(mx)\le L\varphi(x)$$

for all $x\in A$. So, $d(f_i,Ff_i)\le L<\infty$. It follows from the fixed point alternative that there exists a fixed point H_i of F in Ω such that

$$H_i(x)=\lim_{n\to\infty}\frac{1}{m^n}f_i(m^nx) \tag{2.10}$$

for all $x\in A$, since $\lim_{n\to\infty}d(F^nf_i,H_i)=0$. On the other hand, it follows from (2.1), (2.6) and (2.10) that

$$\begin{aligned}\|\Delta_mH_i(x,y)\|&=\lim_{n\to\infty}\frac{1}{|m|^n}\|\Delta_mf_i(m^nx,m^ny)\|\\&\le\lim_{n\to\infty}\frac{1}{|m|^n}\max\{\varphi(m^nx),\varphi(m^ny)\}=0.\end{aligned}$$

Hence, $\Delta_mH_i(x,y)=0$ for all $x,y\in A$. This means that H_i is additive. Using (2.2), (2.7) and (2.10) one can deduce

$$\begin{aligned}||D_{H_1}(x,y,z)||&=\|H_1([x,y,z])-[H_1(x),H_1(y),H_1(z)]\|\\&=\lim_{n\to\infty}\frac{1}{|m|^{3n}}\left\|f_1([m^nx,m^ny,m^nz])-[f_1(m^nx),f_1(m^ny),f_1(m^nz)]\right\|\end{aligned}$$

$$\leq \lim_{n\to\infty} \frac{1}{|m|^{3n}} \psi_1(m^n x, m^n y, m^n z) = 0.$$

So, $H_1([x,y,z]) = [H_1(x), H_1(y), H_1(z)]$ for all $x,y,z \in A$. Therefore, H_1 is a ternary ring homomorphism. Using (2.3), (2.7) and (2.10) one can also deduce

$$\begin{aligned}
||D_{H_1,H_2}(x,y,z)|| &= \Big\| H_2([x,y,z]) - \alpha[H_2(x), H_2(y), H_1(z)] \\
&\quad - \beta[H_2(x), H_1(y), H_2(z)] - \gamma[H_1(x), H_2(y), H_2(z)] \Big\| \\
&= \lim_{n\to\infty} \frac{1}{|m|^{3n}} \Big\| f_2([m^n x, m^n y, m^n z]) - \alpha[f_2(m^n x), f_2(m^n y), f_1(m^n z)] \\
&\quad - \beta[f_2(m^n x), f_1(m^n y), f_2(m^n z)] - \gamma[f_1(m^n x), f_2(m^n y), f_1(m^n z)] \Big\| \\
&= \lim_{n\to\infty} \frac{1}{|m|^{3n}} ||D_{f_1,f_2}(m^n x, m^n y, m^n z)|| \\
&\leq \lim_{n\to\infty} \frac{1}{|m|^{3n}} \psi_2(m^n x, m^n y, m^n z) = 0.
\end{aligned}$$

This indicates that

$$\begin{aligned}
H_2([x,y,z]) &= \alpha[H_2(x), H_2(y), H_1(z)] + \beta[H_2(x), H_1(y), H_2(z)] \\
&\quad + \gamma[H_1(x), H_2(y), H_1(z)]
\end{aligned}$$

for all $x,y,z \in A$. Therefore, H_2 is a generalized ternary ring homomorphism.

Set $x = 0$ in (2.1) to find that

$$\|mf(\frac{y}{m}) - f(y)\| \leq \varphi(0) = 0.$$

This implies $f(y) = \frac{1}{m} f(my)$ for all $y \in A$. According to the fixed point alternative H_i is the unique fixed point of F in the set

$$\Lambda_i = \{g \in \Omega : d(f_i, g) < \infty\},$$

where $i \in \{1,2\}$, so $H_i = f_i$. It follows that f_1 is a ternary ring homomorphism and f_2 is a generalized ternary ring homomorphism. □

The proof of the following theorem is similar to that of Theorem 2, if we define the mapping $F : \Omega \to \Omega$ by $(Fg)(x) = mg(\frac{x}{m})$ for all $x \in A$ and all $g \in \Omega$.

Theorem 3. *Let $f_1 : A \to B$ and $f_2 : A \to B$ be two mappings for which there exist some functions $\varphi : A \to [0,\infty)$ and $\psi_i : A^3 \to [0,\infty)$ such that $\varphi(0) = 0$ and satisfying (2.1), (2.2) and (2.3). If there exists a constant $0 < L < 1$ such that*

$$\varphi(\frac{x}{m}) \leq \frac{1}{|m|} L\varphi(x),$$

$$\psi_i(\frac{x}{m},\frac{y}{m},\frac{z}{m}) \le \frac{1}{|m|^3}L\psi_(x,y,z)$$

for all $x,y,z \in A$ and $i \in \{1,2\}$, then f_2 is a generalized ternary ring homomorphism.

Remark 1. A generalized ternary ring homomorphisms on a non-Archimedean ternary Banach algebra associated with the functional equation $\Delta_2 f(x,y) = 0$ is superstable with an additional assumption $f(0) = 0$.

Corollary 1. *Let λ, p, and s be non-negative real numbers and $f_1\colon A \to B$ and $f_2\colon A \to B$ be two mappings such that $f_i(0) = 0$ and*

$$\|\Delta_m f_i(x,y)\| \le \lambda \max\{\|x\|^p, \|y\|^p\},$$
$$\|D_{f_1}(x,y,z)\| \le \lambda\|x\|^s\|y\|^s\|z\|^s,$$
$$\|D_{f_1,f_2}(x,y,z)\| \le \lambda\|x\|^s\|y\|^s\|z\|^s$$

for all $x,y,z \in A$ and $i \in \{1,2\}$, where $m > 1$ is a natural number. Then each of the following two conditions asserts that f_2 is a generalized ternary ring homomorphism.

(1) $p > 1$ *and* $3s - p > 2$,
(2) $p < 1$ *and* $3s - p < 2$.

Proof. Define

$$\varphi(x) := \lambda\|x\|^p,$$
$$\psi_i(x,y,z) := \lambda\|x\|^s\|y\|^s\|z\|^s$$

for all $x,y,z \in A$ and $i \in \{1,2\}$. If part (i) holds, then choosing $L = |m|^{p-1}$ and applying Theorem 2, we get the desired result. If part (ii) holds, then the result follows from Theorem 3 by letting $L = |m|^{1-p}$. □

3. Stability of generalized ternary ring homomorphisms on non-Archimedean ternary Banach algebras

In this section we consider the generalized Hyers-Ulam stability of generalized ternary ring homomorphisms on non-Archimedean ternary Banach algebras. In the following two theorems we give the conditions which imply the stability of generalized ternary ring homomorphisms on non-Archimedean ternary Banach algebras.

Theorem 4. *Let $f_1\colon A \to B$ and $f_2\colon A \to B$ be two mappings for which there exist some functions $\varphi\colon A^2 \to [0,\infty)$ and $\psi_i\colon A^3 \to [0,\infty)$ such that*

$$\|\Delta_m f_i(x,y)\| \le \varphi(x,y), \tag{3.1}$$
$$\|D_{f_1}(x,y,z)\| \le \psi_1(x,y,z), \tag{3.2}$$
$$\|D_{f_1,f_2}(x,y,z)\| \le \psi_2(x,y,z) \tag{3.3}$$

for all $x,y,z \in A$, $i \in \{1,2\}$ and all natural numbers $m > 2$. If there exists a constant $0 < L < 1$ such that

$$\varphi(mx,my) \le |m|L\varphi(x,y), \tag{3.4}$$

$$\psi_i(mx,my,mz) \le |m|^3 L\psi_i(x,y,z) \tag{3.5}$$

for all $x,y,z \in A$, then there exist a unique ternary ring homomorphism $H_1\colon A \to B$ and a unique generalized ternary ring homomorphism $H_2\colon A \to B$ such that

$$\|f_1(x) - H_1(x)\| \le \frac{L}{1-L}\varphi(x,0), \tag{3.6}$$

$$\|f_2(x) - H_2(x)\| \le \frac{L}{1-L}\varphi(x,0) \tag{3.7}$$

for all $x \in A$.

Proof. It follows from (3.4) and (3.5) that

$$\lim_{n\to\infty} \frac{1}{|m|^n}\varphi(m^n x, m^n y) = 0, \tag{3.8}$$

$$\lim_{n\to\infty} \frac{1}{|m|^{3n}}\psi_i(m^n x, m^n y, m^n z) = 0 \tag{3.9}$$

for all $x,y,z \in A$. By a similar argument as in the proof of Theorem 2 we conclude that $f_i(0) = 0$. Define a generalized metric on Ω, the set of all mappings $g\colon A \to B$, as follows

$$d(g,h) = \inf\{t \in (0,\infty)\colon\ \|g(x) - h(x)\| \le t\varphi(x,0),\ x \in A\}.$$

The space (Ω, d) is a generalized complete metric space [4]. Let $F\colon \Omega \to \Omega$ be a mapping, given by $(Fg)(x) = \frac{1}{m}g(mx)$ for all $x \in A$ and all $g \in \Omega$. The same method as in the proof of Theorem 2 shows that F is a strictly contractive mapping of Ω with the Lipschitz constant L. Put $y = 0$ in (3.1) and apply (3.4) to obtain

$$\|\frac{1}{m}f_i(mx) - f_i(x)\| \le L\varphi(x,0) \tag{3.10}$$

for all $x \in A$. Thus, $d(f_i, Ff_i) \le L < \infty$. From the fixed point alternative it follows that there exists a fixed point H_i of F in Ω such that $\lim_{n\to\infty} d(F^n f_i, H_i) = 0$, so

$$H_i(x) = \lim_{n\to\infty} \frac{1}{m^n} f_i(m^n x) \tag{3.11}$$

for all $x \in A$. Now, by (3.1), (3.8) and (3.11) we conclude that

$$\begin{aligned}\|\Delta_m H_i(x,y)\| &= \lim_{n\to\infty} \frac{1}{|m|^n}\|\Delta_m f_i(m^n x, m^n y)\| \\ &\le \lim_{n\to\infty} \frac{1}{|m|^n}\varphi(m^n x, m^n y) = 0\end{aligned}$$

for all $x,y \in A$ and so H_i is additive. The definition of H_1, (3.2) and (3.9) imply

$$\begin{aligned}||D_{H_1}(x,y,z)|| &= \|H_1([x,y,z]) - [H_1(x), H_1(y), H_1(z)]\| \\ &= \lim_{n\to\infty} \frac{1}{|m|^{3n}}\|f_1([m^n x, m^n y, m^n z]) - [f_1(m^n x), f_1(m^n y), f_1(m^n z)]\|\end{aligned}$$

$$= \lim_{n\to\infty} \frac{1}{|m|^{3n}} ||D_{f_1}(m^n x, m^n y, m^n z)||$$
$$\leq \lim_{n\to\infty} \frac{1}{|m|^{3n}} \psi_1(m^n x, m^n y, m^n z) = 0$$

for all $x,y,z \in A$. Therefore, H_1 is a ternary ring homomorphism. The definition of H_2, (3.3) and (3.9) imply

$$\begin{aligned} ||D_{H_1,H_2}(x,y,z)|| &= \Big\| H_2([x,y,z]) - \alpha[H_2(x),H_2(y),H_1(z)] - \beta[H_2(x),H_1(y),H_2(z)] \\ &\quad - \gamma[H_1(x),H_2(y),H_2(z)] \Big\| \\ &= \lim_{n\to\infty} \frac{1}{|m|^{3n}} \Big\| f_2([m^n x, m^n y, m^n z]) - \alpha[f_2(m^n x), f_2(m^n y), f_1(m^n z)] \\ &\quad - \beta[f_2(m^n x), f_1(m^n y), f_2(m^n z)] - \gamma[f_1(m^n x), f_2(m^n y), f_2(m^n z)] \Big\| \\ &= \lim_{n\to\infty} \frac{1}{|m|^{3n}} ||D_{f_1,f_2}(m^n x, m^n y, m^n z)|| \\ &\leq \lim_{n\to\infty} \frac{1}{|m|^{3n}} \psi_2(m^n x, m^n y, m^n z) = 0 \end{aligned}$$

for all $x,y,z \in A$. Therefore, H_2 is a generalized ternary ring homomorphism. The fixed point alternative yields H_i is the unique mapping such that

$$\|f_i(x) - H_i(x)\| \leq t\varphi(x,0)$$

for all $x \in A$ and $t > 0$. Again using the fixed point alternative, we get

$$d(f_i, H_i) \leq \frac{1}{1-L} d(f_i, F f_i) \leq \frac{L}{1-L}$$

and so we deduce that

$$\|f_i(x) - H_i(x)\| \leq \frac{L}{1-L} \varphi(x,0)$$

for all $x \in A$ and $i \in \{1,2\}$. This completes the proof. □

Theorem 5. *Let $f_1\colon A \to B$ and $f_2\colon A \to B$ be two mappings with $f_i(0) = 0$ for which there exist some functions $\varphi\colon A^2 \to [0,\infty)$ and $\psi_i\colon A^3 \to [0,\infty)$ satisfying (3.1), (3.2), and (3.3). If there exists a constant $0 < L < 1$ such that*

$$\varphi\left(\frac{x}{m}, \frac{y}{m}\right) \leq \frac{1}{|m|} L\varphi(x,y),$$

$$\psi_i\left(\frac{x}{m}, \frac{y}{m}, \frac{z}{m}\right) \leq \frac{1}{|m|^3} L\psi_i(x,y,z)$$

for all $x,y,z \in A$ *and* $i \in \{1,2\}$, *then there exist a unique ternary ring homomorphism* $H_1 \colon A \to B$ *and a unique generalized ternary ring homomorphism* $H_2 \colon A \to B$ *such that*

$$||f_i(x) - H_i(x)|| \leq \frac{1}{1-L}\varphi(x,0)$$

for all $x \in A$.

Proof. Define $(Fg)(x) := mg(\frac{x}{m})$ for all $x \in A$ and $g \in \Omega$ and apply then a similar method as in the proof of Theorem 4 to get the desired result. □

In the following two theorems we show that Theorems 4 and 5 hold for the case where $m = 1,2$.

Theorem 6. *Let* $f_1 \colon A \to B$ *and* $f_2 \colon A \to B$ *be two mappings for which there exist some functions* $\varphi \colon A^2 \to [0,\infty)$ *and* $\psi_i \colon A^3 \to [0,\infty)$ *satisfying* (3.2), (3.3), *and*

$$||\Delta_1 f(x,y)|| \leq \varphi(x,y). \tag{3.12}$$

If there exists a constant $0 < L < 1$ *such that*

$$\varphi(2^r x, 2^r y) \leq |2|^r L\varphi(x,y), \tag{3.13}$$

$$\psi_i(2^r x, 2^r y, 2^r z) \leq |2|^{3r} L\psi_i(x,y,z) \tag{3.14}$$

for all $x,y,z \in A$ *and* $i \in \{1,2\}$, *then there exist a unique ternary ring homomorphism* $H_1 \colon A \to B$ *and a unique generalized ternary ring homomorphism* $H_2 \colon A \to B$ *such that*

$$||f_i(x) - H_i(x)|| \leq \frac{L^{\frac{1-r}{2}}}{|2|(1-L)}\varphi(x,x)$$

for all $x \in A$, *where* $r \in \{-1,1\}$.

Proof. Let $r \in \{-1,1\}$. It follows from (3.13) and (3.14) that

$$\lim_{n\to\infty} \frac{1}{|2|^{nr}}\varphi(2^{nr}x, 2^{nr}y) = \lim_{n\to\infty} \frac{1}{|2|^{3nr}}\psi_i(2^{nr}x, 2^{nr}y, 2^{nr}z) = 0$$

for all $x,y,z \in A$. Let Ω be the set of all mappings $g \colon A \to B$ and define

$$d(g,h) = \inf\{t \in (0,\infty) \colon \ \|g(x) - h(x)\| \leq t\varphi(x,x), \ x \in A\}.$$

Clearly, (Ω, d) is a generalized complete metric space. Define $F \colon \Omega \to \Omega$ by $(Fg)(x) = \frac{1}{2^r} g(2^r x)$ for all $x \in A$ and all $g \in \Omega$. Letting $x = y$ in (3.12) and applying (3.13), we get

$$\|\frac{1}{2^r} f_i(2^r x) - f_i(x)\| \leq \frac{L^{\frac{1-r}{2}}}{|2|}\varphi(x,x)$$

for all $x \in A$. Thus, $d(f_i, Ff_i) \leq \frac{L^{\frac{1-r}{2}}}{|2|} < \infty$. The fixed point alternative implies there exists a fixed point H_i of F in Ω such that $\lim_{n\to\infty} d(F^n f_i, H_i) = 0$, so

$$H_i(x) = \lim_{n\to\infty} \frac{1}{2^{nr}} f_i(2^{nr}x)$$

for all $x \in A$. By a similar method as in the proof of Theorem 4 one can show that H_1 is a ternary ring homomorphism and H_2 is a generalized ternary ring homomorphism. It follows from the fixed point alternative that H_i is the unique mapping such that

$$d(f_i,H_i) \le \frac{1}{1-L} d(f_i,Ff_i) \le \frac{L^{\frac{1-r}{2}}}{1-L}$$

and so we find that

$$\|f_i(x)-H_i(x)\| \le \frac{L^{\frac{1-r}{2}}}{1-L}\varphi(x,x)$$

for all $x \in A$ and $i \in \{1,2\}$. This completes the proof. □

Theorem 7. *Let $f_1 : A \to B$ and $f_2 : A \to B$ be two mappings with $f_i(0) = 0$ for which there exist some functions $\varphi : A^2 \to [0,\infty)$ and $\psi_i : A^3 \to [0,\infty)$ satisfying (3.2), (3.3), and $||\Delta_2 f(x,y)|| \le \varphi(x,y)$. If there exists a constant $0 < L < 1$ such that the inequalities (3.13) and (3.14) hold, then there exist a unique ternary ring homomorphism $H_1 : A \to B$ and a unique generalized ternary ring homomorphism $H_2 : A \to B$ such that*

$$||f_i(x)-H_i(x)|| \le \frac{L^{\frac{r+1}{2}}}{1-L}\varphi(x,0)$$

for all $x \in A$, where $r \in \{-1,1\}$ and $i \in \{1,2\}$.

Proof. The proof is similar to the proof of Theorems 4 and 5 for $m = 2$. □

Corollary 2. *Let $\lambda, \eta, \xi, p, q, s,$ and t be non-negative real numbers with*

$$\max\{2q, s-2, 3t-2\} < p < 1.$$

Let $f_1 : A \to B$ and $f_2 : A \to B$ be two mappings with $f(0) = 0$ and

$$\|\Delta_m f_i(x,y)\| \le \lambda + \eta(\|x\|^p + \|y\|^p) + \xi||x||^q||y||^q,$$
$$\|D_{f_1}(x,y,z)\| \le \lambda + \eta(\|x\|^s + \|y\|^s + \|z\|^s) + \xi||x||^t||y||^t||z||^t,$$
$$\|D_{f_1,f_2}(x,y,z)\| \le \lambda + \eta(\|x\|^s + \|y\|^s + \|z\|^s) + \xi||x||^t||y||^t||z||^t$$

for all $x,y,z \in A$, $i \in \{1,2\}$, and all natural numbers m. Then there exist a unique ternary ring homomorphism $H_1 : A \to B$ and a unique generalized ternary ring homomorphism $H_2 : A \to B$ such that

$$\|f_i(x)-H_i(x)\| \le \begin{cases} \frac{|2|^{1-p}}{|2|(1-|2|^{1-p})}(\lambda + 2\eta||x||^p + \xi||x||^{2q}), & m = 1, \\ \frac{1}{1-|m|^{1-p}}(\lambda + \eta||x||^p), & m \ge 2 \end{cases}$$

for all $x \in A$.

Proof. Define

$$\varphi(x) := \lambda + \eta(\|x\|^p + \|y\|^p) + \xi||x||^q||y||^q,$$
$$\psi_i(x,y,z) := \lambda + \eta(\|x\|^s + \|y\|^s + \|z\|^s) + \xi||x||^t||y||^t||z||^t$$

for all $x,y,z \in A$ and $i \in \{1,2\}$. The proof falls naturally into three parts. For $m=1$ and $m=2$, apply Theorems 6 and 7, respectively by letting $r=-1$ and $L=|2|^{1-p}$. If $m>2$, choose $r=-1$ and $L=|m|^{1-p}$ and apply Theorem 5 to get the desired result. □

Corollary 3. *Let $\eta,\xi,p,q,s,$ and t be non-negative real numbers with*

$$1<p<\min\{2q,s-2,3t-2\}.$$

Let $f_1\colon A\to B$ and $f_2\colon A\to B$ be two mappings such that

$$\|\Delta_m f_i(x,y)\| \le \eta(\|x\|^p+\|y\|^p)+\xi||x||^q||y||^q,$$
$$\|D_{f_1}(x,y,z)\| \le \eta(\|x\|^s+\|y\|^s+\|z\|^s)+\xi||x||^t||y||^t||z||^t,$$
$$\|D_{f_1,f_2}(x,y,z)\| \le \eta(\|x\|^s+\|y\|^s+\|z\|^s)+\xi||x||^t||y||^t||z||^t$$

for all $x,y,z\in A$, $i\in\{1,2\}$, and all natural numbers m. Then there exist a unique ternary ring homomorphism $H_1\colon A\to B$ and a unique generalized ternary ring homomorphism $H_2\colon A\to B$ such that

$$\|f_i(x)-H_i(x)\| \le \begin{cases} \frac{1}{|2|(1-|2|^{p-1})}(2\eta||x||^p+\xi||x||^{2q}), & m=1,\\ \frac{\eta|m|^{p-1}}{1-|m|^{p-1}}||x||^p, & m\ge 2.\end{cases}$$

Proof. Define

$$\varphi(x) := \eta(\|x\|^p+\|y\|^p)+\xi||x||^q||y||^q,$$
$$\psi_i(x,y,z) := \eta(\|x\|^s+\|y\|^s+\|z\|^s)+\xi||x||^t||y||^t||z||^t$$

for all $x,y,z\in A$ and $i\in\{1,2\}$. The proof divides into three parts. For $m=1$ and $m=2$, the result follows from Theorems 6 and 7, respectively by choosing $r=1$ and $L=|2|^{p-1}$. For $m>2$, it is sufficient to choose $r=1$ and $L=|m|^{p-1}$ and then apply Theorem 4 to get the desired result. □

We note that when $p=1$ in Corollaries 2 and 3, the stability result does not hold. Since in this case we have $L=1$ and the condition $0<L<1$ in Theorems 6 and 7 does not fulfill.

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Author’s address

Ismail Nikoufar
Department of Mathematics, Payame Noor University, Tehran, Iran
E-mail address: nikoufar@pnu.ac.ir