Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 591–601

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4545

EXISTENCE OF SOLUTIONS FOR A CLASS OF BVPS ON THE HALF-LINE

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Received 24 January, 2023

Abstract. In this paper, we establish sufficient conditions to guarantee some existence results of nontrivial solutions for a class of nonlinear boundary value problem posed on an unbounded interval of $\mathbb{R}$. According to the behavior of the nonlinear term, an effective operator is considered. Our main tool is especially Schauder's fixed point theorem and the compactness criterion of the Sobolev space $H^1(0,\infty)$ into $C_{l,p}([0,+\infty))$.

2010 *Mathematics Subject Classification:* 47H10; 46E35; 35J25; 46C15

Keywords: fixed point theorem, compact embedding, BVP, Hilbert space

1. INTRODUCTION AND PRELIMINARIES

The existence of solutions of boundary value problems has been studied widely in engineering and other related disciplines. To obtain existence results on the half line is not an easy task and many authors have worked on this subject [2, 4]. Fixed point theory is one of the used tools to overcome the difficulty of proving the existence of solutions on unbounded domains of $\mathbb{R}$. Many researchers have concentred there work in this sense. For new trends on boundary value problems and theory of fixed point theorems, we refer the reader to [8–11]. In this area, there is an important activity among a wide number of investigators. Using available fixed points theorems in Banach space such as Leray-Schauder fixed point theorem and fixed point theorems in cone, authors have proven some existence results in [1, 12, 14]. In this work, we are concerned by some class of nonlinear boundary value problem, where we have used the compactness criterion of the Sobolev space $H^1(0,\infty)$ into $C_{l,p}([0,+\infty))$ to prove some existence results when different conditions on the nonlinear term are assumed.

Let us consider the boundary value problem

$$\begin{cases} -u''(x)+u(x)=f(x,u(x)), & x\in(0,+\infty), \\ u(0)=u(+\infty)=0, \end{cases} \tag{1.1}$$

posed on the half-line, where $f : [0,+\infty] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. According to the behavior of the nonlinear term, the study of Problem (1.1) is investigated. A suitable Hilbert space is chosen and an appropriate compactness criterion which enables us to deal with Schauder's fixed point theorem is used. Firstly, we recall some tools which will used in the sequel.

Theorem 1 (Schauder's fixed point theorem [13]). *Let E be a Banach space and $\mathcal{D} \subset E$ a bounded, closed and convex subset of E. Let $T : \mathcal{D} \to \mathcal{D}$ be a completely continuous operator. Then T has a fixed point in $\mathcal{D}$.*

Let us consider the space $H^1(0,+\infty) = \left\{u \in L^2(0,+\infty),\ u' \in L^2(0,+\infty)\right\}$ which is equipped with the following scalar product

$$\begin{array}{c} (.,.) : H^1(0,+\infty) \times H^1(0,+\infty) \to \mathbb{R} \\ (u,v) = \int_0^{+\infty} u'(x)v'(x)dx + \int_0^{+\infty} u(x)v(x)dx \end{array} \tag{1.2}$$

and the natural norm

$$\|u\| = \left(\int_0^{+\infty} u^2(x)dx + \int_0^{+\infty} u'^2(x)dx \right)^{\frac{1}{2}}.$$

Note that if $u \in H^1(0,+\infty)$, then we get $u(+\infty) = 0$ (see [3], Corollary 8.9). Let

$$C_{l,p}([0,+\infty)) = \left\{ u \in C([0,+\infty),\mathbb{R}) : \lim_{x\to+\infty} p(x)u(x) \text{ exists} \right\}$$

endowed with the norm

$$\|u\|_{\infty,p} = \sup_{x\in[0,+\infty)} p(x)|u(x)|,$$

where $p : [0,+\infty) \to (0,+\infty)$ is continuously differentiable and bounded.

We also need the following results.

Lemma 1 ([6]). *$H^1(0,+\infty)$ embeds continuously in $C_{l,p}([0,+\infty))$, and*

$$\|u\|_{\infty,p} \le M\|u\|,$$

with $M = max(\|p\|_{L^2}, \|p'\|_{L^2}) < \infty$.

Lemma 2 ([6]). *The embedding $H^1(0,+\infty) \hookrightarrow C_{l,p}([0,+\infty))$ is compact.*

2. Existence of Weak Solutions

A weak solution u of Problem (1.1) satisfies

$$\int_0^{+\infty} u'(x)v'(x)dx + \int_0^{+\infty} u(x)v(x)dx = \int_0^{+\infty} f(x,u(x))v(x)dx,\ \ \forall v \in H_0^1(0,+\infty), \tag{2.1}$$

that is,

$$(u,v) = \int_0^{+\infty} f(x,u(x))v(x)dx,\ \ \forall v \in H_0^1(0,+\infty). \tag{2.2}$$

Assume that there exist a constant $\sigma > 0, \sigma \neq 1$ and functions $a(.)$, $b(.)$ such that

(H_1) $|f(x,s)| \leq a(x) + b(x)|s|^{\sigma}$, $\forall x \in (0,+\infty)$, $\forall s \in \mathbb{R}$,

(H_2) $\dfrac{a}{p} \in L^1(0,+\infty)$, and $\dfrac{b}{p^{\sigma+1}} \in L^1(0,+\infty)$,

$$(H_3) \quad \begin{cases} b^*(\frac{1}{\sigma b^*})^{\frac{\sigma}{\sigma-1}} - (\frac{1}{\sigma b^*})^{\frac{1}{\sigma-1}} + a^* \leq 0 \text{ when } \sigma > 1, \\ b^*(\frac{1}{\sigma b^*})^{\frac{\sigma}{\sigma-1}} - (\frac{1}{\sigma b^*})^{\frac{1}{\sigma-1}} + a^* \geq 0 \text{ when } 0 < \sigma < 1, \\ \text{where } a^* := M\|\frac{a}{p}\|_{L^1} \text{ and } b^* := M^{\sigma+1}\|\frac{b}{p^{\sigma+1}}\|_{L^1}. \end{cases} \tag{2.3}$$

For a fixed $u \in H_0^1(0,+\infty)$ and f satisfying (H_1), one can see that

$$\hat{A_u}: v \mapsto (u,v), \quad \hat{T_u}: v \mapsto \int_0^{+\infty} f(x,u(x))v(x)dx$$

are continuous linear functionals on $H_0^1(0,+\infty)$. Since $H_0^1(0,+\infty)$ is a Hilbert space, by Riesz Representation Theorem, there exist Au and Tu in $H_0^1(0,+\infty)$ such that

$$\hat{A_u}(v) = (Au,v) = (u,v) \text{ and } \hat{T_u}(v) = (Tu,v),$$

for all $v \in H_0^1(0,+\infty)$. Hence, from (2.2), we infer that

$$(u,v) = (Tu,v), \text{ for all } v \in H_0^1(0,+\infty),$$

that is, $(u - Tu, v) = 0$, for all $v \in H_0^1(0,+\infty)$. From the last equation, we can say that $u - Tu = 0$, for all $v \in H_0^1(0,+\infty)$, which means that, $Tu = u$ for all $v \in H_0^1(0,+\infty)$. For more details, see [5].

In order to use Schauder's fixed point theorem to prove existence of weak solutions for (1.1) in $H_0^1(0,+\infty)$, let

$$\mathcal{D} = \overline{B}(0,R) = \{u \in H_0^1(0,+\infty);\ \|u\| \leq R\}$$

with a positive number R which will be defined later and consider the operator

$$T : \mathcal{D} \to H_0^1(0,+\infty)$$

defined by

$$\forall v \in H_0^1(0,+\infty),\ (Tu,v) = \int_0^{+\infty} f(x,u(x))v(x)dx.$$

Hence, the fixed points of operator T are weak solutions for Problem (1.1).

Theorem 2. *Under assumptions $(H_1),(H_2)$ and (H_3) Problem (1.1) has at least one solution in $H_0^1(0,+\infty)$.*

Proof. By using assumption (H_3), we can show that according to $\sigma > 1$ and $0 < \sigma < 1$, there exists a suitable $\eta > 0$ which satisfies $a^* + b^*\eta^{\sigma} \leq \eta$. We take $R = \eta$.

Claim-1: T maps $\mathcal{D}$ into itself. Indeed, for any $u \in H_0^1(0,+\infty)$ with $\|u\| \le R$, we have:

$$\begin{aligned}
\|Tu\| &= \sup_{\|v\|\le 1} |(Tu,v)| = \sup_{\|v\|\le 1} \left| \int_0^{+\infty} f(x,u(x))v(x)dx \right| \\
&\le \sup_{\|v\|\le 1} \int_0^{+\infty} |f(x,u(x))||v(x)|dx \\
&\le \sup_{\|v\|\le 1} \|v\|_{\infty,p} \int_0^{+\infty} |f(x,u(x))| \frac{1}{p(x)} dx \\
&\le \sup_{\|v\|\le 1} \|v\|_{\infty,p} \int_0^{+\infty} (a(x)+b(x)|u(x)|^{\sigma}) \frac{1}{p(x)} dx \\
&= \sup_{\|v\|\le 1} \|v\|_{\infty,p} \left[\int_0^{+\infty} a(x) \frac{1}{p(x)} dx + \int_0^{+\infty} b(x)|u(x)|^{\sigma} \frac{1}{p(x)} dx \right] \\
&\le \sup_{\|v\|\le 1} \|v\|_{\infty,p} \left[\int_0^{+\infty} \frac{a(x)}{p(x)} dx + \|u\|_{\infty,p}^{\sigma} \int_0^{+\infty} \frac{b(x)}{(p(x))^{\sigma+1}} dx \right] \\
&\le \sup_{\|v\|\le 1} M\|v\| \left[\int_0^{+\infty} \frac{a(x)}{p(x)} dx + M^{\sigma}\|u\|^{\sigma} \int_0^{+\infty} \frac{b(x)}{(p(x))^{\sigma+1}} dx \right] \\
&\le M \left[\int_0^{+\infty} \frac{a(x)}{p(x)} dx + M^{\sigma}R^{\sigma} \int_0^{+\infty} \frac{b(x)}{(p(x))^{\sigma+1}} dx \right]
\end{aligned}$$

$$\begin{aligned}
\|Tu\| &\le M \int_0^{+\infty} \frac{a(x)}{p(x)} dx + M^{\sigma+1}R^{\sigma} \int_0^{+\infty} \frac{b(x)}{(p(x))^{\sigma+1}} dx \\
&\le M\|\frac{a}{p}\|_{L^1} + M^{\sigma+1}R^{\sigma}\|\frac{b}{p^{\sigma+1}}\|_{L^1} \\
&\le a^* + b^* R^{\sigma} \\
&\le R,
\end{aligned}$$

hence, we have $\|Tu\| \le R$. That is, $T(\mathcal{D}) \subset \mathcal{D}$.

Claim-2: T is compact.

To prove the continuity of T, we consider a sequence $(u_n) \subset \mathcal{D}$ such that $u_n \to u_0$ in $H_0^1(0,+\infty)$ and we show that $Tu_n \to Tu_0$ in $H_0^1(0,+\infty)$. Indeed, if $u_n \to u_0$ in $H_0^1(0,+\infty)$, as $n \to +\infty$, by Lemma 1, we have $u_n \to u_0$ in $C_{l,p}([0,+\infty))$, as $n \to +\infty$ and one can deduce that

$$p(x)|u_n(x) - u_0(x)| \to 0 \text{ as } n \to +\infty$$

for all $x \in [0,+\infty)$, then we have $u_n(x) \to u_0(x)$ as $n \to +\infty$ for all $x \in [0,+\infty)$. By the continuity of f with respect to its second variable, we obtain

$$f(x,u_n(x)) \to f(x,u_0(x)) \text{ as } n \to +\infty.$$

That is,

$$|f(x,u_n(x)) - f(x,u_0(x))| \to 0 \text{ as } n \to +\infty$$

and as $p(x)$ does not vanish on $[0,+\infty)$, we find

$$\frac{1}{p(x)}|f(x,u_n(x)) - f(x,u_0(x))| \to 0 \text{ as } n \to +\infty.$$

Moreover,

$$\frac{M}{p(x)}|f(x,u_n(x)) - f(x,u_0(x))| \le \frac{M}{p(x)}\left(2a(x) + b(x)\left(|u_n(x)|^\sigma + |u_0(x)|^\sigma\right)\right),$$

and one can see that

$$\begin{aligned}
&\int_0^{+\infty} \frac{M}{p(x)}\left(2a(x) + b(x)\left(|u_n(x)|^\sigma + |u_0(x)|^\sigma\right)\right) \\
&= M\int_0^{+\infty} \frac{2a(x)}{p(x)}dx + M\int_0^{+M\infty} \frac{b(x)}{p(x)}|u_n(x)|^\sigma dx + M\int_0^{+\infty} \frac{b(x)}{p(x)}|u_0(x)|^\sigma dx.
\end{aligned}$$

Hence, we get

$$\int_0^{+\infty} \frac{M}{p(x)}|f(x,u_n(x)) - f(x,u_0(x))|dx \le 2a^* + 2b^*R^\sigma < \infty.$$

Then, we have

$$\begin{aligned}
\|Tu_n - Tu_0\| &= \sup_{\|v\|\le 1} |(Tu_n - Tu_0, v)| \\
&= \sup_{\|v\|\le 1} \left|\int_0^{+\infty} (f(x,u_n(x)) - f(x,u_0(x)))\, v(x)dx\right| \\
&\le \sup_{\|v\|\le 1} \int_0^{+\infty} |f(x,u_n(x)) - f(x,u_0(x))||v(x)|dx \\
&\le \sup_{\|v\|\le 1} \|v\|_{\infty,p} \int_0^{+\infty} \frac{1}{p(x)}|f(x,u_n(x)) - f(x,u_0(x))|dx \\
&\le \sup_{\|v\|\le 1} M\|v\| \int_0^{+\infty} \frac{1}{p(x)}|f(x,u_n(x)) - f(x,u_0(x))|dx \\
&\le \int_0^{+\infty} \frac{M}{p(x)}|f(x,u_n(x)) - f(x,u_0(x))|dx.
\end{aligned}$$

By the Lebesgue dominated convergence theorem, we get

$$\|Tu_n - Tu_0\| \to 0 \text{ as } n \to +\infty, \text{ i.e., } Tu_n \to Tu_0 \text{ as } n \to +\infty,$$

which means that T is continuous.

Now, let $(u_n) \subset \mathcal{D}$ be a bounded sequence in $H_0^1(0,+\infty)$ and we show that $(T(u_n))$ is relatively compact. Since (u_n) is bounded in the reflexive space $H_0^1(0,+\infty)$, there exist a subsequence (u_{n_k}) and u_0 such that

$$u_{n_k} \rightharpoonup u_0 \ \text{ in } H_0^1(0,+\infty) \ \text{ as } n \to +\infty \ \text{ with } \ \|u_0\| \le R$$

(see [7]). In view of the compact embedding of $H^1(0,+\infty)$ in $C_{l,p}([0,+\infty))$, we get $u_{n_k} \to u_0$ in $C_{l,p}([0,+\infty))$ as $n \to +\infty$ so $u_{n_k}(x) \to u_0(x)$ as $n \to +\infty$ for all $x \in [0,+\infty)$. Moreover, since f is Carathéodory, we obtain

$$f(x,u_{n_k}(x)) \to f(x,u_0(x)) \text{ as } \ n \to +\infty.$$

Following the same reasoning as above, we show that

$$T(u_{n_k}) \to Tu_0 \text{ as } \ n \to +\infty.$$

Then T is compact and the existence of solutions for Problem (1.1) follows from Schauder's fixed point theorem. □

Example 1. Let us consider the problem

$$\begin{cases} -u''(x)+u(x) = \frac{1}{3}\exp(-2x)+\exp(\frac{-7}{2}x)\dfrac{u^2(x)}{|u(x)|+1}, \ x \in (0,+\infty), \\ u(0) = u(+\infty) = 0. \end{cases} \tag{2.4}$$

This problem has at least one solution. Indeed, if we put

$$f(x,u) = \frac{1}{3}\exp(-2x)+\exp(\frac{-7}{2}x)\frac{u^2}{|u|+1},$$

then we find

$$|f(x,u)| \le \frac{1}{3}\exp(-2x)+\exp(\frac{-7}{2}x)|u|^2.$$

Furthermore, let $p:\ [0,+\infty) \to (0,+\infty)$ be continuously differentiable and bounded mapping defined by $p(x) = \exp(-x)$. It is easily seen that

$$\frac{a}{p} \in L^1(0,+\infty), \ \textit{and} \ \frac{b}{p^{\sigma+1}} \in L^1(0,+\infty),$$

where

$$a(x) = \frac{1}{3}\exp(-2x), \ \ b(x) = \exp(\frac{-7}{2}x) \text{ and } \sigma = 2, \ a^* = \frac{\sqrt{2}}{6}, \ b^* = \frac{\sqrt{2}}{2}.$$

Then assumptions $(H_1),(H_2)$ and (H_3) of Theorem 2 are satisfied.

Now, we prove another existence result of solution for Problem (1.1). Suppose that the nonlinear term f satisfies

$$|f(x,u)| \le p^2(x)H(x,p(x)|u|), \text{ for all } \ x \in (0,+\infty) \text{ and } \ u \in \mathbb{R},$$

with $p \in L^1(0,+\infty) \cap L^2(0,+\infty)$ and $H:(0,+\infty)\times\mathbb{R}^+ \to \mathbb{R}^+$ satisfies

(A_1) H is continuous, nondecreasing with respect to its second variable,

(A_2) There exists $R_0 > 0$ such that $H(x, MR_0) \le \dfrac{R_0}{M\|p\|_{L^1}}$.

Theorem 3. *Under assumptions* (A_1) *and* (A_2)*, Problem (1.1) has at least one solution in* $H_0^1(0,+\infty)$*.*

Proof. Let $\mathcal{D} = \overline{B}(0,R_0) = \left\{u \in H_0^1(0,+\infty);\ \|u\| \le R_0\right\}$ and the operator

$$T:\ \mathcal{D} \to \left(H_0^1(0,+\infty)\right)^*,\ \ u \mapsto Tu$$

with $Tu : H_0^1(0,+\infty) \to \mathbb{R}$ defined by $v \mapsto (Tu,v) = \int_0^{+\infty} f(x,u(x))v(x)dx$.

It can be easily seen that the functional Tu is linear and continuous. To apply Schauder's fixed point theorem, firstly, one can see that T is continuous. Indeed, if a sequence $(u_n) \subset \mathcal{D}$ converges in norm to u_0 as $n \to +\infty$ in $H_0^1(0,+\infty)$, by Lemma 1, we have $u_n \to u_0$ as $n \to +\infty$ in $C_{l,p}([0,+\infty))$, which gives

$$p(x)|u_n(x) - u_0(x)| \to 0 \text{ as } n \to +\infty$$

for all $x \in [0,+\infty)$, implying that $u_n(x) \to u_0(x)$ as $n \to +\infty$ for all $x \in [0,+\infty)$. Moreover, because f is Carathéodory, we get

$$\frac{1}{p(x)}|f(x,u_n(x)) - f(x,u_0(x))| \to 0 \text{ as } n \to +\infty.$$

By the continuity of the embedding of $H_0^1(0,+\infty)$ in $C_{l,p}([0,+\infty))$, we have

$$p(x)|u_n(x| \le \|u_n\|_{\infty,\,p} \le M\|u_n\| \le MR_0$$

and

$$p(x)|u_0(x)| \le \|u_0\|_{\infty,p} \le M\|u_0\| \le MR_0.$$

Then, $[H(x,p(x)|u_n(x)|) + H(x,p(x)|u(x)|)] \le \dfrac{2R_0}{M\|p\|_{L^1}}$. Now, using assumptions on f, we find

$$\int_0^{+\infty} \frac{1}{p(x)}|f(x,u_n(x)) - f(x,u_0(x))|\,dx \le \int_0^{+\infty} p(x)H(x,p(x)|u_n(x)|) + \int_0^{+\infty} p(x)H(x,p(x)|u_0(x)|)dx$$

with

$$\int_0^{+\infty} p(x)\left[H(x,p(x)|u_n(x)|) + H(x,p(x)|u(x)|)\right] \le \frac{2R_0}{M} < \infty.$$

Hence, we can use the Lebesgue dominated convergence theorem to deduce that $Tu_n \to Tu_0$ as $n \to +\infty$, which means that T is continuous. On the other hand, if $(u_n) \subset \mathcal{D}$ is a bounded sequence in $H_0^1(0,+\infty)$, there exist a subsequence (u_{n_k}) and an element u_0 in $H_0^1(0,+\infty)$ such that $u_{n_k} \rightharpoonup u_0$ as $n \to +\infty$ in $H_0^1(0,+\infty)$, and by Lemma 2, we get $u_{n_k} \to u_0$ in $C_{l,p}([0,\ +\infty))$. Then using assumptions on the function H and the Lebesgue dominated convergence theorem, we can show that (Tu_n) has a converging subsequence (Tu_{n_k}) which converges to Tu_0 as $n \to +\infty$. Consequently, T is compact.

Now, we claim that $T(\mathcal{D}) \subset \mathcal{D}$. Indeed, let $u \in H_0^1(0,+\infty)$ such that $\|u\| \leq R_0$, then we have

$$p(x)|u(x)| \leq MR_0$$

and because H is nondecreasing with respect to its second variable, we get

$$H(x, p(x)|u(x)|) \leq H(x, MR_0).$$

Then using (A_1) and (A_2), we obtain

$$\begin{aligned}
\|Tu\| = \sup_{\|v\|\leq 1} |(Tu,v)| &= \sup_{\|v\|\leq 1} |\int_0^{+\infty} f(x,u(x))v(x)dx| \\
&\leq \sup_{\|v\|\leq 1} \int_0^{+\infty} |f(x,u(x))||v(x)|dx \\
&\leq \sup_{\|v\|\leq 1} \|v\|_{\infty,p} \int_0^{+\infty} |f(x,u(x))| \frac{1}{p(x)} dx \\
&\leq \sup_{\|v\|\leq 1} M\|v\| \int_0^{+\infty} |f(x,u(x))| \frac{1}{p(x)} dx \\
&\leq M \int_0^{+\infty} \frac{1}{p(x)} p^2(x) H(x, p(x)|u(x)|)\, dx \\
&\leq M \int_0^{+\infty} p(x) H(x, MR_0)\, dx \\
&\leq M \frac{R_0}{M\|p\|_{L^1}} \int_0^{+\infty} p(x) dx \\
&\leq R_0.
\end{aligned}$$

Hence, Schauder's fixed point theorem ensures the existence of a fixed point of the operator T which is a solution of Problem (1.1). □

This result can be confirmed by the following example.

Example 2. Let us consider the problem

$$\begin{cases} -u''(x) + u(x) = \exp(-3x) \dfrac{u(x)}{\exp(-x)|u(x)|+1} - \exp(-2x), & x \in (0,+\infty), \\ u(0) = u(+\infty) = 0. \end{cases} \tag{2.5}$$

We can show that Problem (2.5) has a nontrivial solution. Let us consider functions p and H defined by

$$p: [0,+\infty) \to (0,+\infty),\ x \mapsto p(x) = \exp(-x)$$

and

$$H:\ (0,+\infty)\times\mathbb{R}^{+}\to\mathbb{R}^{+},\ (x,s)\mapsto H(x,s)=\frac{s}{s+1}+1.$$

It can be easily seen that H is continuous, nondecreasing with respect to its second variable and that $p\in L^{1}(0,+\infty)\cap L^{2}(0,+\infty)$ is continuously differentiable and bounded. Moreover, if we put

$$f(x,u)=\exp(-3x)\frac{u}{\exp(-x)|u|+1}-\exp(-2x),$$

we have

$$|f(x,u)|\le\exp(-2x)\left(\frac{\exp(-x)|u|}{\exp(-x)|u|+1}+1\right),$$

that is,

$$|f(x,u)|\le\exp(-2x)H\left(x,\ \exp(-x)|u|\right).$$

Moreover, for any $R_0\ge 1$, we have $H\left(x,\ \dfrac{R_0}{\sqrt{2}}\right)\le\sqrt{2}R_0$. Then assumptions (A_1) and (A_2) in Theorem 3 are satisfied.

There is an another situation where we can prove the existence of solutions of (1.1). Assume that f satisfies:

(k_1) There exist a nondecreasing function $\Phi\in C(\mathbb{R}^{+},\mathbb{R}^{+})$ and a function $\psi:\mathbb{R}_{+}\to\mathbb{R}_{+}$ such that

$$|f(x,u)|\le\psi(x)\Phi(p(x)|u|),\quad\forall(x,u)\in(0,+\infty)\times\mathbb{R},\tag{2.6}$$

(k_2) $\dfrac{\psi}{p}\in L^{1}(0,+\infty)$ and $\exists r>0$ such that $\Phi(Mr)\le\dfrac{r}{M\|\frac{\psi}{p}\|_{L^1}}$.

Theorem 4. *Under assumptions (k_1) and (k_2), Problem (1.1) has at least one solution.*

Proof. Let $\mathcal{D}=\overline{B}(0,r)=\left\{u\in H_0^1(0,+\infty);\ \|u\|\le r\right\}$ and

$$T:\ \mathcal{D}\to\left(H_0^1(0,+\infty)\right)^{*},\quad u\mapsto\ Tu:H_0^1(0,+\infty)\to\mathbb{R}$$

defined by $(Tu,v)=\int_0^{+\infty}f(x,u(x))v(x)dx$. One can easily check that Tu is linear and continuous. Moreover, when assumptions (k_1) and (k_2) are satisfied, we can prove that the operator T is compact. Indeed, if $(u_n)\subset\mathcal{D}$ is a sequence in $H_0^1(0,+\infty)$ which converges in $H_0^1(0,+\infty)$ to u_0, then we can show that

$$\frac{1}{p(x)}|f(x,u_n(x))-f(x,u_0(x))|\to 0\text{ as }\ n\to+\infty.$$

By the continuous embedding of $H_0^1(0,+\infty)$ in $C_{l,p}([0,+\infty))$, we find

$$\begin{aligned}p(x)|u_n(x)|&\le\|u_n\|_{\infty,\,p}\le M\|u_n\|\le Mr\\ p(x)|u_0(x)|&\le\|u\|_{\infty,\,p}\le M\|u_0\|\le Mr.\end{aligned}\tag{2.7}$$

Moreover, by using $(k_1),(k_2)$ and the Lebesgue dominated convergence theorem, we can deduce that T is continuous. The compactness of T follows in a same manner as in theorems above. Now, it remains to show that T maps $\mathcal{D}$ into itself. Let $u \in \mathcal{D}$, using $(k_1),(k_2)$, we get

$$\begin{aligned}
\|Tu\| = \sup_{\|v\|\leq 1} |(Tu,v)| &= \sup_{\|v\|\leq 1} |\int_0^{+\infty} f(x,u(x))v(x)dx| \\
&\leq \sup_{\|v\|\leq 1} \int_0^{+\infty} |f(x,u(x))||v(x)|dx \\
&\leq \sup_{\|v\|\leq 1} \|v\|_{\infty,p} \int_0^{+\infty} |f(x,u(x))| \frac{1}{p(x)} dx \\
&\leq \sup_{\|v\|\leq 1} M\|v\| \int_0^{+\infty} |f(x,u(x))| \frac{1}{p(x)} dx \\
&\leq M \int_0^{+\infty} \frac{1}{p(x)} \psi(x).\Phi(p(x)|u(x)|)dx \\
&\leq M \int_0^{+\infty} \frac{1}{p(x)} \psi(x).\Phi(Mr)dx \\
&\leq M \frac{r}{M\|\frac{\psi}{p}\|_{L^1}} \int_0^{+\infty} \frac{\psi(x)}{p(x)} dx \\
&\leq M \frac{r}{M\|\frac{\psi}{p}\|_{L^1}} \int_0^{+\infty} \frac{|\psi(x)|}{p(x)} dx \\
&\leq r.
\end{aligned}$$

Hence, Schauder's fixed point theorem gives the required result. □

Example 3. In the following, we show the existence of solution for the problem

$$\begin{cases} -u''(x) + u(x) = \dfrac{\exp(\frac{-5}{2}x)u(x)}{\exp(-x)|u(x)|+2} - \exp(\frac{-3}{2}x), \quad x \in (0,+\infty), \\ u(0) = u(+\infty) = 0. \end{cases} \tag{2.8}$$

If we put $f(x,u) = \dfrac{\exp(\frac{-5}{2}x)u}{\exp(-x)|u|+2} - \exp(\frac{-3}{2}x)$, then

$$\begin{aligned}
|f(x,u)| &\leq \left| \frac{\exp(\frac{-5}{2}x)u}{\exp(-x)|u|+2} \right| + |\exp(\frac{-3}{2}x)| \leq \frac{\exp(\frac{-5}{2}x)|u|}{\exp(-x)|u|+1} + \exp(\frac{-3}{2}x) \\
&\leq \exp(\frac{-3}{2}x) \left(\frac{\exp(-x|u|}{\exp(-x)|u|+1} + 1 \right) \leq \psi(x)\Phi(x, \exp(-x)|u|),
\end{aligned}$$

with

$$\psi(x) = \exp(\frac{-3}{2}x),\ \Phi(x,s) = \frac{s}{s+1}+1,\ p(x) = \exp(-x), \forall x \in (0,+\infty), \forall s \in \mathbb{R}^+.$$

In addition, $\frac{\psi}{p} \in L^1(0,+\infty)$ with $\|\frac{\psi}{p}\|_{L^1} = 2$ and for any $r \geq \frac{1+\sqrt{5}}{\sqrt{2}}$, we have

$$\Phi\left(\frac{r}{\sqrt{2}}\right) \leq \frac{r}{\sqrt{2}}.$$

Then assumptions (k_1), (k_2) in Theorem 4 are satisfied.

Remark 1. In all the above theorems, to obtain the solutions of problems is not trivial. So we must for example suppose that there exists $x_0 \in (0,+\infty)$ such that $f(x_0,0) \neq 0$.

ACKNOWLEDGEMENTS

The authors express their gratitude to the anonymous referees for their helpful suggestions and corrections.

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