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APPROXIMATE SOLUTION OF THE CONFORMABLE INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, fractional linear and nonlinear integro-differential equations are solved by using an iteration method. Fractional derivative and fractional integral are considered in the conformable sense. The conformable integro-differential equation is converted to a conformable integral equation. Then, the conformable integral equation leads to an iteration sequence, the limit of which is a solution of the conformable integro-differential equation. In addition, stability and convergence analysis of the presented method are investigated. The applicability of the presented method is also shown by using numerical examples.

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1. INTRODUCTION

In recent years, the theory of fractional calculus has attracted the attention of many researchers in various disciplines due to the fact that fractional calculus is more useful for formulation of the problems in physics, chemistry and engineering such as the field of electromagnetic waves, viscoelasticity, dielectric polarization and diffusion equations [7, 10, 12, 13, 16, 18, 19, 21, 24].

Conformable fractional derivative has been defined by Khalil et al. ([14]) and has also some applications in the real world (see, for example, [3, 15, 20, 23]). In the last decades, solutions of the conformable fractional integro-differential equations have been investigated by researchers. In [4], existence of the solutions of periodic boundary value problems for impulsive fractional integro-differential equations has been investigated. In [5], fractional linear Volterra-Fredholm integro-differential equations have been solved by using the Sinc-Collocation Method. In [17], fractional linear integral and integro-differential equations have been solved by using the conformable Laplace transform. Laguerre polynomial solutions of the linear fractional integro differential equations have been obtained in [9]. Conformable linear and nonlinear fractional integro-differential systems of second order have been solved by using the reproducing kernel Hilbert space method in [6]. Variational iteration method has been applied to the conformable linear and nonlinear fractional partial integro-differential equation in [8]. A computational approach based on the shifted Legendre polynomials has been presented for solving the conformable linear and nonlinear fractional partial integro-differential equation in [2].

In the present paper, we consider conformable fractional integro-differential equations in the following form

$$T_\alpha^a y(t) = f(t) + \lambda\,(I_\alpha^a F(t,x,y(x)))\,(t), \qquad 0<\alpha<1, \tag{1.1}$$

with initial condition

$$y(a) = c, \tag{1.2}$$

where $f(t)$ is continuous for $\forall t \in [a,T]$ and $F(x,t,y(t))$ is continuous for $\forall (x,t) \in [a,T]\times[a,T]$, $\forall y \in \Re$, $T>0$, T_α^a is conformable fractional derivative of order α and I_α^a is conformable fractional integral of order α.

The rest of the paper is organized as follows. In Section 2, we present the basic properties of the conformable fractional derivative and integral. In Section 3, we construct an iteration method. In addition, we present stability and convergence analysis of the iteration method. In Section 4, we show accuracy of the method by using numerical examples. We give conclusion in the last section.

2. Description of the conformable fractional derivative and integrals

For a function $f\colon (a,\infty)\to R$, the conformable fractional derivative of f of order $0<\alpha<1$ in variable t is defined as follows: (see, for example, [14])

$$T_\alpha^a f(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon(t-a)^{1-\alpha}) - f(t)}{\varepsilon}.$$

Lemma 1 ([14, Theorem 2.2]). *Let $\alpha \in (0,1]$ and f,g be α differentiable at a point $t>0$. Then*

$$\begin{aligned} T_\alpha^a(c_1 f + c_2 g)(t) &= c_1 T_\alpha^a f(t) + c_2 T_\alpha^a g(t), \qquad \forall c_1,c_2 \in R,\\ T_\alpha^a(fg)(t) &= f(t)T_\alpha^a g(t) + g(t) T_\alpha^a f(t),\\ T_\alpha^a(f)(t) &= (t-a)^{1-\alpha} f'(t). \end{aligned}$$

Conformable fractional integral of f of order $0<\alpha<1$ is defined by (see, for example, [1])

$$(I_\alpha^a f)(t) = \int_a^t f(x)\,\mathrm{d}_\alpha(x,a) = \int_a^t f(x)(x-a)^{\alpha-1}\mathrm{d}x.$$

Note that for $a = 0$, the conformable fractional derivative of f is denoted by $T_\alpha f(t)$ and the conformable fractional integral of f is denoted by $(I_\alpha f)(t) = \int_0^t f(x) \mathrm{d}_\alpha(x)$.

Lemma 2 ([1, Lemma 2.8]). *Let $f : (a,b) \to R$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > a$ we have*

$$(I_\alpha^a T_\alpha^a f)(t) = f(t) - f(a).$$

3. Iterative Method

Theorem 1. *Consider the following conformable integro-differential equation with initial condition*

$$\begin{aligned} T_\alpha^a y(t) &= f(t) + \lambda \left(I_\alpha^a F(t,x,y(x))\right)(t), \qquad 0 < \alpha < 1, \\ y(a) &= c. \end{aligned} \tag{3.1}$$

Let $y(t)$ be differentiable for $\forall t \in (a,T)$. Then the general solution of the above equation is equivalent to the general solution of the following equation

$$y(t) = y(a) + (I_\alpha^a f)(t) + \lambda \left(I_\alpha^a \int_a^u F(u,x,y(x))\, d_\alpha(x,a) \right)(t), \qquad 0 < \alpha < 1. \tag{3.2}$$

Proof. Applying the operator I_α^a to Eq. (3.1) and using Lemma 2, we have

$$y(t) - y(a) = (I_\alpha^a f)(t) + \lambda \left(I_\alpha^a \int_a^u F(u,x,y(x))\, d_\alpha(x,a) \right)(t).$$

□

Using Eq. (3.2), the iteration formula for the solution of the problem (1.1)-(1.2) is constructed as follows

$$\begin{aligned} y_k(t) &= y_0(t) + \lambda \left(I_\alpha^a \int_a^u F(u,x,y_{k-1}(x))\, d_\alpha(x,a) \right)(t), \quad k = 1,2,\ldots, \\ y_0(t) &= y(a) + (I_\alpha^a f)(t). \end{aligned} \tag{3.3}$$

Theorem 2. *The sequence $y_k(t)$ defined by (3.3) is convergent for $\forall t \in [a,T]$ when $F(x,t,y(t))$ is continuous; $\| F(x,t,y(t)) \|_\infty < M$; $\| F(x,t,y_1(t)) - F(x,t,y_2(t)) \|_\infty < L. \| y_1(t) - y_2(t) \|_\infty$, $\forall y_1(t), y_2(t) \in \Re$, $\forall (x,t) \in [a,T] \times [a,T]$ and some $L, M > 0$.*

Proof. Taking $k = 1$ in the iteration formula (3.3), we have

$$|y_1(t) - y_0(t)| \leq |\lambda| \left(I_\alpha^a \int_a^u |F(u,x,y_0(x))| |d_\alpha(x,a)| \right)(t) \leq |\lambda| M \frac{|t-a|^{2\alpha}}{2\alpha^2}$$

and by induction

$$|y_k(t)-y_{k-1}(t)| \leq \frac{M}{L}|\lambda|^k L^k.\frac{|t-a|^{2k\alpha}}{(2k)!\alpha^{2k}}$$

is obtained. Since $y_k(t) = y_0(t) + \sum_{i=1}^{k}(y_i(t)-y_{i-1}(t))$ and

$$\sum_{k=0}^{\infty}|\lambda|^k L^k.\frac{|t-a|^{2k\alpha}}{(2k)!\alpha^{2k}} = E_{2,1}\left(L|\lambda|.\frac{|t-a|^{2\alpha}}{\alpha^2}\right)$$

in the whole line, the sequence $y_k(t)$ is convergent for $t \in [a,T]$. Here $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{Z^k}{\Gamma(\alpha k+\beta)}$, $\alpha,\beta > 0$, is the Mittag-Leffler function (see, for example, [22]). □

Theorem 3. *Let* $\varphi(t) = \lim_{k\to\infty} y_k(t)$ *for* $\forall t \in [a,T]$. *Assume that* $F(x,t,y(t))$ *is continuous;* $\| F(x,t,y(t)) \|_\infty < M$; $\quad \| F(x,t,y_1(t)) - F(x,t,y_2(t)) \|_\infty < L. \| y_1(t) - y_2(t) \|_\infty$, $\quad \forall y_1(t), y_2(t) \in \Re$, $\quad \forall(x,t) \in [a,T]\times[a,T]$ *and some* $L,M > 0$. *Then for* $\forall t \in [a,T]$

$$\lim_{k\to\infty}\left(I_\alpha^a\int_a^u F(u,x,y_k(x))\,d_\alpha(x,a)\right)(t) = \left(I_\alpha^a\int_a^u F(u,x,\varphi(x))\,d_\alpha(x,a)\right)(t). \tag{3.4}$$

Proof. Since

$$y_k(t) = y_0(t) + \sum_{i=1}^{k}(y_i(t)-y_{i-1}(t))$$

and

$$\varphi(t) = y_0(t) + \sum_{i=1}^{\infty}(y_i(t)-y_{i-1}(t))$$

we can write

$$|y_k(t)-\varphi(t)| \leq \sum_{i=k+1}^{\infty}|y_i(t)-y_{i-1}(t)| \leq \sum_{i=k+1}^{\infty}\frac{M}{L}|\lambda|^i.L^i.\frac{|t-a|^{2i\alpha}}{(2i)!\alpha^{2i}}. \tag{3.5}$$

Let $z = |\lambda| L\frac{|t-a|^{2\alpha}}{\alpha^2}$. Since $E_{2,1}(z) = \frac{\exp(\sqrt{z})+\exp(-\sqrt{z})}{2} = \sum_{i=0}^{\infty}\frac{z^i}{(2i)!}$ (see, for example, [11]) and $E_{2,1}(z) = 1+z+\frac{z^2}{2!}+\cdots+\frac{z^k}{(2k)!}+\exp(\theta)\frac{z^{k+1}}{(2k+2)!}$, $|\theta| < \sqrt{z}$, we have

$$|E_{2,1}(z)-(1+z+\frac{z^2}{2!}+\cdots+\frac{z^k}{(2k)!})| \leq \exp(\sqrt{z})\frac{|z|^{k+1}}{(2k+2)!}. \tag{3.6}$$

From (3.6), (3.5) can be written as

$$|y_k(t)-\varphi(t)| \leq \frac{M}{L}\exp\left(\sqrt{|\lambda| L}\frac{|T-a|^\alpha}{|\alpha|}\right)\frac{L^{k+1}|\lambda|^{k+1}|T-a|^{2\alpha(k+1)}}{\alpha^{2k+2}(2k+2)!}. \tag{3.7}$$

From (3.7), we have

$$\left|\left(I_\alpha^a\int_a^u F(u,x,y_k(x))\,d_\alpha(x,a)\right)(t)-\left(I_\alpha^a\int_a^u F(u,x,\varphi(x))\,d_\alpha(x,a)\right)(t)\right|$$
$$\leq M.\exp(\sqrt{|\lambda|L}\frac{|T-a|^\alpha}{|\alpha|})\frac{L^{k+1}|\lambda|^{k+1}|T-a|^{2\alpha(k+2)}}{2\alpha^{2k+4}(2k+2)!}.$$

Since $\exp(\sqrt{|\lambda|L}\frac{|T-a|^\alpha}{|\alpha|})\frac{L^{k+1}|\lambda|^{k+1}|T-a|^{2\alpha(k+2)}}{2\alpha^{2k+4}(2k+2)!}\to 0$ as $k\to\infty$, for $k\to\infty$

$$\left(I_\alpha^a\int_a^u F(u,x,y_k(x))\,d_\alpha(x,a)\right)(t)\to\left(I_\alpha^a\int_a^u F(u,x,\varphi(x))\,d_\alpha(x,a)\right)(t).$$

□

Theorem 4. *Assume that $F(x,t,y(t))$ is continuous; $\| F(x,t,y(t)) \|_\infty < M$; $\| F(x,t,y_1(t))-F(x,t,y_2(t)) \|_\infty < L.\ \| y_1(t)-y_2(t) \|_\infty$, $\forall y_1(t),y_2(t)\in\Re$, $\forall (x,t)\in [a,T]\times[a,T]$ and some $L,M>0$. If the sequence $y_k(t)$ defined by (3.3) converges to $\varphi(t)$, then $\varphi(t)$ is the exact solution of problem (1.1)-(1.2) for $\forall t\in[a,T]$.*

Proof. Taking limit as k approaches to infinity in the iteration formula (3.3) and using (3.4) we have

$$\varphi(t)=y_0(t)+\lambda\left(I_\alpha^a\int_a^u F(u,x,\varphi(x))\,d_\alpha(x,a)\right)(t).$$

□

Theorem 5. *Assume that the exact solution of the problem (1.1)-(1.2) is $\varphi(t)$ for $\forall t\in[a,T]$ and the approximate solution is $y_k(t)$ for $\forall t\in[a,T]$, where $y_k(t)$ is the k_{th} order approximation, $F(x,t,y(t))$ is continuous; $\| F(x,t,y(t)) \|_\infty < M$; $\| F(x,t,y_1(t))-F(x,t,y_2(t)) \|_\infty < L.\ \| y_1(t)-y_2(t) \|_\infty$, $\forall y_1(t),y_2(t)\in\Re$, $\forall (x,t)\in[a,T]\times[a,T]$ and some $L,M>0$. Then for $\forall t\in[a,T]$ the maximum error is estimated as*

$$|y_k(t)-\varphi(t)|\leq\frac{M}{L}\exp\left(\sqrt{|\lambda|L}\frac{|T-a|^\alpha}{|\alpha|}\right).E$$

where

$$E=\frac{L^{k+1}|\lambda|^{k+1}|T-a|^{2\alpha(k+1)}}{\alpha^{2k+2}(2k+2)!}.$$

Proof. The proof can be shown from (3.7). □

Theorem 6. *Let $y(t)$, $\bar{y}(t)$ be differentiable for $\forall t\in(a,T)$ and be the solutions of the following problems*

$$T_\alpha^a y(t)=f(t)+\lambda\,(I_\alpha^a F(t,x,y(x)))\,(t),\qquad 0<\alpha<1,$$

$$y(a) = c. \tag{3.8}$$

$$T^a_\alpha \bar{y}(t) = g(t) + \lambda\,(I^a_\alpha F(t,x,\bar{y}(x)))\,(t), \qquad 0 < \alpha < 1,$$
$$\bar{y}(a) = d. \tag{3.9}$$

Assume that there exists constants H and L such that $\| f(t) - g(t) \|_\infty \leq H$, $\forall t \in [a,T]$; $\| F(x,t,y_1(t)) - F(x,t,y_2(t)) \|_\infty < L.\ \| y_1(t) - y_2(t) \|_\infty$ *for* $\forall y_1(t), y_2(t) \in \Re$ *and* $\forall (x,t) \in [a,T] \times [a,T]$, *then the following inequality is satisfied*

$$|y(t) - \bar{y}(t)| \leq C.E_{2,1}\left(L|\lambda| \frac{(t-a)^{2\alpha})}{\alpha^2}\right), \qquad \forall t \in [a,T],$$

where $C = |c-d| + H\frac{|T-a|^\alpha}{\alpha}$.

Proof. From Thm.(1), the iteration formulas for Eq. (3.8) and Eq. (3.9) can be written in the following form

$$y_k(t) = y_0(t) + \lambda \left(I^a_\alpha \int_a^u F(u,x,y_{k-1}(x))\mathrm{d}_\alpha(x,a) \right)(t), \quad k = 1,2,\ldots, \tag{3.10}$$

$$y_0(t) = c + (I^a_\alpha f)(t) \tag{3.11}$$

and

$$\bar{y}_k(t) = \bar{y}_0(t) + \lambda \left(I^a_\alpha \int_a^u F(u,x,\bar{y}_{k-1}(x))\,d_\alpha(x,a) \right)(t), \quad k = 1,2,\ldots, \tag{3.12}$$

$$\bar{y}_0(t) = d + (I^a_\alpha g)(t). \tag{3.13}$$

From Eq. (3.11) and Eq. (3.13), we have

$$|y_0(t) - \bar{y}_0(t)| \leq |c-d| + H\frac{|t-a|^\alpha}{\alpha} \leq |c-d| + H\frac{|T-a|^\alpha}{\alpha} = C.$$

From Eq. (3.10) and Eq. (3.12) for $k = 1$ we have

$$\begin{aligned} |y_1(t) - \bar{y}_1(t)| &\leq |y_0(t) - \bar{y}_0(t)| \\ &\quad + |\lambda| \Big(I^a_\alpha \int_a^u |F(u,x,y_0(x)) - F(u,x,\bar{y}_0(x))|\,d_\alpha(x,a) \Big)(t) \\ &\leq C + |\lambda|.L.C\frac{|t-a|^{2\alpha}}{2\alpha^2}. \end{aligned}$$

From Eq. (3.10) and Eq. (3.12) for $k = 2$ we have

$$\begin{aligned} |y_2(t) - \bar{y}_2(t)| &\leq |y_0(t) - \bar{y}_0(t)| \\ &\quad + |\lambda| \Big(I^a_\alpha \int_a^u |F(u,x,y_1(x)) - F(u,x,\bar{y}_1(x))|\,d_\alpha(x,a) \Big)(t) \end{aligned}$$

$$\leq C + |\lambda|.L.C\frac{|t-a|^{2\alpha}}{2\alpha^2} + |\lambda|^2.L^2.C\frac{|t-a|^{4\alpha}}{4!\alpha^4}.$$

Similarly, k^{th} term is written as follows

$$|y_k(t) - \bar{y}_k(t)| \leq C\sum_{i=0}^{k} |\lambda|^i.L^i.\frac{|t-a|^{2i\alpha}}{(2i)!\alpha^{2i}}. \tag{3.14}$$

Since $y(t) = \lim_{k\to\infty} y_k(t)$ and $\bar{y}(t) = \lim_{k\to\infty} \bar{y}_k(t)$, taking the limit of (3.14) as $k \to \infty$ we have

$$|y(t) - \bar{y}(t)| \leq C\sum_{i=0}^{\infty} |\lambda|^i.L^i.\frac{|t-a|^{2i\alpha}}{(2i)!\alpha^{2i}} = C.E_{2,1}\left(L|\lambda|\frac{(t-a)^{2\alpha})}{\alpha^2}\right).$$

□

It concludes from this theorem that small changes in initial condition and non-homogenous term cause only small changes of the obtained solution.

4. Applications

Example 1. Let us consider the following conformable integro-differential equation

$$T_{\frac{1}{2}} y(t) = 1 + \int_0^t y(x)\, d_{\frac{1}{2}}(x) \tag{4.1}$$

with initial condition

$$y(0) = 1. \tag{4.2}$$

Applying the operator $I_{\frac{1}{2}}$ to Eq. (4.1) we have the following equation

$$y(t) = 1 + (I_{\frac{1}{2}} 1)(t) + \left(I_{\frac{1}{2}} \int_0^u y(x)\, d_{\frac{1}{2}}(x)\right)(t)$$

and its iteration formula is obtained by

$$y_k(t) = y_0(t) + \left(I_{\frac{1}{2}} \int_0^u y_{k-1}(x)\, d_{\frac{1}{2}}(x)\right)(t), \quad k = 1, 2, \ldots,$$

$$y_0(t) = 1 + 2\sqrt{t}.$$

Therefore, we have the following approximations

$$y_1(t) = 1 + 2\sqrt{t} + 2t + \frac{4}{3}t^{3/2},$$

$$y_2(t) = 1 + 2\sqrt{t} + 2t + \frac{4}{3}t^{3/2} + \frac{4}{15}t^{5/2} + \frac{2}{3}t^2,$$

$$y_3(t) = 1 + 2\sqrt{t} + 2t + \frac{4}{3}t^{3/2} + \frac{4}{15}t^{5/2} + \frac{2}{3}t^2 + \frac{4}{45}t^3 + \frac{8}{315}t^{7/2},$$
$$y_4(t) = 1 + 2\sqrt{t} + 2t + \frac{4}{3}t^{3/2} + \frac{4}{15}t^{5/2} + \frac{2}{3}t^2 + \frac{4}{45}t^3 + \frac{8}{315}t^{7/2} + \frac{2}{315}t^4 + \frac{4}{2835}t^{9/2}.$$

The k^{th} approximation is obtained as follows

$$y_k(t) = \sum_{i=0}^{2k+1} \frac{2^i t^{i/2}}{i!}.$$

Then $y(t) = \lim_{k\to\infty} y_k(t) = \sum_{i=0}^{\infty} \frac{2^i t^{i/2}}{i!} = \exp(2\sqrt{t})$ (see, for example, [1]). Note that the $y(t) = \exp(2\sqrt{t})$ is the exact solution of the problem (4.1)-(4.2).

Example 2. Let us consider the following conformable integro-differential equation

$$T_{\frac{1}{3}} y(t) = \frac{1}{3} + \frac{2}{3}t^{1/3} + \frac{9}{10}t^{5/3} + \frac{1}{2}t^2 + \int_0^t (x-t)y(x)\,d_{\frac{1}{3}}(x) \tag{4.3}$$

with initial condition

$$y(0) = 0. \tag{4.4}$$

Applying the operator $I_{\frac{1}{3}}$ to Eq. (4.3) we have the following equation

$$y(t) = \left(I_{\frac{1}{3}}(\frac{1}{3} + \frac{2}{3}u^{1/3} + \frac{9}{10}u^{5/3} + \frac{1}{2}u^2)\right)(t) + \left(I_{\frac{1}{3}} \int_0^u (x-u)y(x)\,d_{\frac{1}{3}}(x)\right)(t)$$

and its iteration formula is obtained by

$$y_k(t) = y_0(t) + \left(I_{\frac{1}{3}} \int_0^u (x-u)y_{k-1}(x)\,d_{\frac{1}{3}}(x)\right)(t), \quad k = 1, 2, \ldots,$$
$$y_0(t) = t^{1/3} + t^{2/3} + \frac{9}{20}t^2 + \frac{3}{14}t^{7/3}.$$

Therefore, we have the following approximations

$$y_1(t) = t^{1/3} + t^{2/3} - \frac{243}{15400}t^{11/3} - \frac{27}{4928}t^4,$$
$$y_2(t) = t^{1/3} + t^{2/3} + \frac{729}{4928000}t^{16/3} + \frac{729}{17425408}t^{17/3},$$
$$y_3(t) = t^{1/3} + t^{2/3} - \frac{6561}{11728640000}t^7 - \frac{729}{5367025664}t^{22/3},$$

$$y_4(t) = t^{1/3} + t^{2/3} + \frac{177147}{167719552000000} t^{26/3} + \frac{729}{3209481347072} t^9,$$
$$y_5(t) = t^{1/3} + t^{2/3} - \frac{59049}{51993061120000000} t^{31/3} - \frac{19683}{89146553896271872} t^{32/3},$$
$$y_6(t) = t^{1/3} + t^{2/3} + \frac{177147}{2329289138176000000000} t^{12} + \frac{19683}{14513058974313060761 6} t^{37/3}.$$

The k^{th} approximation can be written in the following form

$$y_k(t) = t^{1/3} + t^{2/3} + 10^{-2k}.c_1 t^{c_2}, \qquad c_1, c_2 \in R.$$

Then $y(t) = \lim_{k\to\infty} y_k(t) = t^{1/3} + t^{2/3}$. Note that the $y(t) = t^{1/3} + t^{2/3}$ is the exact solution of the problem (4.3)-(4.4).

Example 3. Let us consider the following conformable integro-differential equation

$$T_{\frac{1}{2}} y(t) = 2t\sqrt{t}\cos(2\sqrt{t}) - \sin(2\sqrt{t}) - t\sin(2\sqrt{t}) + 2t^2\sin(2\sqrt{t}) - 2\int_0^t xty(x)\, d_{\frac{1}{2}}(x) \tag{4.5}$$

with initial condition

$$y(0) = 1. \tag{4.6}$$

Applying the operator $I_{\frac{1}{2}}$ to Eq. (4.5) we have the following equation

$$y(t) = 1 + \left(I_{\frac{1}{2}}(2u\sqrt{u}\cos(2\sqrt{u}) - \sin(2\sqrt{u}) - u\sin(2\sqrt{u}) + 2u^2\sin(2\sqrt{u}))\right)(t) - 2\left(I_{\frac{1}{2}} \int_0^u xty(x)\, d_{\frac{1}{2}}(x)\right)(t)$$

and its iteration formula is obtained by

$$y_k(t) = y_0(t) + \left(I_{\frac{1}{2}} \int_0^u xty_{k-1}(x)\, d_{\frac{1}{2}}(x)\right)(t), \quad k = 1, 2, \ldots,$$
$$y_0(t) = 1 - 2t + \frac{2t^2}{3} + \frac{16t^3}{45} - \frac{124t^4}{315} + \frac{1076x^5}{14175} - \frac{3076x^6}{467775} + \cdots .$$

Therefore, we have the following approximations

$$y_1(t) = 1 - 2t + \frac{2t^2}{3} - \frac{4t^3}{45} + \frac{2t^4}{315} - \frac{4t^5}{14175} - \frac{5132t^6}{155925} + \cdots$$

$$y_2(t) = 1 - 2t + \frac{2t^2}{3} - \frac{4t^3}{45} + \frac{2t^4}{315} - \frac{4t^5}{14175} + \frac{4t^6}{467775} - \frac{8t^7}{42567525}$$
$$+ \frac{2t^8}{638512875} + \frac{95295196t^9}{97692469875} + \cdots .$$

Note that $\cos(2\sqrt{t}) = \sum_{i=0}^{\infty} (-1)^i \frac{t^i 2^{2i}}{(2i)!}$ (see, for example, [1]). Then $y(t) \simeq \cos(2\sqrt{t})$ is the exact solution of the problem (4.5)-(4.6).

Example 4. Let us consider the following conformable integro-differential equation

$$T_{\frac{1}{2}} y(t) = 2t\sqrt{t} + \frac{2}{45}\sqrt{t}(45 + 36t^2 + 20t^4)$$
$$- \int_0^t (x^2 + y(x))^2 \, d_{\frac{1}{2}}(x) \tag{4.7}$$

with initial condition

$$y(0) = 1. \tag{4.8}$$

Applying the operator $I_{\frac{1}{2}}$ to Eq. (4.7) we have the following equation

$$y(t) = 1 + \left(I_{\frac{1}{2}} (2u\sqrt{u} + \frac{2}{45}\sqrt{u}(45 + 36u^2 + 20u^4)) \right)(t)$$
$$- \left(I_{\frac{1}{2}} \int_0^u (x^2 + y(x))^2 \, d_{\frac{1}{2}}(x) \right)(t)$$

and its iteration formula is obtained by

$$y_k(t) = y_0(t) + \left(I_{\frac{1}{2}} \int_0^u (x^2 + y_{k-1}(x))^2 \, d_{\frac{1}{2}}(x) \right)(t), \quad k = 1, 2, \ldots,$$
$$y_0(t) = 1 + 2t + t^2 + \frac{8}{15}t^3 + \frac{8}{45}t^5.$$

Therefore, we have the following approximations

$$y_1(t) = 1 - \frac{t^2}{2} - \frac{8}{15}t^3 - \frac{68}{105}t^4 - \frac{64}{675}t^5 + \cdots - \frac{128}{467775}t^{11},$$
$$y_2(t) = 1 + t^2 + \frac{16}{45}t^3 + \frac{8}{105}t^4 + \frac{3056}{14175}t^5 + \cdots - \frac{32768}{226471921396875}t^{23},$$
$$y_3(t) = 1 + t^2 - \frac{16}{315}t^4 - \frac{32}{4725}t^5 - \frac{26272}{467775}t^6 + \cdots + 10^{-23}t^{47},$$
$$y_4(t) = 1 + t^2 + \frac{64}{14175}t^5 + \frac{64}{155925}t^6 + \frac{295168}{42567525}t^7 + \cdots + 10^{-50}t^{95},$$

$$
\begin{aligned}
y_5(t) &= 1+t^2-\frac{128}{467775}t^6-\frac{256}{14189175}t^7+\cdots-3.10^{-105}t^{191},\\
y_6(t) &= 1+t^2+10^{-5}t^7+10^{-7}t^8+\cdots,\\
y_7(t) &= 1+t^2+4.10^{-7}t^8+10^{-8}t^9+\cdots,\\
y_8(t) &= 1+t^2+10^{-8}t^9+10^{-10}t^{10}+\cdots .
\end{aligned}
$$

$y(t)=\lim_{k\to\infty} y_k(t)=1+t^2$. Note that the $y(t)=1+t^2$ is the exact solution of the problem (4.7)-(4.8).

Example 5. Let us consider the following conformable integro-differential equation

$$
T_{\frac{1}{2}}y(t)=\sqrt{t}+\frac{4}{315}t^{\frac{5}{2}}(35t^3-135t^2+189t-105) -2\int_0^t xty^3(x)\,d_{\frac{1}{2}}(x) \tag{4.9}
$$

with initial condition

$$
y(0)=-1. \tag{4.10}
$$

Applying the operator $I_{\frac{1}{2}}$ to Eq. (4.9) we have the following equation

$$
y(t)=-1+\left(I_{\frac{1}{2}}(\sqrt{u}+\frac{4}{315}u^{\frac{5}{2}}(35u^3-135u^2+189u-105))\right)(t) -2\left(I_{\frac{1}{2}}\int_0^u xty^3(x)\,d_{\frac{1}{2}}(x)\right)(t)
$$

and its iteration formula is obtained by

$$
\begin{aligned}
y_k(t) &= y_0(t)+\left(I_{\frac{1}{2}}\int_0^u xt(y_{k-1}(x))^3(x)\,d_{\frac{1}{2}}(x)\right)(t), \quad k=1,2,\ldots,\\
y_0(t) &= -1+t-\frac{4}{9}t^3+\frac{3}{5}t^4-\frac{12}{35}t^5+\frac{2}{27}t^6.
\end{aligned}
$$

Therefore, we have the following approximations

$$
\begin{aligned}
y_1(t) &= -1+t+0.099t^6-0.23t^7+0.23t^8+\cdots-2.10^{-6}t^{21},\\
y_2(t) &= -1+t-0.0088t^9+0.03t^{10}-0.045t^{11}+\cdots+4.10^{-21}t^{66},\\
y_3(t) &= -1+t-0.00042t^{12}-0.0019t^{13}+0.0039t^{14}+\cdots-10^{-66}t^{201},\\
y_4(t) &= -1+t-0.000012t^{15}+0.000071t^{16}+\cdots,\\
y_5(t) &= -1+t+2.510^{-7}t^{18}-1.710^{-6}t^{19}+\cdots .
\end{aligned}
$$

$y(t) = \lim_{k\to\infty} y_k(t) = t-1$. Note that the $y(t) = t-1$ is the exact solution of the problem (4.9)-(4.10).

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