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SOME INEQUALITIES INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS VIA DISCRETE JENSEN'S TYPE INEQUALITIES

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Abstract. Recently, an enormous amount of effort has been devoted to extending the gamma and beta functions because of their nice properties and interesting applications. The contribution of this paper falls within this framework. We devote our attention here to investigate some approximations for a class of Gauss hypergeometric functions by the use of some discrete Jensen's type inequalities.

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1. Introduction

In various areas of applied mathematics, several types of special functions become essential tools for scientists and engineers. For instance, we cite the gamma and beta functions that are very useful in modeling many phenomena arising in mathematical physics, probability theory, information theory and so on. Thus, many researchers have been interested by generalizing and extending these special functions, as we can find in a wide bibliography as for instance in [2, 3, 6–10] and in the related references cited therein.

In the current work, we are interested in giving further approximations for some classes of these generalizations. So, we will provide an overview on some generalized Gauss hypergeometric functions that will be used. We begin by recalling the Gauss hypergeometric function (GHF) that was defined in [9, p.1]

$$ {}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}, \ \ \Re e(a) > 0,\ \Re e(b) > 0,\ \Re e(c) > 0, \tag{1.1} $$

provided that this series is convergent. Here, the notation $(\lambda)_n$, for $\Re e(\lambda) > 0$, refers to the Pochhammer symbol defined by

$$(\lambda)_n := \lambda(\lambda+1)...(\lambda+n-1) = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \text{ and } (\lambda)_0 = 1,$$

where $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$, $\Re e(x) > 0$, is the standard gamma function.

The (GHF) can be written in terms of the classical beta function as follows [3]

$$_2F_1(a,b;c;z) := \frac{1}{B(b,c-b)}\sum_{n=0}^{\infty}(a)_n B(b+n,c-b)\frac{z^n}{n!}, \tag{1.2}$$

provided that $0 < \Re e(b) < \Re e(c)$, where $B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$, $\Re e(x) > 0$, $\Re e(y) > 0$, is the standard beta function. The formula (1.2) was deduced from the following integral representation, [9, p.20]

$$_2F_1(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t, \quad |z| \le 1. \tag{1.3}$$

The confluent hypergeometric function (CHF), also called Kummer's function, was defined in [4] by the following formula

$$_1F_1(b;c;z) := \sum_{n=0}^{\infty}\frac{(b)_n}{(c)_n}\frac{z^n}{n!}, \quad \Re e(b) > 0,\ \Re e(c) > 0, \tag{1.4}$$

provided that this series is convergent. It can be written in terms of the classical beta function as follows [1, p.504]

$$_1F_1(b;c;z) = \frac{1}{B(b,c-b)}\sum_{n=0}^{\infty}B(b+n,c-b)\frac{z^n}{n!}, \quad 0 < \Re e(b) < \Re e(c). \tag{1.5}$$

In integral form we have, [6]

$$_1F_1(b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 u^{b-1}(1-u)^{c-b-1}e^{zu}\mathrm{d}u. \tag{1.6}$$

The extension of Euler's beta function given by Chaudhy et al. in [2] has motivated Özergin et al. to introduce in [6] the following generalized beta function

$$B_p^{(\alpha,\beta)}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t, \tag{1.7}$$

where $\Re e(\alpha) > 0, \Re e(\beta) > 0, \Re e(p) > 0$.

Based on (1.7), Özergin et al. defined in [6] the generalized Gauss hypergeometric function (GGHF) and the generalized confluent hypergeometric function (GCHF), respectively, as follows

$$_2F_1^{(\alpha,\beta;p)}(a,b;c;z) := \frac{1}{B(b,c-b)}\sum_{n=0}^{\infty}(a)_n B_p^{(\alpha,\beta)}(b+n,c-b)\frac{z^n}{n!}, \tag{1.8}$$

$$ {}_1F_1^{(\alpha,\beta;p)}(b;c;z) := \frac{1}{B(b,c-b)} \sum_{n=0}^{\infty} B_p^{(\alpha,\beta)}(b+n,c-b)\,\frac{z^n}{n!}. \tag{1.9} $$

It is clear that

$$ {}_2F_1^{(\alpha,\beta;0)}(a,b;c;z) = {}_2F_1(a,b;c;z) \text{ and } {}_1F_1^{(\alpha,\beta;0)}(b;c;z) = {}_1F_1(b;c;z). \tag{1.10} $$

In [6], the authors gave integral representations of (GGHF) and (GCHF) as follows

$$ \begin{aligned} &{}_2F_1^{(\alpha,\beta;p)}(a,b;c;z) && (1.11)\\ &= \frac{1}{B(b,c-b)} \times \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t, \quad |z|\le 1, \\ &{}_1F_1^{(\alpha,\beta;p)}(b;c;z) && (1.12)\\ &= \frac{1}{B(b,c-b)} \times \int_0^1 t^{b-1}(1-t)^{c-b-1}e^{zt}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t. \end{aligned} $$

The fundamental goal of this paper is to state some inequalities concerning the special functions evoked above by the use of discrete Jensen's type inequalities.

2. Statement of main results

To establish our main results, we will need the following lemma which concerns the so-called discrete Jensen's inequality [5, p.6].

Lemma 1. *Let I be a nonempty interval of $\mathbb{R}$ and let $f\colon I \longrightarrow \mathbb{R}$ be a convex function. For any $x_0,x_1,x_2,\ldots,x_n \in I$ and $m_0,m_1,m_2,\ldots,m_n \ge 0$, with $\sum_{k=0}^n m_k > 0$, we have*

$$ f\left(\frac{\sum_{k=0}^n m_k x_k}{\sum_{k=0}^n m_k}\right) \le \frac{\sum_{k=0}^n m_k f(x_k)}{\sum_{k=0}^n m_k}. \tag{2.1} $$

If f is concave then inequality (2.1) is reversed.

Our first main result reads as follows.

Theorem 1. *Let $\alpha > 0, \beta > 0, 0 < b < c$, $z_1 > 0$ and $z_2 \in \mathbb{R}$. Then the following inequality*

$$ \left({}_1F_1^{(\alpha,\beta;p)}(b;c;z_1z_2)\right)^r \le {}_1F_1^{(\alpha,\beta;p)}(b;c;z_1^r z_2) \times \left({}_1F_1^{(\alpha,\beta;p)}(b;c;z_2)\right)^{r-1} \tag{2.2} $$

holds for any $r \in (-\infty,0]\cup[1,\infty)$. In particular, we have

$$ \left({}_1F_1(b;c;z_1z_2)\right)^r \le {}_1F_1(b;c;z_1^r z_2) \times \left({}_1F_1(b;c;z_2)\right)^{r-1}. \tag{2.3} $$

If $r \in [0,1]$ then (2.2) and (2.3) are reversed.

Proof. We consider the following positive sequences

$$x_k = z_1^k \text{ and } m_k = B_p^{(\alpha,\beta)}(b+k, c-b)\frac{z_2^k}{k!},\ \ k \geq 0,$$

and the function f defined on $(0,+\infty)$ by $f(t) = t^r$ which is convex if $r \in (-\infty, 0] \cup [1,\infty)$, and concave for $r \in [0,1]$.

Assume that $r \in (-\infty, 0] \cup [1,\infty)$. Applying (2.1), we get

$$\left(\sum_{k=0}^{n} m_k x_k\right)^r \leq \left(\sum_{k=0}^{n} m_k x_k^r\right) \times \left(\sum_{k=0}^{n} m_k\right)^{r-1}. \tag{2.4}$$

In another part, we have

$$\sum_{k=0}^{\infty} m_k = \sum_{k=0}^{\infty} B_p^{(\alpha,\beta)}(b+k, c-b)\frac{z_2^k}{k!} = B(b, c-b)\,{}_1F_1^{(\alpha,\beta;p)}(b; c; z_2), \tag{2.5}$$

$$\sum_{k=0}^{\infty} m_k x_k = \sum_{k=0}^{\infty} B_p^{(\alpha,\beta)}(b+k, c-b)\,\frac{(z_1 z_2)^k}{k!} = B(b, c-b)\,{}_1F_1^{(\alpha,\beta;p)}(b; c; z_1 z_2) \tag{2.6}$$

and

$$\sum_{k=0}^{\infty} m_k x_k^r = \sum_{k=0}^{\infty} B_p^{(\alpha,\beta)}(b+k, c-b)\,\frac{(z_1^r z_2)^k}{k!} = B(b, c-b)\,{}_1F_1^{(\alpha,\beta;p)}(b; c; z_1^r z_2).$$

Letting $n \to \infty$ in (2.4) and then substituting these three latter expressions in (2.4) we get the desired inequality. Taking $p = 0$ in (2.2), we obtain (2.3).

If $r \in [0,1]$, f is concave and Lemma 1 asserts that (2.2) and (2.3) should be reversed. □

Note that, if $r \in [1,\infty)$ is integer then (2.2) is still valid for any $z_1 \in \mathbb{R}$. In another part, a simple mathematical induction shows that (2.2) can be generalized as recited in the following result.

Corollary 1. *Let $\alpha > 0, \beta > 0, 0 < b < c$, $z_1 > 0$ and $z_2, z_3, \ldots, z_n \in \mathbb{R}$. Then we have*

$$\left({}_1F_1^{(\alpha,\beta;p)}\left(b; c; \prod_{i=1}^{n} z_i\right)\right)^r \leq {}_1F_1^{(\alpha,\beta;p)}\left(b; c; z_1^r \prod_{i=2}^{n} z_i\right) \times \left({}_1F_1^{(\alpha,\beta;p)}\left(b; c; \prod_{i=2}^{n} z_i\right)\right)^{r-1} \tag{2.7}$$

for any $r \in (-\infty, 0] \cup [1,\infty)$. In particular, we have

$$\left({}_1F_1\left(b; c; \prod_{i=1}^{n} z_i\right)\right)^r \leq {}_1F_1\left(b; c; z_1^r \prod_{i=2}^{n} z_i\right) \times \left({}_1F_1\left(b; c; \prod_{i=2}^{n} z_i\right)\right)^{r-1}. \tag{2.8}$$

If $r \in [0,1]$ then (2.7) and (2.8) are reversed.

It is worth mentioning that (2.7) and (2.8) can be reiterated many times as explained in the following example.

Example 1. Let $r=3$ and $z_1,z_2,z_3\in\mathbb{R}$. If we apply (2.7) two times, we get

$$\begin{aligned}\left({}_1F_1^{(\alpha,\beta;p)}(b;c;z_1z_2z_3)\right)^3 &\leq {}_1F_1^{(\alpha,\beta;p)}(b;c;z_1^3z_2z_3)\times\left({}_1F_1^{(\alpha,\beta;p)}(b;c;z_2z_3)\right)^2\\ &\leq {}_1F_1^{(\alpha,\beta;p)}(b;c;z_1^3z_2z_3)\times{}_1F_1^{(\alpha,\beta;p)}(b;c;z_2^2z_3)\\ &\quad\times{}_1F_1^{(\alpha,\beta;p)}(b;c;z_3).\end{aligned}$$

If moreover $0\leq z_1\leq 1$, $0\leq z_2\leq 1$ and $z_3\geq 0$ then there holds

$$\begin{aligned}\left({}_1F_1^{(\alpha,\beta;p)}(b;c;z_1z_2z_3)\right)^3 &\leq {}_1F_1^{(\alpha,\beta;p)}(b;c;z_1z_2z_3)\times{}_1F_1^{(\alpha,\beta;p)}(b;c;z_2z_3)\\ &\quad\times{}_1F_1^{(\alpha,\beta;p)}(b;c;z_3).\end{aligned}$$

Now, we state our second main result.

Theorem 2. *Let $\alpha>0,\beta>0,a>0,0<b<c$, $0<z_1<1$ and $|z_2|<1$. Then the following inequality*

$$\left({}_2F_1^{(\alpha,\beta;p)}(a,b;c;z_1z_2)\right)^r\leq{}_2F_1^{(\alpha,\beta;p)}(a,b;c;z_1^rz_2)\times\left({}_2F_1^{(\alpha,\beta;p)}(a,b;c;z_2)\right)^{r-1}\tag{2.9}$$

holds for any $r\in(-\infty,0]\cup[1,\infty)$. In particular, we have

$$\left({}_2F_1(a,b;c;z_1z_2)\right)^r\leq{}_2F_1(a,b;c;z_1^rz_2)\times\left({}_2F_1(a,b;c;z_2)\right)^{r-1}.\tag{2.10}$$

If $r\in[0,1]$ then (2.9) and (2.10) are reversed.

Proof. Here, we consider the following positive sequences,

$$x_k=z_1^k\text{ and }m_k=(a)_kB_p^{(\alpha,\beta)}(b+k,c-b)\frac{z_2^k}{k!},\ k\geq 0,$$

with the same function $f(t)=t^r$, $t\in(0,\infty)$. The details are similar to those of the proof of Theorem 2 and therefore omitted here for the reader. □

We also left to the reader the routine task for formulating analogous statements as those of Corollary 1 and Example 1 when ${}_1F_1(b;c;z)$ is replaced by ${}_2F_1(a,b;c;z)$. Otherwise, another main result is stated in the following.

Theorem 3. *Let $\alpha>0,\beta>0,0<b<c$ and $z_1,z_2>0$. Then we have*

$$\frac{{}_1F_1^{(\alpha,\beta;p)}(b;c;z_1z_2)}{{}_1F_1^{(\alpha,\beta;p)}(b;c;z_2)}\geq\exp\left(\frac{b}{c}z_2(\log z_1)\frac{{}_1F_1^{(\alpha,\beta;p)}(b+1;c+1;z_2)}{{}_1F_1^{(\alpha,\beta;p)}(b;c;z_2)}\right).\tag{2.11}$$

Proof. We also consider the following sequences

$$x_k=z_1^k\ \text{ and }\ m_k=B_p^{(\alpha,\beta)}(b+k,c-b)\frac{z_2^k}{k!},\ k\geq 0$$

and the concave function defined on $(0,+\infty)$ by $f(t)=\log t$. Applying the reverse of (2.1), we get (after a simple manipulation)

$$\log\left(\sum_{k=0}^{n} m_k x_k\right) \geq \frac{\sum_{k=0}^{n} m_k \log x_k}{\sum_{k=0}^{n} m_k} + \log\left(\sum_{k=0}^{n} m_k\right). \tag{2.12}$$

In another part, we have

$$\sum_{k=0}^{\infty} m_k \log x_k = (\log z_1)\sum_{k=1}^{\infty} B_p^{(\alpha,\beta)}(b+k,c-b)\,\frac{z_2^k}{(k-1)!}$$

$$= z_2(\log z_1)\sum_{k=1}^{\infty} B_p^{(\alpha,\beta)}(b+k,c-b)\,\frac{z_2^{k-1}}{(k-1)!} = z_2(\log z_1)\sum_{k=0}^{\infty} B_p^{(\alpha,\beta)}(b+k+1,c-b)\,\frac{z_2^k}{k!}.$$

This, with (1.9) and the fact that $B(b+1,c-b)=\frac{b}{c}B(b,c-b)$, respectively yields

$$\begin{aligned}\sum_{k=0}^{\infty} m_k \log x_k &= z_2(\log z_1)B(b+1,c-b)\,{}_1F_1^{(\alpha,\beta;p)}(b+1;c+1;z_2)\\ &= \frac{b}{c} z_2(\log z_1)B(b,c-b)\,{}_1F_1^{(\alpha,\beta;p)}(b+1;c+1;z_2).\end{aligned} \tag{2.13}$$

Letting $n\to\infty$ in (2.12) and then utilizing (2.5),(2.6) and (2.13), we get

$$\begin{aligned}&\log\Big(B(b,c-b)\,{}_1F_1^{(\alpha,\beta;p)}(b;c;z_1z_2)\Big)\\ &\geq \frac{b}{c}\,\frac{z_2(\log z_1)\,{}_1F_1^{(\alpha,\beta;p)}(b+1;c+1;z_2)}{{}_1F_1^{(\alpha,\beta;p)}(b;c;z_2)} + \log\Big(B(b,c-b)\,{}_1F_1^{(\alpha,\beta;p)}(b;c;z_2)\Big),\end{aligned}$$

which, after simple algebraic operations, is equivalent to (2.11). □

From the previous theorem we can deduce the following result.

Corollary 2. *Let $\alpha>0,\beta>0,0<b<c$ and $z_1,z_2,\dots,z_n>0$. Then we have*

$$\frac{{}_1F_1^{(\alpha,\beta;p)}\left(b;c;\prod_{i=1}^n z_i\right)}{{}_1F_1^{(\alpha,\beta;p)}(b;c;z_n)} \geq \prod_{i=1}^{n}\exp\left(\frac{b}{c} z_n(\log z_i)\frac{{}_1F_1^{(\alpha,\beta;p)}(b+1;c+1;z_n)}{{}_1F_1^{(\alpha,\beta;p)}(b;c;z_n)}\right). \tag{2.14}$$

Proof. According to (2.11) we have

$$\frac{{}_1F_1^{(\alpha,\beta;p)}\left(b;c;\prod_{i=1}^n z_i\right)}{{}_1F_1^{(\alpha,\beta;p)}(b;c;z_n)} \geq \exp\left(\frac{b}{c} z_n\left(\log\prod_{i=1}^{n-1} z_i\right)\frac{{}_1F_1^{(\alpha,\beta;p)}(b+1;c+1;z_n)}{{}_1F_1^{(\alpha,\beta;p)}(b;c;z_n)}\right),$$

from which the desired inequality follows by simple elementary properties of logarithm and exponential. □

Remark 1. It is worth noting that inequalities similar to (2.11) can be established by considering logarithmic functions of different bases.

Finally, the following main result is also stated.

Theorem 4. *Let* $\alpha > 0, \beta > 0, a > 0, 0 < b < c$ *and* $0 < z_1, z_2 < 1$. *Then we have*

$$\frac{{}_2F_1^{(\alpha,\beta;p)}(a,b;c;z_1z_2)}{{}_2F_1^{(\alpha,\beta;p)}(a,b;c;z_2)} \geq \exp\left(\frac{ab}{c}z_2(\log z_1)\frac{{}_2F_1^{(\alpha,\beta;p)}(a+1,b+1;c+1;z_2)}{cB(b,c-b)\,{}_2F_1^{(\alpha,\beta;p)}(a,b;c;z_2)}\right). \tag{2.15}$$

Proof. We consider here the following positive sequences

$$x_k = z_1^k \text{ and } m_k = (a)_k B_p^{(\alpha,\beta)}(b+k,c-b)\frac{z_2^k}{k!},\ k \geq 0$$

and also the concave function $f(t) = \log t, t \in (0,\infty)$. The remainder of the proof is similar to that of the previous theorem and we therefore omitted here. □

Remark 2. By similar way, we can deduce from (2.15) a generalized inequality analogous to (2.14). We left it to the reader.

3. Conclusion

In this paper, we focused on establishing some approximations for the generalized Gauss hypergeometric and the generalized confluent hypergeometric functions. Employing the discrete Jensen's inequality, several analytic inequalities involving these functions have been provided. Related to these inequalities, we consider that it will be interesting to study in a future work the following two subjects. The tightness of the bounds in the inequalities pointed out throughout this paper and their applications in studying the behavior of the evoked hypergeometric functions.

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