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ON SOME INEQUALITIES RELATED TO OPEN NEWTON-COTES FORMULAS FOR DIFFERENTIABLE CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, first, we prove a novel integral identity involving a single time-differentiable function. Then, we prove some new inequalities associated with one of the open Newton-Cotes formulas for differentiable convex functions. The newly established inequalities can be helpful in finding the error bounds of one of the open Newton-Cotes formulas. Finally, some applications of the inequalities are also presented in the context of open Newton-Cotes formulas.

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1. INTRODUCTION

The area of mathematics known as mathematical analysis covers the theory of measure, limits, differentiation, integration, and convex functions. Convex functions are fundamental as positive or increasing functions, and they have emerged as a key topic in the field of mathematical analysis research.

Inequalities are at the core of mathematical analysis, and they have developed into a crucial tool in that process up until the early 20th century, when we started to view them as a separate field of modern mathematics. The pioneering work in this field was the book "Inequalities" [12] by Hardy, Littlewood and Pólya. Other books (see, e.g., [19, 20]) are of great value in this field as well.

In recent years, many researchers have developed numerical integration formulas and found their error bounds using different techniques. To determine the error bounds of numerical integration formulas, mathematical inequalities are used, and the authors used various functions such as convex functions, bounded functions, Lipschitzian functions, and so on. For example, some error bounds for trapezoidal

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and midpoint formulas of numerical integration using the convex functions were found in [6, 15]. A number of papers have been published on the error bounds of Simpson's formula using the convex functions in different calculi and some of these bounds can be found in [1,3,5,7–9,17,21]. Some error bounds for Newton's formula in numerical integration have also been established by using the convex functions in different calculi and these bounds can be found in [10, 11, 13, 14, 16, 22, 23]. In open Newton-Cotes formulas, Milne's formula is very important and its error bounds for four times twice differentiable functions were found in [2]. In [18], the authors used a general form of the convexity and established some new Maclaurin's formula type inequalities, and discussed their applications.

Inspired by the ongoing studies, we use differentiable convex functions and prove some integral inequalities related to one of the open Newton-Cotes formulas for $n=1$ (see, [4, p. 200]). These inequalities can help us to find the error bounds for the discussed formula. Finally, we give some applications to quadrature formulas.

2. Main Results

In this section, we establish and prove some important integral inequalities for differentiable convex functions. For this, we will prove the following lemma first.

Lemma 1. *Let $F:\ [\pi_1,\pi_2]\to\mathbb{R}$ be a differentiable function over (π_1,π_2). If $F'\in L_1[\pi_1,\pi_2]$, then the following equality holds:*

$$\frac{1}{2}\left[F\left(\frac{2\pi_1+\pi_2}{3}\right)+F\left(\frac{\pi_1+2\pi_2}{3}\right)\right]-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}F(\tau)\,d\tau \tag{2.1}$$
$$=(\pi_2-\pi_1)\left[\int_0^{\frac{1}{3}}\varkappa F'(\varkappa\pi_2+(1-\varkappa)\pi_1)\,d\varkappa\right.$$
$$\left.+\int_{\frac{1}{3}}^{\frac{2}{3}}\left(\varkappa-\frac{1}{2}\right)F'(\varkappa\pi_2+(1-\varkappa)\pi_1)\,d\varkappa+\int_{\frac{2}{3}}^{1}(\varkappa-1)F'(\varkappa\pi_2+(1-\varkappa)\pi_1)\,d\varkappa\right]$$

Proof. With simple computations, we have

$$\begin{aligned} I_1 &= (\pi_2-\pi_1)\int_0^{\frac{1}{3}}\varkappa F'(\varkappa\pi_2+(1-\varkappa)\pi_1)\,d\varkappa \\ &= \frac{1}{3}F\left(\frac{2\pi_1+\pi_2}{3}\right)-\int_0^{\frac{1}{3}}F(\varkappa\pi_2+(1-\varkappa)\pi_1)\,d\varkappa, \end{aligned} \tag{2.2}$$

$$\begin{aligned} I_2 &= (\pi_2-\pi_1)\int_{\frac{1}{3}}^{\frac{2}{3}}\left(\varkappa-\frac{1}{2}\right)F'(\varkappa\pi_2+(1-\varkappa)\pi_1)\,d\varkappa \\ &= \frac{1}{6}F\left(\frac{\pi_1+2\pi_2}{3}\right)+\frac{1}{6}F\left(\frac{2\pi_1+\pi_2}{3}\right)-\int_0^{1}F(\varkappa\pi_2+(1-\varkappa)\pi_1)\,d\varkappa \end{aligned} \tag{2.3}$$

and

$$I_3 = (\pi_2-\pi_1)\int_{\frac{2}{3}}^{1}(\varkappa-1)F'(\varkappa\pi_2+(1-\varkappa)\pi_1)\,d\varkappa \tag{2.4}$$
$$= \frac{1}{3}F\left(\frac{\pi_1+2\pi_2}{3}\right)-\int_{\frac{2}{3}}^{1}F(\varkappa\pi_2+(1-\varkappa)\pi_1)\,d\varkappa.$$

Thus, we have the following required equality by adding (2.2)-(2.4):

$$I_1+I_2+I_3=\frac{1}{2}\left[F\left(\frac{2\pi_1+\pi_2}{3}\right)+F\left(\frac{\pi_1+2\pi_2}{3}\right)\right]-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}F(\tau)\,d\tau.$$

The proof is completed. □

Theorem 1. *If all conditions of Lemma 1 hold and $|F'|$ is a convex function, then the following inequality holds:*

$$\left|\frac{1}{2}\left[F\left(\frac{2\pi_1+\pi_2}{3}\right)+F\left(\frac{\pi_1+2\pi_2}{3}\right)\right]-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}F(\tau)\,d\tau\right|$$
$$\leq \frac{5(\pi_2-\pi_1)}{72}\left[|F'(\pi_1)|+|F'(\pi_2)|\right].$$

Proof. By taking modulus in (2.1) and using convexity of $|F'|$, we have

$$\left|\frac{1}{2}\left[F\left(\frac{2\pi_1+\pi_2}{3}\right)+F\left(\frac{\pi_1+2\pi_2}{3}\right)\right]-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}F(\tau)\,d\tau\right|$$
$$\leq (\pi_2-\pi_1)\left[\int_0^{\frac{1}{3}}\varkappa\left|F'(\varkappa\pi_2+(1-\varkappa)\pi_1)\right|d\varkappa\right.$$
$$+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\varkappa-\frac{1}{2}\right|\left|F'(\varkappa\pi_2+(1-\varkappa)\pi_1)\right|d\varkappa$$
$$\left.+\int_{\frac{2}{3}}^{1}(1-\varkappa)\left|F'(\varkappa\pi_2+(1-\varkappa)\pi_1)\right|d\varkappa\right]$$
$$\leq (\pi_2-\pi_1)\left[\int_0^{\frac{1}{3}}\varkappa\left[\varkappa|F'(\pi_2)|+(1-\varkappa)|F'(\pi_1)|\right]d\varkappa\right.$$
$$+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\varkappa-\frac{1}{2}\right|\left[\varkappa|F'(\pi_2)|+(1-\varkappa)|F'(\pi_1)|\right]d\varkappa$$
$$\left.+\int_{\frac{2}{3}}^{1}(1-\varkappa)\left[\varkappa|F'(\pi_2)|+(1-\varkappa)|F'(\pi_1)|\right]d\varkappa\right]$$
$$=\frac{5(\pi_2-\pi_1)}{72}\left[|F'(\pi_1)|+|F'(\pi_2)|\right].$$

Thus, the proof is completed. □

Theorem 2. *If all conditions of Lemma 1 hold and $|F'|^q$, $q \geq 1$ is a convex function, then the following inequality holds:*

$$\left|\frac{1}{2}\left[F\left(\frac{2\pi_1+\pi_2}{3}\right)+F\left(\frac{\pi_1+2\pi_2}{3}\right)\right]-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}F(\tau)\,\mathrm{d}\tau\right|$$
$$\leq \frac{(\pi_2-\pi_1)}{18}\left[\left(\frac{2|F'(\pi_2)|^q+7|F'(\pi_1)|^q}{9}\right)^{\frac{1}{q}}+\frac{1}{2}\left(\frac{|F'(\pi_2)|^q+|F'(\pi_1)|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{7|F'(\pi_2)|^q+2|F'(\pi_1)|^q}{9}\right)^{\frac{1}{q}}\right].$$

Proof. Taking modulus of equality (2.1) and using power mean inequality, we have

$$\left|\frac{1}{2}\left[F\left(\frac{2\pi_1+\pi_2}{3}\right)+F\left(\frac{\pi_1+2\pi_2}{3}\right)\right]-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}F(\tau)\,\mathrm{d}\tau\right|$$
$$\leq (\pi_2-\pi_1)\left[\left(\int_0^{\frac{1}{3}}\varkappa \mathrm{d}\varkappa\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}\varkappa\left|F'(\varkappa\pi_2+(1-\varkappa)\pi_1)\right|^q\mathrm{d}\varkappa\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\varkappa-\frac{1}{2}\right|\mathrm{d}\varkappa\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\varkappa-\frac{1}{2}\right|\left|F'(\varkappa\pi_2+(1-\varkappa)\pi_1)\right|^q\mathrm{d}\varkappa\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}(1-\varkappa)\,\mathrm{d}\varkappa\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}(1-\varkappa)\left|F'(\varkappa\pi_2+(1-\varkappa)\pi_1)\right|^q\mathrm{d}\varkappa\right)^{\frac{1}{q}}\right].$$

We have the following relation by using the convexity of $|F'|^q$

$$\left|\frac{1}{2}\left[F\left(\frac{2\pi_1+\pi_2}{3}\right)+F\left(\frac{\pi_1+2\pi_2}{3}\right)\right]-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}F(\tau)\,\mathrm{d}\tau\right|$$
$$\leq (\pi_2-\pi_1)\left[\left(\int_0^{\frac{1}{3}}\varkappa \mathrm{d}\varkappa\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}\varkappa\left[\varkappa\left|F'(\pi_2)\right|^q+(1-\varkappa)\left|F'(\pi_1)\right|^q\right]\mathrm{d}\varkappa\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\varkappa-\frac{1}{2}\right|\mathrm{d}\varkappa\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\varkappa-\frac{1}{2}\right|\left[\varkappa\left|F'(\pi_2)\right|^q+(1-\varkappa)\left|F'(\pi_1)\right|^q\right]\mathrm{d}\varkappa\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}(1-\varkappa)\,\mathrm{d}\varkappa\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}(1-\varkappa)\left[\varkappa\left|F'(\pi_2)\right|^q+(1-\varkappa)\left|F'(\pi_1)\right|^q\right]\mathrm{d}\varkappa\right)^{\frac{1}{q}}\right]$$
$$=(\pi_2-\pi_1)\left[\left(\frac{1}{18}\right)^{1-\frac{1}{q}}\left(\frac{|F'(\pi_2)|^q}{81}+\frac{7|F'(\pi_1)|^q}{162}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{1}{36}\right)^{1-\frac{1}{q}}\left(\frac{|F'(\pi_2)|^q}{72}+\frac{|F'(\pi_1)|^q}{72}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{1}{18}\right)^{1-\frac{1}{q}}\left(\frac{7|F'(\pi_2)|^q}{162}+\frac{|F'(\pi_1)|^q}{81}\right)^{\frac{1}{q}}\right].$$

Thus, the proof is completed. □

Theorem 3. *If all conditions of Lemma 1 hold and $|F'|^q$, $q>1$ is a convex function, then the following inequality holds:*

$$\left|\frac{1}{2}\left[F\left(\frac{2\pi_1+\pi_2}{3}\right)+F\left(\frac{\pi_1+2\pi_2}{3}\right)\right]-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}F(\tau)\,\mathrm{d}\tau\right|$$
$$\leq(\pi_2-\pi_1)\left[\left(\frac{1}{3^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{|F'(\pi_2)|^q+5|F'(\pi_1)|^q}{18}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{2}{6^{p+1}(p+1)}\right)\left(\frac{|F'(\pi_2)|^q+|F'(\pi_1)|^q}{6}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{1}{3^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{5|F'(\pi_2)|^q+|F'(\pi_1)|^q}{18}\right)^{\frac{1}{q}}\right],$$

where $p+q=pq$.

Proof. Taking modulus of equality (2.1) and using Hölder's inequality, we have

$$\left|\frac{1}{2}\left[F\left(\frac{2\pi_1+\pi_2}{3}\right)+F\left(\frac{\pi_1+2\pi_2}{3}\right)\right]-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}F(\tau)\,\mathrm{d}\tau\right|$$
$$\leq(\pi_2-\pi_1)\left[\left(\int_0^{\frac{1}{3}}\varkappa^p\mathrm{d}\varkappa\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}|F'(\varkappa\pi_2+(1-\varkappa)\pi_1)|^q\,\mathrm{d}\varkappa\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\varkappa-\frac{1}{2}\right|^p\mathrm{d}\varkappa\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}|F'(\varkappa\pi_2+(1-\varkappa)\pi_1)|^q\,\mathrm{d}\varkappa\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}(1-\varkappa)^p\,\mathrm{d}\varkappa\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^{1}|F'(\varkappa\pi_2+(1-\varkappa)\pi_1)|^q\,\mathrm{d}\varkappa\right)^{\frac{1}{q}}\right].$$

We have the following relation by using the convexity of $|F'|^q$

$$\left|\frac{1}{2}\left[F\left(\frac{2\pi_1+\pi_2}{3}\right)+F\left(\frac{\pi_1+2\pi_2}{3}\right)\right]-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}F(\tau)\,\mathrm{d}\tau\right|$$
$$\leq(\pi_2-\pi_1)\left[\left(\int_0^{\frac{1}{3}}\varkappa^p\mathrm{d}\varkappa\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left[\varkappa|F'(\pi_2)|^q+(1-\varkappa)|F'(\pi_1)|^q\right]\mathrm{d}\varkappa\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\varkappa-\frac{1}{2}\right|^{p}\mathrm{d}\varkappa\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left[\varkappa\left|F'(\pi_2)\right|^{q}+(1-\varkappa)\left|F'(\pi_1)\right|^{q}\right]\mathrm{d}\varkappa\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_{\frac{2}{3}}^{1}(1-\varkappa)^{p}\mathrm{d}\varkappa\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^{1}\left[\varkappa\left|F'(\pi_2)\right|^{q}+(1-\varkappa)\left|F'(\pi_1)\right|^{q}\right]\mathrm{d}\varkappa\right)^{\frac{1}{q}}\right]$$

$$=(\pi_2-\pi_1)\left[\left(\frac{1}{3^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{|F'(\pi_2)|^{q}+5|F'(\pi_1)|^{q}}{18}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{2}{6^{p+1}(p+1)}\right)\left(\frac{|F'(\pi_2)|^{q}+|F'(\pi_1)|^{q}}{6}\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{1}{3^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{5|F'(\pi_2)|^{q}+|F'(\pi_1)|^{q}}{18}\right)^{\frac{1}{q}}\right].$$

Thus, the proof is completed. □

3. Applications

Let P be a partition of the interval $[\pi_1,\pi_2]$ as:

$$\pi_1=\tau_0<\tau_1<\tau_2<...<\tau_n=\pi_2,$$

and we consider the quadrature formula

$$\int_{\pi_1}^{\pi_2}F(\tau)\,\mathrm{d}\tau=\Lambda(F,P)+R(F,P), \tag{3.1}$$

where

$$\Lambda(F,P)=\sum_{i=0}^{n-1}\frac{\tau_{i+1}-\tau_i}{2}\left[F\left(\frac{2\tau_i+\tau_{i+1}}{3}\right)+F\left(\frac{\tau_i+2\tau_{i+1}}{3}\right)\right] \tag{3.2}$$

and $R(F,P)$ is the associated approximation error.

Proposition 1. *Let $F:\ [\pi_1,\pi_2]\to\mathbb{R}$ be a differentiable function over (π_1,π_2) with $0\le\pi_1<\pi_2$ and $F'\in L_1[\pi_1,\pi_2]$. If $|F'|$ is a convex function, then we have*

$$|R(F,P)|\le\sum_{i=0}^{n-1}\frac{5(\tau_{i+1}-\tau_i)^2}{144}\left[\left|F'(\tau_i)\right|+\left|F'(\tau_{i+1})\right|\right]. \tag{3.3}$$

Proof. From Theorem 1 over the subintervals $[\tau_i,\tau_{i+1}]$ of partition P, we get

$$\left|\frac{1}{2}\left[F\left(\frac{2\tau_i+\tau_{i+1}}{3}\right)+F\left(\frac{\tau_i+2\tau_{i+1}}{3}\right)\right]-\frac{1}{\tau_{i+1}-\tau_i}\int_{\tau_i}^{\tau_{i+1}}F(\varkappa)\,\mathrm{d}\varkappa\right| \tag{3.4}$$
$$\le\frac{5(\tau_{i+1}-\tau_i)}{144}\left[\left|F'(\tau_i)\right|+\left|F'(\tau_{i+1})\right|\right].$$

Thus, we obtain the required inequality by multiplying (3.4) with $(\tau_{i+1}-\tau_i)$ and then summing the resultant inequality for all $i=0,1,2,\ldots,n-1$. □

Proposition 2. *Let $F : [\pi_1,\pi_2] \to \mathbb{R}$ be a differentiable function over (π_1,π_2) with $0 \le \pi_1 < \pi_2$ and $F' \in L_1[\pi_1,\pi_2]$. If $|F'|^q$, $q \ge 1$ is a convex function, then we have*

$$|R(F,P)| \le \sum_{i=0}^{n-1} \frac{(\tau_{i+1}-\tau_i)^2}{18} \left[\left(\frac{2|F'(\tau_{i+1})|^q + 7|F'(\tau_i)|^q}{9} \right)^{\frac{1}{q}} + \frac{1}{2} \left(\frac{|F'(\tau_{i+1})|^q + |F'(\tau_i)|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{7|F'(\tau_{i+1})|^q + 2|F'(\tau_i)|^q}{9} \right)^{\frac{1}{q}} \right]. \tag{3.5}$$

Proof. From Theorem 2 and method used in the proof of (3.3), we get the desired inequality (3.5). □

Proposition 3. *Let $F : [\pi_1,\pi_2] \to \mathbb{R}$ be a differentiable function over (π_1,π_2) with $0 \le \pi_1 < \pi_2$ and $F' \in L_1[\pi_1,\pi_2]$. If $|F'|^q$, $q > 1$ is a convex function, then we have*

$$\begin{aligned} |R(F,P)| \le (\tau_{i+1}-\tau_i) & \left[\left(\frac{1}{3^{p+1}(p+1)} \right)^{\frac{1}{p}} \left(\frac{|F'(\tau_{i+1})|^q + 5|F'(\tau_i)|^q}{18} \right)^{\frac{1}{q}} \right. \\ & + \left(\frac{2}{6^{p+1}(p+1)} \right) \left(\frac{|F'(\tau_{i+1})|^q + |F'(\tau_i)|^q}{6} \right)^{\frac{1}{q}} \\ & \left. + \left(\frac{1}{3^{p+1}(p+1)} \right)^{\frac{1}{p}} \left(\frac{5|F'(\tau_{i+1})|^q + |F'(\tau_i)|^q}{18} \right)^{\frac{1}{q}} \right], \end{aligned} \tag{3.6}$$

where $p+q = pq$.

Proof. From Theorem 3 and method used in the proof of (3.3), we get the desired inequality (3.6). □

4. Conclusion

In this work, we have proved some error bounds for one of the open Newton-Cotes formulas for differentiable convex functions. We also gave some applications of newly established results in the context of open Newton-Cotes formulas. It is an interesting and new problem that the upcoming researchers can obtain similar inequalities for fractional integrals and coordinated convex functions.

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