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SOME NEWTON'S INEQUALITIES ON CO-ORDINATES FOR RIEMANN–LIOUVILLE FRACTIONAL INTEGRALS

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Abstract. In this paper, we give an identity involving Riemann–Liouville fractional integrals and partially differentiable function of two variables. Then, we use the newly established identity and prove some new Newton's type inequalities for differentiable co-ordinated convex functions. We also recapture some existing inequalities and obtain some new inequalities in the special cases of newly established results. Furthermore, we present an example with graphs to illustrate the obtained main result.

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1. Introduction

The Hermite–Hadamard and Simpson's type inequalities have been studied widely in recent years. Simpson's inequalities are inequalities that are created from Simpson's following rules:

i. Simpson's quadrature formula (Simpson's 1/3 rule):

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

ii. Simpson's second formula or Newton-Cotes quadrature formula (Simpson's 3/8 rule):

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right].$$

Simpson's second rule has the rule of three-point Newton-Cotes quadrature, hence evaluations for three steps quadratic kernel are sometimes called Newton-type results. These results are known as Newton-type inequalities in the literature. Many mathematicians have been investigated to Newton-type inequalities. For instance, Gao and Shi investigated Newton's type inequalities to the case of functions whose second derivatives are convex in the paper [4]. Erden et al. established some error estimates of Newton-type quadrature formula by bounded variation and Lipschitzian mappings in [3]. Newton's type inequalities for quantum differentiable convex mappings were given in the paper [1]. Luangboon et al. established Newton-type inequalities by post-quantum integrals in the paper [11]. In addition, Noor et al. gave Newton's type inequalities based on harmonic convex and p-harmonic convex functions in [13] and [14], respectively. The reader is referred to [7,8,12,21] and the references therein for more information and unexplained subjects about Newton's type inequalities including convex differentiable functions.

Fractional calculus has grown in popularity in recent years. Because of the significance of fractional calculus, mathematicians have investigated different fractional integral inequalities. Riemann-Lioville fractional integrals are the most commonly used integrals among fractional integrals. Sarikaya et al. presented some inequalities of Hermite–Hadamard type and trapezoid type by using the first time generalised the Riemann–Liouville fractional integrals in [18]. Ostrowski-type inequalities using Riemann–Liouville fractional for differentiable functions are investigated in the paper [20]. İşcan and Wu proved Hermite–Hadamard type inequalities to the case of harmonic convexity in the paper [9]. Sarikaya and Yıldırım presented new Hermite–Hadamard type inequalities and midpoint type inequalities via Riemann–Liouville fractional integrals in [19]. Chen and Huang investigated to $1/3$ rule type inequality for s-convex functions involving Riemann–Liouville fractional integrals. Moreover, several fractional Simpson type inequalities were investigated to functions whose second derivatives in absolute value are convex in the paper [6]. The reader is referred to [2,15,16,22] and the references therein for further information about Simpson type inequalities and varied properties of Riemann–Liouville fractional integrals and various fractional integral operators.

2. Preliminaries

In this section, we present the basic definitions and facts about the Riemann–Liouville integrals for one and two variables.

Definition 1 ([5, 10])**.** Let us consider that $f \in L_1[a,b]$ is a function and $\alpha > 0$ and $a \geq 0$. Then, Riemann–Liouville fractional integrals $J_{a+}^{\alpha} f(x)$ and $J_{b-}^{\alpha} f(x)$ are defined as follows:

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(a)} \int_a^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,\mathrm{d}t \quad x < b.$$

Here, Γ is the well-known Gamma function.

Definition 2 ([17]). Let $f \in L_1(\Delta = [a,b] \times [c,d])$ be a function and $\alpha, \beta > 0$ and $a, c \geq 0$. Then, Riemann–Liouville fractional integrals $J^{\alpha,\beta}_{a+,c+}, J^{\alpha,\beta}_{a+,d-}, +J^{\alpha,\beta}_{b-,c+}$ and $J^{\alpha,\beta}_{b-,d-}$ on co-ordinates are described as follows:

$$J^{\alpha,\beta}_{a+,c+} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x-t)^{\alpha-1} (y-s)^{\beta-1} f(t,s)\,\mathrm{d}s\mathrm{d}t, \ \ x > a, \ y > c,$$

$$J^{\alpha,\beta}_{a+,d-} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d (x-t)^{\alpha-1} (s-y)^{\beta-1} f(t,s)\,\mathrm{d}s\mathrm{d}t, \ \ x > a, \ y > d,$$

$$J^{\alpha,\beta}_{b-,c+} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y (t-x)^{\alpha-1} (y-s)^{\beta-1} f(t,s)\,\mathrm{d}s\mathrm{d}t, \ \ x < b, \ y > c,$$

$$J^{\alpha,\beta}_{b-,d-} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_y^d (t-x)^{\alpha-1} (s-y)^{\beta-1} f(t,s)\,\mathrm{d}s\mathrm{d}t, \ \ x < b, \ y < d,$$

where Γ is the well-known Gamma function.

3. AN IDENTITY

In this section, we prove an identity which help us to obtain the main results of this paper.

Lemma 1. *If $f : \Delta \to \mathbb{R}$ is a partially differentiable function on Δ°, then the following Riemann–Liouville fractional integrals identity hold:*

$$\Theta^{\alpha,\beta}(a,b;c,d) = \frac{(b-a)(d-c)}{4} \sum_{i=1}^{9} I_i, \tag{3.1}$$

where

$$\begin{aligned}
\Theta^{\alpha,\beta}(a,b;c,d) &:= \frac{[f(a,c)+f(a,d)+f(b,c)+f(b,d)]}{64} \\
&+ \frac{3}{64}\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right] \\
&+ \frac{3}{64}\left[f\left(\frac{2a+b}{3},c\right)+f\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},c\right)+f\left(\frac{a+2b}{3},d\right)\right] \\
&+ \frac{9}{64}\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)\right. \\
&\left.+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right] - \frac{3\Gamma(\alpha+1)}{16(b-a)^{\alpha}}\left[J^{\alpha}_{b-}f\left(a,\frac{2c+d}{3}\right)+J^{\alpha}_{b-}f\left(a,\frac{c+2d}{3}\right)\right.
\end{aligned} \tag{3.2}$$

$$+J_{a+}^{\alpha}f\left(b,\frac{2c+d}{3}\right)+J_{a+}^{\alpha}f\left(b,\frac{c+2d}{3}\right)\bigg]-\frac{\Gamma(\alpha+1)}{16(b-a)^{\alpha}}\left[J_{b-}^{\alpha}f(a,c)+J_{b-}^{\alpha}f(a,d)\right.$$
$$+J_{a+}^{\alpha}f(b,c)+J_{a+}^{\alpha}f(b,d)\big]-\frac{3\Gamma(\beta+1)}{16(d-c)^{\beta}}\left[J_{d-}^{\beta}f\left(\frac{2a+b}{3},c\right)+J_{d-}^{\beta}f\left(\frac{a+2b}{3},c\right)\right.$$
$$+J_{c+}^{\beta}f\left(\frac{2a+b}{3},d\right)+J_{c+}^{\beta}f\left(\frac{a+2b}{3},d\right)\bigg]-\frac{\Gamma(\beta+1)}{16(d-c)^{\beta}}\left[J_{d-}^{\beta}f(a,c)+J_{c+}^{\beta}f(a,d)\right.$$
$$+J_{d-}^{\beta}f(b,c)+J_{c+}^{\beta}f(b,d)\Big]+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\left[J_{b-,d-}^{\alpha,\beta}f(a,c)+J_{b-,c+}^{\alpha,\beta}f(a,d)\right.$$
$$\left.+J_{a+,d-}^{\alpha,\beta}f(b,c)+J_{a+,c+}^{\alpha,\beta}f(b,d)\right],$$

$$\begin{cases} I_1=\int_0^{\frac13}\int_0^{\frac13}\left(t^{\alpha}-\frac18\right)\left(s^{\beta}-\frac18\right)\Omega(t,s)\,\mathrm{d}s\mathrm{d}t, & I_6=\int_{\frac13}^{\frac23}\int_{\frac23}^{1}\left(t^{\alpha}-\frac12\right)\left(s^{\beta}-\frac78\right)\Omega(t,s)\,\mathrm{d}s\mathrm{d}t,\\ I_2=\int_0^{\frac13}\int_{\frac13}^{\frac23}\left(t^{\alpha}-\frac18\right)\left(s^{\beta}-\frac12\right)\Omega(t,s)\,\mathrm{d}s\mathrm{d}t, & I_7=\int_{\frac23}^{1}\int_0^{\frac13}\left(t^{\alpha}-\frac78\right)\left(s^{\beta}-\frac18\right)\Omega(t,s)\,\mathrm{d}s\mathrm{d}t,\\ I_3=\int_0^{\frac13}\int_{\frac23}^{1}\left(t^{\alpha}-\frac18\right)\left(s^{\beta}-\frac78\right)\Omega(t,s)\,\mathrm{d}s\mathrm{d}t, & I_8=\int_{\frac23}^{1}\int_{\frac13}^{\frac23}\left(t^{\alpha}-\frac78\right)\left(s^{\beta}-\frac12\right)\Omega(t,s)\,\mathrm{d}s\mathrm{d}t,\\ I_4=\int_{\frac13}^{\frac23}\int_0^{\frac13}\left(t^{\alpha}-\frac12\right)\left(s^{\beta}-\frac18\right)\Omega(t,s)\,\mathrm{d}s\mathrm{d}t, & I_9=\int_{\frac23}^{1}\int_{\frac23}^{1}\left(t^{\alpha}-\frac78\right)\left(s^{\beta}-\frac78\right)\Omega(t,s)\,\mathrm{d}s\mathrm{d}t,\\ I_5=\int_{\frac13}^{\frac23}\int_{\frac13}^{\frac23}\left(t^{\alpha}-\frac12\right)\left(s^{\beta}-\frac12\right)\Omega(t,s)\,\mathrm{d}s\mathrm{d}t, & \end{cases}$$

and

$$\Omega(t,s)=\frac{\partial^2}{\partial s\partial t}f(tb+(1-t)a,sd+(1-s)c)-\frac{\partial^2}{\partial s\partial t}f(tb+(1-t)a,sc+(1-s)d)$$
$$-\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sd+(1-s)c)+\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d).$$

Proof. From the fact of the fundamental rules of integration by parts, we obtain

$$I_1=\frac{1}{(b-a)(d-c)}\left\{\left(\frac{1}{3^{\alpha}}-\frac18\right)\left(\frac{1}{3^{\beta}}-\frac18\right)\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)\right.\right.$$
$$\left.+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right]$$
$$+\frac18\left(\frac{1}{3^{\beta}}-\frac18\right)\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right]$$
$$+\frac18\left(\frac{1}{3^{\alpha}}-\frac18\right)\left[f\left(\frac{2a+b}{3},c\right)+f\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},c\right)+f\left(\frac{a+2b}{3},d\right)\right]$$

$$
\begin{aligned}
&+\frac{1}{64}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right] \\
&-\alpha\left(\frac{1}{3^{\beta}}-\frac{1}{8}\right)\int_{0}^{\frac{1}{3}} t^{\alpha-1}\left[f\left(tb+(1-t)a,\frac{2c+d}{3}\right)+f\left(tb+(1-t)a,\frac{c+2d}{3}\right)\right. \\
&\left.+f\left(ta+(1-t)b,\frac{2c+d}{3}\right)+f\left(ta+(1-t)b,\frac{c+2d}{3}\right)\right]\mathrm{d}t \\
&-\frac{\alpha}{8}\int_{0}^{\frac{1}{3}} t^{\alpha-1}\left[f(tb+(1-t)a,c)+f(tb+(1-t)a,d)+f(ta+(1-t)b,c)\right. \\
&\left.+f(ta+(1-t)b,d)\right]\mathrm{d}t-\beta\left(\frac{1}{3^{\alpha}}-\frac{1}{8}\right)\int_{0}^{\frac{1}{3}} s^{\beta-1}\left[f\left(\frac{2a+b}{3},sd+(1-s)c\right)\right. \\
&\left.+f\left(\frac{2a+b}{3},sc+(1-s)d\right)+f\left(\frac{a+2b}{3},sd+(1-s)c\right)+f\left(\frac{a+2b}{3},sc+(1-s)d\right)\right]\mathrm{d}s \\
&-\frac{\beta}{8}\int_{0}^{\frac{1}{3}} s^{\beta-1}\left[f(a,sd+(1-s)c)+f(a,sc+(1-s)d)+f(b,sd+(1-s)c)\right. \\
&\left.+f(b,sc+(1-s)d)\right]\mathrm{d}s \\
&+\alpha\beta\int_{0}^{\frac{1}{3}}\int_{0}^{\frac{1}{3}} t^{\alpha-1}s^{\beta-1}\left[f(tb+(1-t)a,sd+(1-s)c)+f(tb+(1-t)a,sc+(1-s)d)\right. \\
&\left.\left.+f(ta+(1-t)b,sd+(1-s)c)+f(ta+(1-t)b,sc+(1-s)d)\right]\mathrm{d}s\mathrm{d}t\right\}. \qquad (3.3)
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
I_2=&\ \frac{1}{(b-a)(d-c)}\left\{\left(\frac{2^{\beta}-1}{3^{\beta}}\right)\left(\frac{1}{3^{\alpha}}-\frac{1}{8}\right)\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)\right.\right. \\
&\left.+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right] \\
&+\frac{1}{8}\left(\frac{2^{\beta}-1}{3^{\beta}}\right)\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right] \\
&+\alpha\left(\frac{1-2^{\beta}}{3^{\beta}}\right)\int_{0}^{\frac{1}{3}} t^{\alpha-1}\left[f\left(tb+(1-t)a,\frac{2c+d}{3}\right)+f\left(tb+(1-t)a,\frac{c+2d}{3}\right)\right. \\
&\left.+f\left(ta+(1-t)b,\frac{2c+d}{3}\right)+f\left(ta+(1-t)b,\frac{c+2d}{3}\right)\right]\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&-\beta\left(\frac{1}{3^{\alpha}}-\frac{1}{8}\right)\int_{\frac{1}{3}}^{\frac{2}{3}} s^{\beta-1}\left[f\left(\frac{2a+b}{3},sd+(1-s)c\right)+f\left(\frac{2a+b}{3},sc+(1-s)d\right)\right.\\
&\left.+f\left(\frac{a+2b}{3},sd+(1-s)c\right)+f\left(\frac{a+2b}{3},sc+(1-s)d\right)\right]\mathrm{d}s\\
&-\frac{\beta}{8}\int_{\frac{1}{3}}^{\frac{2}{3}} s^{\beta-1}\left[f(a,sd+(1-s)c)+f(a,sc+(1-s)d)+f(b,sd+(1-s)c)\right.\\
&\left.+f(b,sc+(1-s)d)\right]\mathrm{d}s\\
&+\alpha\beta\int_{0}^{\frac{1}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}} t^{\alpha-1}s^{\beta-1}\left[f(tb+(1-t)a,sd+(1-s)c)+f(tb+(1-t)a,sc+(1-s)d)\right.\\
&\left.\left.+f(ta+(1-t)b,sd+(1-s)c)+f(ta+(1-t)b,sc+(1-s)d)\right]\mathrm{d}s\mathrm{d}t\right\}, \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
I_3=&\frac{1}{(b-a)(d-c)}\left\{\left(\frac{7}{8}-\left(\frac{2}{3}\right)^{\beta}\right)\left(\frac{1}{3^{\alpha}}-\frac{1}{8}\right)\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)\right.\right.\\
&\left.+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right]\\
&+\frac{1}{8}\left(\frac{7}{8}-\left(\frac{2}{3}\right)^{\beta}\right)\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right]\\
&+\frac{1}{8}\left(\frac{1}{3^{\alpha}}-\frac{1}{8}\right)\left[f\left(\frac{2a+b}{3},c\right)+f\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},c\right)+f\left(\frac{a+2b}{3},d\right)\right]\\
&+\frac{1}{64}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]\\
&+\alpha\left(\left(\frac{2}{3}\right)^{\beta}-\frac{7}{8}\right)\int_{0}^{\frac{1}{3}} t^{\alpha-1}\left[f\left(tb+(1-t)a,\frac{2c+d}{3}\right)+f\left(tb+(1-t)a,\frac{c+2d}{3}\right)\right.\\
&\left.+f\left(ta+(1-t)b,\frac{2c+d}{3}\right)+f\left(ta+(1-t)b,\frac{c+2d}{3}\right)\right]\mathrm{d}t\\
&-\frac{\alpha}{8}\int_{0}^{\frac{1}{3}} t^{\alpha-1}\left[f(tb+(1-t)a,d)+f(tb+(1-t)a,c)+f(ta+(1-t)b,d)\right.\\
&\left.+f(ta+(1-t)b,c)\right]\mathrm{d}t
\end{aligned}
$$

$$-\beta\left(\frac{1}{3^{\alpha}}-\frac{1}{8}\right)\int_{\frac{2}{3}}^{1}s^{\beta-1}\left[f\left(\frac{2a+b}{3},sd+(1-s)c\right)+f\left(\frac{2a+b}{3},sc+(1-s)d\right)\right.$$
$$\left.+f\left(\frac{a+2b}{3},sd+(1-s)c\right)+f\left(\frac{a+2b}{3},sc+(1-s)d\right)\right]\mathrm{d}s$$
$$-\frac{\beta}{8}\int_{\frac{2}{3}}^{1}s^{\beta-1}\left[f(a,sd+(1-s)c)+f(a,sc+(1-s)d)+f(b,sd+(1-s)c)\right.$$
$$\left.+f(b,sc+(1-s)d)\right]\mathrm{d}s$$
$$+\alpha\beta\int_{0}^{\frac{1}{3}}\int_{\frac{2}{3}}^{1}t^{\alpha-1}s^{\beta-1}\left[f(tb+(1-t)a,sd+(1-s)c)+f(tb+(1-t)a,sc+(1-s)d)\right.$$
$$\left.\left.+f(ta+(1-t)b,sd+(1-s)c)+f(ta+(1-t)b,sc+(1-s)d)\right]\mathrm{d}s\mathrm{d}t\right\}, \tag{3.5}$$

$$I_4=\frac{1}{(b-a)(d-c)}\left\{\left(\frac{2^{\alpha}-1}{3^{\alpha}}\right)\left(\frac{1}{3^{\beta}}-\frac{1}{8}\right)\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)\right.\right.$$
$$\left.+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right]$$
$$+\frac{1}{8}\left(\frac{2^{\alpha}-1}{3^{\alpha}}\right)\left[f\left(\frac{2a+b}{3},c\right)+f\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},c\right)+f\left(\frac{a+2b}{3},d\right)\right]$$
$$-\alpha\left(\frac{1}{3^{\beta}}-\frac{1}{8}\right)\int_{\frac{1}{3}}^{\frac{2}{3}}t^{\alpha-1}\left[f\left(tb+(1-t)a,\frac{2c+d}{3}\right)+f\left(tb+(1-t)a,\frac{c+2d}{3}\right)\right.$$
$$\left.+f\left(ta+(1-t)b,\frac{2c+d}{3}\right)+f\left(ta+(1-t)b,\frac{c+2d}{3}\right)\right]\mathrm{d}t$$
$$-\frac{\alpha}{8}\int_{\frac{1}{3}}^{\frac{2}{3}}t^{\alpha-1}\left[f(tb+(1-t)a,d)+f(tb+(1-t)a,c)+f(ta+(1-t)b,d)\right.$$
$$\left.+f(ta+(1-t)b,c)\right]\mathrm{d}t$$
$$-\beta\left(\frac{2^{\alpha}-1}{3^{\alpha}}\right)\int_{0}^{\frac{1}{3}}s^{\beta-1}\left[f\left(\frac{2a+b}{3},sd+(1-s)c\right)+f\left(\frac{2a+b}{3},sc+(1-s)d\right)\right.$$
$$\left.+f\left(\frac{a+2b}{3},sd+(1-s)c\right)+f\left(\frac{a+2b}{3},sc+(1-s)d\right)\right]\mathrm{d}s$$

$$+\alpha\beta\int_{\frac{1}{3}}^{\frac{2}{3}}\int_{0}^{\frac{1}{3}} t^{\alpha-1}s^{\beta-1}\left[f\left(tb+(1-t)a,sd+(1-s)c\right)+f\left(tb+(1-t)a,sc+(1-s)d\right)\right.$$

$$\left.\left.+f\left(ta+(1-t)b,sd+(1-s)c\right)+f\left(ta+(1-t)b,sc+(1-s)d\right)\right]\mathrm{d}s\mathrm{d}t\right\}, \tag{3.6}$$

$$I_5=\frac{1}{(b-a)(d-c)}\left\{\left(\frac{2^{\alpha}-1}{3^{\alpha}}\right)\left(\frac{2^{\beta}-1}{3^{\beta}}\right)\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)\right.\right.$$

$$\left.+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right]$$

$$-\alpha\left(\frac{2^{\beta}-1}{3^{\beta}}\right)\int_{\frac{1}{3}}^{\frac{2}{3}} t^{\alpha-1}\left[f\left(tb+(1-t)a,\frac{2c+d}{3}\right)+f\left(tb+(1-t)a,\frac{c+2d}{3}\right)\right.$$

$$\left.+f\left(ta+(1-t)b,\frac{2c+d}{3}\right)+f\left(ta+(1-t)b,\frac{c+2d}{3}\right)\right]\mathrm{d}t$$

$$-\beta\left(\frac{2^{\alpha}-1}{3^{\alpha}}\right)\int_{\frac{1}{3}}^{\frac{2}{3}} s^{\beta-1}\left[f\left(\frac{2a+b}{3},sd+(1-s)c\right)+f\left(\frac{2a+b}{3},sc+(1-s)d\right)\right.$$

$$\left.+f\left(\frac{a+2b}{3},sd+(1-s)c\right)+f\left(\frac{a+2b}{3},sc+(1-s)d\right)\right]\mathrm{d}s$$

$$+\alpha\beta\int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}} t^{\alpha-1}s^{\beta-1}\left[f\left(tb+(1-t)a,sd+(1-s)c\right)+f\left(tb+(1-t)a,sc+(1-s)d\right)\right.$$

$$\left.\left.+f\left(ta+(1-t)b,sd+(1-s)c\right)+f\left(ta+(1-t)b,sc+(1-s)d\right)\right]\mathrm{d}s\mathrm{d}t\right\}, \tag{3.7}$$

$$I_6=\frac{1}{(b-a)(d-c)}\left\{-\left(\frac{2^{\alpha}-1}{3^{\alpha}}\right)\left(\left(\frac{2}{3}\right)^{\beta}-\frac{7}{8}\right)\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)\right.\right.$$

$$\left.+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right]$$

$$+\frac{1}{8}\left(\frac{2^{\alpha}-1}{3^{\alpha}}\right)\left[f\left(\frac{2a+b}{3},c\right)+f\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},c\right)+f\left(\frac{a+2b}{3},d\right)\right]$$

$$-\frac{\alpha}{8}\int_{\frac{1}{3}}^{\frac{2}{3}} t^{\alpha-1}\left[f\left(tb+(1-t)a,d\right)+f\left(tb+(1-t)a,c\right)+f\left(ta+(1-t)b,d\right)\right.$$

$$\left.+f\left(ta+(1-t)b,c\right)\right]\mathrm{d}t$$

$$+\alpha\left(\left(\frac{2}{3}\right)^{\beta}-\frac{7}{8}\right)\int_{\frac{1}{3}}^{\frac{2}{3}} t^{\alpha-1}\left[f\left(tb+(1-t)a,\frac{2c+d}{3}\right)+f\left(tb+(1-t)a,\frac{c+2d}{3}\right)\right.$$
$$\left.+f\left(ta+(1-t)b,\frac{2c+d}{3}\right)+f\left(ta+(1-t)b,\frac{c+2d}{3}\right)\right]\mathrm{d}t$$
$$-\beta\left(\frac{2^{\alpha}-1}{3^{\alpha}}\right)\int_{\frac{2}{3}}^{1} s^{\beta-1}\left[f\left(\frac{2a+b}{3},sd+(1-s)c\right)+f\left(\frac{2a+b}{3},sc+(1-s)d\right)\right.$$
$$\left.+f\left(\frac{a+2b}{3},sd+(1-s)c\right)+f\left(\frac{a+2b}{3},sc+(1-s)d\right)\right]\mathrm{d}s$$
$$+\alpha\beta\int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{2}{3}}^{1} t^{\alpha-1}s^{\beta-1}\left[f\left(tb+(1-t)a,sd+(1-s)c\right)+f\left(tb+(1-t)a,sc+(1-s)d\right)\right.$$
$$\left.\left.+f\left(ta+(1-t)b,sd+(1-s)c\right)+f\left(ta+(1-t)b,sc+(1-s)d\right)\right]\mathrm{d}s\mathrm{d}t\right\}, \tag{3.8}$$

$$I_7=\frac{1}{(b-a)(d-c)}\left\{-\left(\frac{1}{3^{\beta}}-\frac{1}{8}\right)\left(\left(\frac{2}{3}\right)^{\alpha}-\frac{7}{8}\right)\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)\right.\right.$$
$$\left.+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right]$$
$$+\frac{1}{8}\left(\frac{1}{3^{\beta}}-\frac{1}{8}\right)\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right]$$
$$-\frac{1}{8}\left(\left(\frac{2}{3}\right)^{\alpha}-\frac{7}{8}\right)\left[f\left(\frac{2a+b}{3},c\right)+f\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},c\right)+f\left(\frac{a+2b}{3},d\right)\right]$$
$$+\frac{1}{64}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]$$
$$-\frac{\alpha}{8}\int_{\frac{2}{3}}^{1} t^{\alpha-1}\left[f\left(tb+(1-t)a,d\right)+f\left(tb+(1-t)a,c\right)+f\left(ta+(1-t)b,d\right)\right.$$
$$\left.+f\left(ta+(1-t)b,c\right)\right]\mathrm{d}t$$
$$-\alpha\left(\frac{1}{3^{\beta}}-\frac{1}{8}\right)\int_{\frac{2}{3}}^{1} t^{\alpha-1}\left[f\left(tb+(1-t)a,\frac{2c+d}{3}\right)+f\left(tb+(1-t)a,\frac{c+2d}{3}\right)\right.$$
$$\left.+f\left(ta+(1-t)b,\frac{2c+d}{3}\right)+f\left(ta+(1-t)b,\frac{c+2d}{3}\right)\right]\mathrm{d}t$$
$$+\beta\left(\left(\frac{2}{3}\right)^{\alpha}-\frac{7}{8}\right)\int_{0}^{\frac{1}{3}} s^{\beta-1}\left[f\left(\frac{2a+b}{3},sd+(1-s)c\right)+f\left(\frac{2a+b}{3},sc+(1-s)d\right)\right.$$

$$+f\left(\frac{a+2b}{3},sd+(1-s)c\right)+f\left(\frac{a+2b}{3},sc+(1-s)d\right)\Bigg]\mathrm{d}s$$
$$-\frac{\beta}{8}\int_{0}^{\frac{1}{3}}s^{\beta-1}\left[f(a,sd+(1-s)c)+f(a,sc+(1-s)d)+f(b,sd+(1-s)c)\right.$$
$$\left.+f(b,sc+(1-s)d)\right]\mathrm{d}s$$
$$+\alpha\beta\int_{\frac{2}{3}}^{1}\int_{0}^{\frac{1}{3}}t^{\alpha-1}s^{\beta-1}\left[f(tb+(1-t)a,sd+(1-s)c)+f(tb+(1-t)a,sc+(1-s)d)\right.$$
$$\left.+f(ta+(1-t)b,sd+(1-s)c)+f(ta+(1-t)b,sc+(1-s)d)\right]\mathrm{d}s\mathrm{d}t\Bigg\}, \tag{3.9}$$

$$I_8=\frac{1}{(b-a)(d-c)}\Bigg\{-\left(\frac{2^{\beta}-1}{3^{\beta}}\right)\left(\left(\frac{2}{3}\right)^{\alpha}-\frac{7}{8}\right)\Bigg[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)$$
$$+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\Bigg]$$
$$+\frac{1}{8}\left(\frac{2^{\beta}-1}{3^{\beta}}\right)\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right]$$
$$-\alpha\left(\frac{2^{\beta}-1}{3^{\beta}}\right)\int_{\frac{2}{3}}^{1}t^{\alpha-1}\Bigg[f\left(tb+(1-t)a,\frac{2c+d}{3}\right)+f\left(tb+(1-t)a,\frac{c+2d}{3}\right)$$
$$+f\left(ta+(1-t)b,\frac{2c+d}{3}\right)+f\left(ta+(1-t)b,\frac{c+2d}{3}\right)\Bigg]\mathrm{d}t$$
$$+\beta\left(\left(\frac{2}{3}\right)^{\alpha}-\frac{7}{8}\right)\int_{\frac{1}{3}}^{\frac{2}{3}}s^{\beta-1}\Bigg[f\left(\frac{2a+b}{3},sd+(1-s)c\right)+f\left(\frac{2a+b}{3},sc+(1-s)d\right)$$
$$+f\left(\frac{a+2b}{3},sd+(1-s)c\right)+f\left(\frac{a+2b}{3},sc+(1-s)d\right)\Bigg]\mathrm{d}s$$
$$-\frac{\beta}{8}\int_{\frac{1}{3}}^{\frac{2}{3}}s^{\beta-1}\left[f(a,sd+(1-s)c)+f(a,sc+(1-s)d)+f(b,sd+(1-s)c)\right.$$
$$\left.+f(b,sc+(1-s)d)\right]\mathrm{d}s$$
$$+\alpha\beta\int_{\frac{2}{3}}^{1}\int_{\frac{1}{3}}^{\frac{2}{3}}t^{\alpha-1}s^{\beta-1}\left[f(tb+(1-t)a,sd+(1-s)c)+f(tb+(1-t)a,sc+(1-s)d)\right.$$

$$\left.+f\left(ta+(1-t)b,sd+(1-s)c\right)+f\left(ta+(1-t)b,sc+(1-s)d\right)\right]\mathrm{d}s\mathrm{d}t\Bigg\} \tag{3.10}$$

and

$$\begin{aligned}
I_9 = \frac{1}{(b-a)(d-c)}\Bigg\{ & \left(\left(\frac{2}{3}\right)^{\beta}-\frac{7}{8}\right)\left(\left(\frac{2}{3}\right)^{\alpha}-\frac{7}{8}\right)\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)\right.\\
& \left.+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right]\\
& -\frac{1}{8}\left(\left(\frac{2}{3}\right)^{\beta}-\frac{7}{8}\right)\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right]\\
& -\frac{1}{8}\left(\left(\frac{2}{3}\right)^{\alpha}-\frac{7}{8}\right)\left[f\left(\frac{2a+b}{3},c\right)+f\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},c\right)+f\left(\frac{a+2b}{3},d\right)\right]\\
& +\frac{1}{64}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]\\
& +\alpha\left(\left(\frac{2}{3}\right)^{\beta}-\frac{7}{8}\right)\int_{\frac{2}{3}}^{1}t^{\alpha-1}\left[f\left(tb+(1-t)a,\frac{2c+d}{3}\right)+f\left(tb+(1-t)a,\frac{c+2d}{3}\right)\right.\\
& \left.+f\left(ta+(1-t)b,\frac{2c+d}{3}\right)+f\left(ta+(1-t)b,\frac{c+2d}{3}\right)\right]\mathrm{d}t\\
& -\frac{\alpha}{8}\int_{\frac{2}{3}}^{1}t^{\alpha-1}\left[f(tb+(1-t)a,d)+f(tb+(1-t)a,c)+f(ta+(1-t)b,d)\right.\\
& \left.+f(ta+(1-t)b,c)\right]\mathrm{d}t\\
& +\beta\left(\left(\frac{2}{3}\right)^{\alpha}-\frac{7}{8}\right)\int_{\frac{2}{3}}^{1}s^{\beta-1}\left[f\left(\frac{2a+b}{3},sd+(1-s)c\right)+f\left(\frac{2a+b}{3},sc+(1-s)d\right)\right.\\
& \left.+f\left(\frac{a+2b}{3},sd+(1-s)c\right)+f\left(\frac{a+2b}{3},sc+(1-s)d\right)\right]\mathrm{d}s\\
& -\frac{\beta}{8}\int_{\frac{1}{3}}^{\frac{2}{3}}s^{\beta-1}\left[f(a,sd+(1-s)c)+f(a,sc+(1-s)d)+f(b,sd+(1-s)c)\right.\\
& \left.+f(b,sc+(1-s)d)\right]\mathrm{d}s\\
& +\alpha\beta\int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}}t^{\alpha-1}s^{\beta-1}\left[f(tb+(1-t)a,sd+(1-s)c)+f(tb+(1-t)a,sc+(1-s)d)\right.\\
& \left.+f(ta+(1-t)b,sd+(1-s)c)+f(ta+(1-t)b,sc+(1-s)d)\right]\mathrm{d}s\mathrm{d}t\Bigg\}.
\end{aligned} \tag{3.11}$$

By adding the equalities (3.3)–(3.11) and by using the change of the variable, it can be rewritten as follows:

$$\begin{aligned}
\sum_{i=1}^{9} I_i = \frac{1}{(b-a)(d-c)} \Bigg\{ & \frac{1}{16}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right] \\
& + \frac{3}{16}\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right] \\
& + \frac{3}{16}\left[f\left(\frac{2a+b}{3},c\right)+f\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},c\right)+f\left(\frac{a+2b}{3},d\right)\right] \\
& + \frac{9}{16}\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)\right. \\
& \left. + f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right] - \frac{3\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\left[J_{b-}^{\alpha} f\left(a,\frac{2c+d}{3}\right)+J_{b-}^{\alpha} f\left(a,\frac{c+2d}{3}\right)\right. \\
& \left. + J_{a+}^{\alpha} f\left(b,\frac{2c+d}{3}\right)+J_{a+}^{\alpha} f\left(b,\frac{c+2d}{3}\right)\right] \\
& - \frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\left[J_{b-}^{\alpha} f(a,c)+J_{b-}^{\alpha} f(a,d)+J_{a+}^{\alpha} f(b,c)+J_{a+}^{\alpha} f(b,d)\right] \\
& - \frac{3\Gamma(\beta+1)}{4(d-c)^{\beta}}\left[J_{d-}^{\beta} f\left(\frac{2a+b}{3},c\right)+J_{d-}^{\beta} f\left(\frac{a+2b}{3},c\right)+J_{c+}^{\beta} f\left(\frac{2a+b}{3},d\right)\right. \\
& \left. + J_{c+}^{\beta} f\left(\frac{a+2b}{3},d\right)\right] - \frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}\left[J_{d-}^{\beta} f(a,c)+J_{c+}^{\beta} f(a,d)+J_{d-}^{\beta} f(b,c)+J_{c+}^{\beta} f(b,d)\right] \\
& + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}}\left[J_{b-,d-}^{\alpha,\beta} f(a,c)+J_{b-,c+}^{\alpha,\beta} f(a,d)+J_{a+,d-}^{\alpha,\beta} f(b,c)+J_{a+,c+}^{\alpha,\beta} f(b,d)\right] \Bigg\}.
\end{aligned} \tag{3.12}$$

Multiplying the both sides of equality (3.12) by $\frac{(b-a)(d-c)}{4}$, equality (3.1) is obtained. This finishes the proof of Lemma 1. □

4. Newton inequalities for Riemann–Liouville fractional integrals

In this section, we will give fractional Newton-type inequalities in the case of differentiable co-ordinated convex function.

Theorem 1. *Let us note that the conditions of Lemma 1 are valid. If* $\left|\frac{\partial^2 f}{\partial s \partial t}\right|$ *is co-ordinated convex function, then we obtain the following inequality*

$$\begin{aligned}
\left|\Theta^{\alpha,\beta}(a,b;c,d)\right| \leq & \frac{(b-a)(d-c)}{4}\left[\varphi_1(\alpha)+\varphi_2(\alpha)+\varphi_3(\alpha)\right]\left[\varphi_1(\beta)+\varphi_2(\beta)+\varphi_3(\beta)\right] \\
& \times \left[\left|\frac{\partial^2 f}{\partial s \partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s \partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s \partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s \partial t}(b,d)\right|\right],
\end{aligned} \tag{4.1}$$

where

$$\begin{cases} \varphi_1(\eta) = \int\limits_0^{\frac{1}{3}} \left|t^\eta - \frac{1}{8}\right| \mathrm{d}t = \frac{2\eta}{\eta+1}\left(\frac{1}{8}\right)^{1+\frac{1}{\eta}} + \frac{1}{(\eta+1)3^{\eta+1}} - \frac{1}{24}, \\ \varphi_2(\eta) = \int\limits_{\frac{1}{3}}^{\frac{2}{3}} \left|t^\eta - \frac{1}{2}\right| \mathrm{d}t = \frac{2\eta}{\eta+1}\left(\frac{1}{2}\right)^{1+\frac{1}{\eta}} + \frac{1+2^{\eta+1}}{(\eta+1)3^{\eta+1}} - \frac{1}{2}, \\ \varphi_3(\eta) = \int\limits_{\frac{2}{3}}^{1} \left|t^\eta - \frac{7}{8}\right| \mathrm{d}t = \frac{2\eta}{\eta+1}\left(\frac{7}{8}\right)^{1+\frac{1}{\eta}} + \frac{2^{\eta+1}+3^{\eta+1}}{(\eta+1)3^{\eta+1}} - \frac{35}{24}, \end{cases}$$

and $\Theta^{\alpha,\beta}(a,b;c,d)$ *is given in equality* (3.2).

Proof. If we take absolute value of (3.1), then the following inequality holds:

$$\left|\Theta^{\alpha,\beta}(a,b;c,d)\right| \le \frac{(b-a)(d-c)}{4}\sum_{k=1}^{9}|I_k|. \tag{4.2}$$

From the fact that $\left|\frac{\partial^2 f}{\partial s \partial t}\right|$ is co-ordinated convex function, we have

$$\begin{aligned} |I_1| \le & \int\limits_0^{\frac{1}{3}}\int\limits_0^{\frac{1}{3}} \left|t^\alpha - \frac{1}{8}\right|\left|s^\beta - \frac{1}{8}\right| \left|\frac{\partial^2}{\partial s\partial t} f(tb+(1-t)a, sd+(1-s)c)\right. \\ & - \frac{\partial^2}{\partial s\partial t} f(tb+(1-t)a, sc+(1-s)d) - \frac{\partial^2}{\partial s\partial t} f(ta+(1-t)b, sd+(1-s)c) \\ & \left. + \frac{\partial^2}{\partial s\partial t} f(ta+(1-t)b, sc+(1-s)d)\right| \mathrm{d}s\mathrm{d}t \\ \le & \int\limits_0^{\frac{1}{3}}\int\limits_0^{\frac{1}{3}} \left|t^\alpha - \frac{1}{8}\right|\left|s^\beta - \frac{1}{8}\right| \left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right] \mathrm{d}s\mathrm{d}t \\ = & \varphi_1(\alpha)\varphi_1(\beta)\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right]. \end{aligned} \tag{4.3}$$

Similarly, we state the following inequalities

$$|I_2| \le \varphi_1(\alpha)\varphi_2(\beta)\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right], \tag{4.4}$$

$$|I_3| \le \varphi_1(\alpha)\varphi_3(\beta)\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right], \tag{4.5}$$

$$|I_4| \le \varphi_2(\alpha)\varphi_1(\beta)\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right], \tag{4.6}$$

$$|I_5| \leq \varphi_2(\alpha)\varphi_2(\beta)\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right], \tag{4.7}$$

$$|I_6| \leq \varphi_2(\alpha)\varphi_3(\beta)\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right], \tag{4.8}$$

$$|I_7| \leq \varphi_3(\alpha)\varphi_1(\beta)\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right], \tag{4.9}$$

$$|I_8| \leq \varphi_3(\alpha)\varphi_2(\beta)\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right], \tag{4.10}$$

and

$$|I_9| \leq \varphi_3(\alpha)\varphi_3(\beta)\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right]. \tag{4.11}$$

Hence, we reach the required inequality by adding from (4.3) to (4.11) and then multiplying by $\frac{(b-a)(d-c)}{4}$. □

Example 1. Let us consider that the function $f\colon [0,1]\times[0,1]\to\mathbb{R}$, $f(t,s)=t^2s^2$ such that $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is co-ordinated convex function on $[0,1]\times[0,1]$. From the facts of assumptions, the following inequality holds:

$$\begin{aligned}&\left|\frac{1}{9}-\frac{1}{6}\left[\frac{\alpha^2+\alpha+2}{(\alpha+1)(\alpha+2)}+\frac{\beta^2+\beta+2}{(\beta+1)(\beta+2)}\right]+\frac{[\alpha(\alpha+1)+2][\beta(\beta+1)+2]}{4(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\right|\\&\leq[\varphi_1(\alpha)+\varphi_2(\alpha)+\varphi_3(\alpha)][\varphi_1(\beta)+\varphi_2(\beta)+\varphi_3(\beta)],\end{aligned}$$

where φ_1,φ_2 and φ_3 functions are as given in Theorem 1.

With easy computations, we obtain the following equalities

$$\frac{[f(a,c)+f(a,d)+f(b,c)+f(b,d)]}{64}=\frac{1}{64},$$

$$\frac{3}{64}\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right]=\frac{5}{192},$$

$$\frac{3}{64}\left[f\left(\frac{2a+b}{3},c\right)+f\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},c\right)+f\left(\frac{a+2b}{3},d\right)\right]=\frac{5}{192},$$

$$\frac{9}{64}\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)\right.$$

$$+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\Bigg]=\frac{25}{576},$$

and

$$\frac{3\Gamma(\alpha+1)}{16(b-a)^{\alpha}}\left[J_{b-}^{\alpha}f\left(a,\frac{2c+d}{3}\right)+J_{b-}^{\alpha}f\left(a,\frac{c+2d}{3}\right)+J_{a+}^{\alpha}f\left(b,\frac{2c+d}{3}\right)\right.$$
$$\left.+J_{a+}^{\alpha}f\left(b,\frac{c+2d}{3}\right)\right]=\frac{5\left(\alpha^2+\alpha+2\right)}{48(\alpha+2)(\alpha+1)}.$$

From the facts of the definitions of Riemann–Liouville fractional integrals, we obtain

$$\frac{\Gamma(\alpha+1)}{16(b-a)^{\alpha}}\left[J_{b-}^{\alpha}f(a,c)+J_{b-}^{\alpha}f(a,d)+J_{a+}^{\alpha}f(b,c)+J_{a+}^{\alpha}f(b,d)\right]=\frac{\left(\alpha^2+\alpha+2\right)}{16(\alpha+2)(\alpha+1)},$$

$$\frac{3\Gamma(\beta+1)}{16(d-c)^{\beta}}\left[J_{d-}^{\beta}f\left(\frac{2a+b}{3},c\right)+J_{d-}^{\beta}f\left(\frac{a+2b}{3},c\right)+J_{c+}^{\beta}f\left(\frac{2a+b}{3},d\right)\right.$$
$$\left.+J_{c+}^{\beta}f\left(\frac{a+2b}{3},d\right)\right]=\frac{5\left(\beta^2+\beta+2\right)}{48(\beta+2)(\beta+1)},$$

$$\frac{\Gamma(\beta+1)}{16(d-c)^{\beta}}\left[J_{d-}^{\beta}f(a,c)+J_{c+}^{\beta}f(a,d)+J_{d-}^{\beta}f(b,c)+J_{c+}^{\beta}f(b,d)\right]=\frac{\left(\beta^2+\beta+2\right)}{16(\beta+2)(\beta+1)},$$

and

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\left[J_{b-,d-}^{\alpha,\beta}f(a,c)+J_{b-,c+}^{\alpha,\beta}f(a,d)+J_{a+,d-}^{\alpha,\beta}f(b,c)+J_{a+,c+}^{\alpha,\beta}f(b,d)\right]$$
$$=\frac{5\left(\beta^2+\beta+2\right)}{48(\beta+2)(\beta+1)}.$$

The left hand side of the inequality (4.1) is

$$\left|\frac{1}{9}-\frac{1}{6}\left[\frac{\alpha^2+\alpha+2}{(\alpha+1)(\alpha+2)}+\frac{\beta^2+\beta+2}{(\beta+1)(\beta+2)}\right]+\frac{[\alpha(\alpha+1)+2][\beta(\beta+1)+2]}{4(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\right|:=\Psi_1$$

and the right hand side of the inequality (4.1) is

$$\left[\varphi_1(\alpha)+\varphi_2(\alpha)+\varphi_3(\alpha)\right]\left[\varphi_1(\beta)+\varphi_2(\beta)+\varphi_3(\beta)\right]:=\Psi_2.$$

As you can see in Figure 1 above, the result of our appropriate choices in Example 1 is satisfied in the inequality (4.1).

Remark 1. If we choose $\alpha=\beta=1$ in Theorem 1, then we get

$$\left|\Theta^{1,1}(a,b;c,d)\right|$$
$$\leq(b-a)(d-c)\frac{625}{331776}\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right],$$

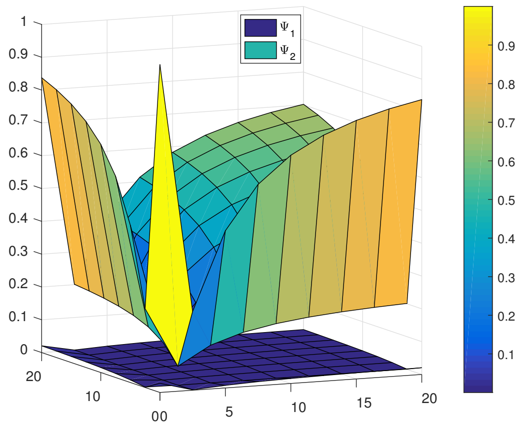

FIGURE 1. Graph for the result of Example 1 computed and plotted in MATLAB program.

which is given by Iftikhar et al. in [12] and $\Theta^{1,1}(a,b;c,d)$ equals to

$$\begin{aligned}
\Theta^{1,1}(a,b;c,d) &:= \frac{[f(a,c)+f(a,d)+f(b,c)+f(b,d)]}{64} \qquad (4.12)\\
&+\frac{3}{64}\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right.\\
&\left.+f\left(\frac{2a+b}{3},c\right)+f\left(\frac{a+2b}{3},c\right)+\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},d\right)\right]\\
&+\frac{9}{64}\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)\right.\\
&\left.+f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right]-\frac{1}{8(b-a)}\left[\int_a^b f(x,c)\,\mathrm{d}x\right.\\
&\left.+\int_a^b f(x,d)\,\mathrm{d}x+3\int_a^b f\left(x,\frac{2c+d}{3}\right)\mathrm{d}x+3\int_a^b f\left(x,\frac{c+2d}{3}\right)\mathrm{d}x\right]\\
&-\frac{1}{8(d-c)}\left[\int_c^d f(a,y)\,\mathrm{d}y+\int_c^d f(b,y)\,\mathrm{d}y+3\int_c^d f\left(\frac{2a+b}{3},y\right)\mathrm{d}y\right.\\
&\left.+3\int_c^d f\left(\frac{a+2b}{3},y\right)\mathrm{d}y\right]+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

Theorem 2. *Suppose that the conditions of Lemma 1 are valid. Suppose also that* $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$*, is a co-ordinated convex function for* $q > 1$*. Then, we obtain the following Newton inequality:*

$$\left|\Theta^{\alpha,\beta}(a,b;c,d)\right| \leq \frac{(b-a)(d-c)}{4}\Bigg\{ (\varphi_4(\alpha,p)+\varphi_6(\alpha,p))(\varphi_4(\beta,p)+\varphi_6(\beta,p))$$

$$\times\left[\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}$$

$$+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{25\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}\right]$$

$$+2\left[\varphi_4(\alpha,p)+\varphi_6(\alpha,p)\right]\varphi_5(\beta,p)$$

$$\times\left[\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}\right]$$

$$+2\varphi_5(\alpha,p)\left[\varphi_4(\beta,p)+\varphi_6(\beta,p)\right]$$

$$\times\left[\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}\Bigg]$$

$$+4\varphi_5(\alpha,p)\varphi_5(\beta,p)$$

$$\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}\times f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{36}\right)^{\frac{1}{q}}\Bigg\}.$$

Here $q^{-1}+p^{-1}=1$, $\Theta^{\alpha,\beta}(a,b;c,d)$ *is given by* (3.2), *and*

$$\begin{cases}\varphi_4(\eta,p)=\left(\int\limits_0^{\frac{1}{3}}\left|t^\eta-\frac{1}{8}\right|^p \mathrm{d}t\right)^{\frac{1}{p}},\\ \varphi_5(\eta,p)=\left(\int\limits_{\frac{1}{3}}^{\frac{2}{3}}\left|t^\eta-\frac{1}{2}\right|^p \mathrm{d}t\right)^{\frac{1}{p}},\\ \varphi_6(\eta,p)=\left(\int\limits_{\frac{2}{3}}^{1}\left|t^\eta-\frac{7}{8}\right|^p \mathrm{d}t\right)^{\frac{1}{p}}.\end{cases}$$

Proof. From the facts of the Hölder inequality and co-ordinated convexity of $\left|\frac{\partial^2 f}{\partial s\partial t}(t,s)\right|^q$, the following inequalities hold:

$$\begin{aligned}|I_1|\leq&\int\limits_0^{\frac{1}{3}}\int\limits_0^{\frac{1}{3}}\left|t^\alpha-\frac{1}{8}\right|\left|s^\beta-\frac{1}{8}\right|\left|\frac{\partial^2}{\partial s\partial t}f(tb+(1-t)a,sd+(1-s)c)\right|\mathrm{d}s\mathrm{d}t\\&+\int\limits_0^{\frac{1}{3}}\int\limits_0^{\frac{1}{3}}\left|t^\alpha-\frac{1}{8}\right|\left|s^\beta-\frac{1}{8}\right|\left|\frac{\partial^2}{\partial s\partial t}f(tb+(1-t)a,sc+(1-s)d)\right|\mathrm{d}s\mathrm{d}t\\&+\int\limits_0^{\frac{1}{3}}\int\limits_0^{\frac{1}{3}}\left|t^\alpha-\frac{1}{8}\right|\left|s^\beta-\frac{1}{8}\right|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sd+(1-s)c)\right|\mathrm{d}s\mathrm{d}t\\&+\int\limits_0^{\frac{1}{3}}\int\limits_0^{\frac{1}{3}}\left|t^\alpha-\frac{1}{8}\right|\left|s^\beta-\frac{1}{8}\right|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|\mathrm{d}s\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\int_0^{\frac{1}{3}}\int_0^{\frac{1}{3}}\left|t^{\alpha}-\frac{1}{8}\right|^{p}\left|s^{\beta}-\frac{1}{8}\right|^{p}\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{p}}\\
&\quad\times\left[\left(\int_0^{\frac{1}{3}}\int_0^{\frac{1}{3}}\left|\frac{\partial^2}{\partial s\partial t}f(tb+(1-t)a,sd+(1-s)c)\right|^{q}\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\int_0^{\frac{1}{3}}\int_0^{\frac{1}{3}}\left|\frac{\partial^2}{\partial s\partial t}f(tb+(1-t)a,sc+(1-s)d)\right|^{q}\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^{\frac{1}{3}}\int_0^{\frac{1}{3}}\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sd+(1-s)c)\right|^{q}\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_0^{\frac{1}{3}}\int_0^{\frac{1}{3}}\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|^{q}\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{q}}\right]\\
&\leq \varphi_4(\alpha,p)\,\varphi_4(\beta,p)\\
&\quad\times\left[\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^{q}+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^{q}+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^{q}+25\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^{q}}{324}\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^{q}+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^{q}+25\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^{q}+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^{q}}{324}\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^{q}+25\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^{q}+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^{q}+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^{q}}{324}\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\frac{25\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^{q}+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^{q}+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^{q}+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^{q}}{324}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Similarly,

$$|I_2| \leq 2\varphi_4(\alpha,p)\,\varphi_5(\beta,p)$$

$$\times\left[\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}\right],$$

$$|I_3|\le\varphi_4(\alpha,p)\,\varphi_6(\beta,p)$$
$$\times\left[\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}$$
$$+\left(\frac{25\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}\right],$$

$$|I_4|\le 2\varphi_5(\alpha,p)\,\varphi_4(\beta,p)$$
$$\times\left[\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}\right],$$

$$|I_5|\le 4\varphi_5(\alpha,p)\,\varphi_5(\beta,p)$$

$$\times\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{36}\right)^{\frac{1}{q}},$$

$$\begin{aligned}|I_6|\leq{}&2\varphi_5(\alpha,p)\,\varphi_6(\beta,p)\\&\times\left[\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}\right.\\&\left.+\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}\right],\end{aligned}$$

$$\begin{aligned}|I_7|\leq{}&\varphi_6(\alpha,p)\,\varphi_4(\beta,p)\\&\times\left[\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}\right.\\&+\left(\frac{25\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}\\&+\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}\\&\left.+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}\right],\end{aligned}$$

$$\begin{aligned}|I_8|\leq{}&2\varphi_6(\alpha,p)\,\varphi_5(\beta,p)\\&\times\left[\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}\right.\end{aligned}$$

$$+\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}\Bigg],$$

and

$$|I_9|\leq \varphi_6(\alpha,p)\,\varphi_6(\beta,p)$$
$$\times\Bigg[\left(\frac{25\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}$$
$$+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}$$
$$+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}$$
$$+\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}\Bigg].$$

This finishes the proof of Theorem 2. □

Corollary 1. *Let us consider $\alpha=\beta=1$ in Theorem 2. Then, it follows*

$$\left|\Theta^{1,1}(a,b;c,d)\right|\leq\frac{(b-a)(d-c)}{4}\left\{4\left(\frac{1}{p+1}\left(\left(\frac{1}{8}\right)^{p+1}+\left(\frac{5}{24}\right)^{p+1}\right)\right)^{\frac{2}{p}}\right.$$
$$\times\Bigg[\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}$$
$$+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}$$
$$+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}$$

$$+\left(\frac{25\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}\Bigg]$$

$$+2\left[2\left(\frac{1}{p+1}\left(\left(\frac{1}{8}\right)^{p+1}+\left(\frac{5}{24}\right)^{p+1}\right)\right)^{\frac{1}{p}}\right]\left(\frac{2}{p+1}\left(\frac{1}{6}\right)^{p+1}\right)^{\frac{1}{p}}$$

$$\times\left[\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}$$

$$+\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}\right]$$

$$+4\left(\frac{2}{p+1}\left(\frac{1}{6}\right)^{p+1}\right)^{\frac{2}{p}}$$

$$\left.\times\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{36}\right)^{\frac{1}{q}}\right\}.$$

Here, $\Theta^{1,1}(a,b;c,d)$ *is given by* (4.12).

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