Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 645–658

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4511

HERMITE–HADAMARD- AND PACHPATTE-TYPE INTEGRAL INEQUALITIES FOR GENERALIZED SUBADDITIVE FUNCTIONS IN THE FRACTAL SENSE

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Received 02 December, 2022

Abstract. In this paper, we establish the generalized Hermite–Hadamard- and Pachpatte-type integral inequalities for local fractional integrals via the generalized subadditive functions. In particular, we put forward a refined version of the generalized Hermite–Hadamard-type inequalities in the framework of fractal space.

2010 *Mathematics Subject Classification:* 26D07; 26D10; 26D15; 26A33

Keywords: Hermite–Hadamard inequality, Pachpatte inequalities, generalized subadditive functions

1. INTRODUCTION

A function $f\colon [0,\infty)\subset\mathbb{R}\to\mathbb{R}$ is said to be subadditive if the following inequality

$$f(x+y)\leq f(x)+f(y)$$

holds true for each $x,y\in[0,\infty)$. If equality holds, then the function f is called additive, and if the inequality is reversed, then the function f is named superadditive.

Subadditivity usually appears in thermodynamic properties of nonideal solutions and mixtures, such as excess molar volume, heat of mixing, or excess enthalpy. In addition, inequalities and subadditive functions can be occurred in electrical networks, quantum relative entropy, purification, ergodic theory and dynamical systems, equilibrium and repulsive perturbation. Here, we mention some findings of the published articles [6, 7, 12, 13, 16] and the bibliographies quoted therein.

A function $f\colon I\subset\mathbb{R}\to\mathbb{R}$ is said to be convex if the following inequality

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)$$

holds true for each $x,y\in I$ and $t\in[0,1]$.

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The concept of convexity is well known in the literature. The classical Hermite–Hadamard's integral inequality is closely related to convexity, which can be noted as below:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2},$$

in which $f\colon I \to \mathbb{R}$ is a convex function as well as $a, b \in I$ with $a \neq b$.

The inequality above has been studied extensively by many scholars on Euclidean space. Quite a number of generalizations, refinements, extensions and variants have been existed in the literature, see for instance the published articles [3, 4, 10, 11] and the bibliographies quoted therein.

In [19], Tseng et al. acquired the following modified Hermite–Hadamard-type inequalities for convex functions.

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &\le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \\ &\le \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right] \le \frac{f(a)+f(b)}{2}. \end{aligned}$$

In [14], Pachpatte developed the estimates regarding the products of two nonnegative convex functions.

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}f(x)g(x)\mathrm{d}x + \frac{1}{6}M(a,b)+\frac{1}{3}N(a,b)$$

and

$$\frac{1}{b-a}f(x)g(x)\mathrm{d}x \le \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b),$$

where $M(a,b) = f(a)g(a)+f(b)g(b)$ and $N(a,b) = f(a)g(b)+f(b)g(a)$.

2. Preliminaries

Let us retrospect the theoretical knowledge in connection with local fractional operators over $\mathbb{R}^{\alpha}$ $(0<\alpha\le 1)$ advanced by Yang in [21]. Here, we want to emphasize that α denotes the fractal dimension of Cantor set, not an exponential symbol.

Let Υ_1^{α}, Υ_2^{α} and Υ_3^{α} belong to the set $\mathbb{R}^{\alpha}$ $(0<\alpha\le 1)$, then

(1) $\Upsilon_1^{\alpha}+\Upsilon_2^{\alpha}$ and $\Upsilon_1^{\alpha}\Upsilon_2^{\alpha}$ belong to the set $\mathbb{R}^{\alpha}$;
(2) $\Upsilon_1^{\alpha}+\Upsilon_2^{\alpha} = \Upsilon_2^{\alpha}+\Upsilon_1^{\alpha} = (\Upsilon_1+\Upsilon_2)^{\alpha} = (\Upsilon_2+\Upsilon_1)^{\alpha}$;
(3) $\Upsilon_1^{\alpha}+(\Upsilon_2^{\alpha}+\Upsilon_3^{\alpha}) = (\Upsilon_1^{\alpha}+\Upsilon_2^{\alpha})+\Upsilon_3^{\alpha}$;
(4) $\Upsilon_1^{\alpha}\Upsilon_2^{\alpha} = \Upsilon_2^{\alpha}\Upsilon_1^{\alpha} = (\Upsilon_1\Upsilon_2)^{\alpha} = (\Upsilon_2\Upsilon_1)^{\alpha}$;
(5) $\Upsilon_1^{\alpha}(\Upsilon_2^{\alpha}\Upsilon_3^{\alpha}) = (\Upsilon_1^{\alpha}\Upsilon_2^{\alpha})\Upsilon_3^{\alpha}$;
(6) $\Upsilon_1^{\alpha}(\Upsilon_2^{\alpha}+\Upsilon_3^{\alpha}) = \Upsilon_1^{\alpha}\Upsilon_2^{\alpha}+\Upsilon_1^{\alpha}\Upsilon_3^{\alpha}$;

(7) $\Upsilon_1^\alpha + 0^\alpha = 0^\alpha + \Upsilon_1^\alpha = \Upsilon_1^\alpha$ and $\Upsilon_1^\alpha 1^\alpha = 1^\alpha \Upsilon_1^\alpha = \Upsilon_1^\alpha$;
(8) $\Upsilon_1^\alpha \geq \Upsilon_2^\alpha$ when and only when $\Upsilon_1 \geq \Upsilon_2$, $\Upsilon_1, \Upsilon_2 \in \mathbb{R}$;
(9) $(\Upsilon_1^\alpha)^t = (\Upsilon_1^t)^\alpha, t > 0$ and $\Upsilon_1 > 0$;
(10) $\Upsilon_1^\alpha - \Upsilon_2^\alpha = (\Upsilon_1 - \Upsilon_2)^\alpha$;
(11) For any $\Upsilon_1^\alpha \in \mathbb{R}^\alpha$, its inverse element $(-\Upsilon_1)^\alpha$ may be noted as $-\Upsilon_1^\alpha$; for any $\Upsilon_2^\alpha \in \mathbb{R}^\alpha \setminus \{0^\alpha\}$, its inverse element $(1/\Upsilon_2)^\alpha$ may be noted as $1^\alpha/\Upsilon_2^\alpha$, but not $1/\Upsilon_2^\alpha$.

Definition 1 ([21])**.** The non-differentiable function $f\colon \mathbb{R} \to \mathbb{R}^\alpha, x \to f(x)$ is named as local fractional continuous at x_0, if for any $\varepsilon > 0$, there exists $\sigma > 0$, such that the following inequality

$$|f(x) - f(x_0)| < \varepsilon^\alpha$$

is valid for $|x - x_0| < \sigma$. If the function $f(x)$ is local fractional continuous defined over the interval (a,b), then one denotes it by $f(x) \in C_\alpha(a,b)$.

Definition 2 ([21])**.** Suppose that $f(x) \in C_\alpha[a,b]$, and if $\triangle = \{\xi_0, \xi_1, \cdots, \xi_N\}$, $(N \in \mathbb{N})$ is a partition with regard to the interval $[a,b]$ which satisfies that $a = \xi_0 < \xi_1 < \cdots < \xi_N = b$, then the local fractional integral of f defined over the interval $[a,b]$ of order α is defined by

$${}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(\xi)(\mathrm{d}\xi)^\alpha := \frac{1}{\Gamma(\alpha+1)} \lim_{\triangle\xi \to 0} \sum_{j=0}^{N-1} f(\xi_j)(\Delta\xi_j)^\alpha,$$

where $\Delta\xi\colon = \max\{\Delta\xi_1, \Delta\xi_2, \cdots, \Delta\xi_{N-1}\}$, $\Delta\xi_j\colon = \xi_{j+1} - \xi_j$, $j = 0, \cdots, N-1$. For all $x \in [a,b]$, if there exists ${}_aI_x^{(\alpha)} f(x)$, then it is denoted by $f(x) \in I_x^\alpha[a,b]$.

Certain properties widely utilized in the fractal space regarding the integral ${}_aI_b^{(\alpha)} f(x)$ are established as below.

Assume that $f(x) = g^{(\alpha)}(x) \in C_\alpha[a,b]$, then one achieves that

$${}_aI_b^{(\alpha)} f(x) = g(b) - g(a).$$

In accordance with this, the following two formulas are deduced:

$${}_aI_b^{(\alpha)}(1^\alpha) = \frac{1^\alpha}{\Gamma(1+\alpha)}(b^\alpha - a^\alpha)$$

and

$${}_aI_b^{(\alpha)}(x^{\tau\alpha}) = \frac{\Gamma(1+\tau\alpha)}{\Gamma(1+(\tau+1)\alpha)}\left(b^{(\tau+1)\alpha} - a^{(\tau+1)\alpha}\right),\ \tau > 0.$$

In [21], Yang presented the concept with regard to the generalized convexity.

A function $f\colon I \subset \mathbb{R} \to \mathbb{R}^\alpha$ is said to be generalized convex if the following inequality

$$f(tx + (1-t)y) \leq t^\alpha f(x) + (1-t)^\alpha f(y)$$

holds true for each $x, y \in I$ and $t \in [0,1]$. Here, we present the concept of generalized subadditive functions on fractal sets as follows.

Definition 3. The function $f\colon [0,\infty) \subset \mathbb{R} \to \mathbb{R}^{\alpha}$ is said to be generalized subadditive if the following inequality

$$f(x+y) \le f(x) + f(y)$$

holds true for each $x, y \in [0,\infty)$. If equality holds, then f is called generalized additive, and if the inequality is reversed, then f is named generalized superadditive.

In 2019, Du et al. used the local fractional calculus to introduce the following generalized starshaped functions.

Definition 4 ([5])**.** The functions $f\colon [0,\infty) \subset \mathbb{R} \to \mathbb{R}^{\alpha}$ is referred to as generalized starshaped, if $f(tx) \le t^{\alpha} f(x)$ holds true for each $x \in [0,\infty)$ and $t \in [0,1]$.

Remark 1. If the generalized subadditive function $f\colon [0,\infty) \to \mathbb{R}^{\alpha}$ is generalized starshaped, then the function f is also generalized convex. Moreover, if the function f is generalized subadditive and generalized convex over the interval $[0,\infty)$, as well as $f(0) = 0^{\alpha}$, then the function f is generalized additive.

In [9], Mo et al. gave the following generalized Hermite–Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} {}_aI_b^{(\alpha)} f(x) \le \frac{f(a)+f(b)}{2^{\alpha}},$$

in which $f(x) \in C_{\alpha}[a,b]$ is a generalized convex function over the interval $[a,b]$ with $a < b$.

For recent outcomes in accordance with inequalities in the fractal sense, one may consults the published articles [1, 2, 8, 15, 17, 18, 20] and the bibliographies quoted therein.

In this paper we present certain local fractional Hermite–Hadamard-type integral inequalities via generalized subadditive functions, and the estimates of the products of two generalized subadditive functions are also considered.

3. Main results

Firstly, we construct the following generalized Hermite–Hadamard-type inequality under the generalized subadditivity in the settings of fractal sets.

Theorem 1. *Assume that $f\colon I = [0,\infty) \to \mathbb{R}^{\alpha}$ is a generalized subadditive function. If $a < b$ $(a, b \in I^{\circ})$ and $f \in C_{\alpha}[0,\infty)$, then the following Hermite–Hadamard-type inequalities in the settings of fractal sets hold*

$$\left(\frac{1}{2}\right)^{\alpha} f(a+b) \le \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} {}_aI_b^{(\alpha)} f(x) \le \frac{\Gamma(1+\alpha)}{a^{\alpha}} {}_0I_a^{(\alpha)} f(x) + \frac{\Gamma(1+\alpha)}{b^{\alpha}} {}_0I_b^{(\alpha)} f(x).$$

Proof. Assume that $x = ta + (1-t)b \in [a,b]$, or $x = (1-t)a + tb \in [a,b]$ for any $t \in [0,1]$. On account of the generalized subadditivity of the function f, we can figure out that

$$f(ta + (1-t)b) \le f(ta) + f((1-t)b) \tag{3.1}$$

and

$$f((1-t)a + tb) \le f((1-t)a) + f(tb). \tag{3.2}$$

Combining the inequalities (3.1) and (3.2), and in terms of the generalized subadditivity of the function f again, we obtain that

$$\begin{aligned} f(a+b) &\le f(ta + (1-t)b) + f((1-t)a + tb) \\ &\le f(ta) + f((1-t)b) + f((1-t)a) + f(tb). \end{aligned} \tag{3.3}$$

Integrating both sides of the inequalities (3.3) in the fractal sense, we deduce that

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_0^1 f(a+b)(\mathrm{d}t)^\alpha \\ &\le \frac{1}{\Gamma(1+\alpha)} \int_0^1 f(ta + (1-t)b)(\mathrm{d}t)^\alpha + \frac{1}{\Gamma(1+\alpha)} \int_0^1 f((1-t)a + tb)(\mathrm{d}t)^\alpha \\ &\le \frac{2^\alpha}{\Gamma(1+\alpha)} \int_0^1 f(ta)(\mathrm{d}t)^\alpha + \frac{2^\alpha}{\Gamma(1+\alpha)} \int_0^1 f(tb)(\mathrm{d}t)^\alpha. \end{aligned} \tag{3.4}$$

Taking advantage of the appropriate substitutions, we attain that

$$\begin{aligned} &\left(\frac{1}{2}\right)^\alpha \frac{1^\alpha}{\Gamma(1+\alpha)} f(a+b) \\ &\le \left(\frac{1}{b-a}\right)^\alpha \frac{1}{\Gamma(1+\alpha)} \int_a^b f(x)(\mathrm{d}x)^\alpha \\ &\le \left(\frac{1}{a}\right)^\alpha \frac{1}{\Gamma(1+\alpha)} \int_0^a f(x)(\mathrm{d}x)^\alpha + \left(\frac{1}{b}\right)^\alpha \frac{1}{\Gamma(1+\alpha)} \int_0^b f(x)(\mathrm{d}x)^\alpha. \end{aligned}$$

As a consequence, we acquire that

$$\left(\frac{1}{2}\right)^\alpha f(a+b) \le \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_aI_b^{(\alpha)} f(x) \le \frac{\Gamma(1+\alpha)}{a^\alpha} {}_0I_a^{(\alpha)} f(x) + \frac{\Gamma(1+\alpha)}{b^\alpha} {}_0I_b^{(\alpha)} f(x).$$

This fulfills the proof. □

Corollary 1. *If one attempts to take $f(tx) \le t^\alpha f(x)$ in Theorem 1, then one acquires the following Hermite–Hadamard-type inequality for the generalized subadditive functions:*

$$f\left(\frac{a+b}{2}\right) \le \left(\frac{1}{2}\right)^\alpha f(a+b) \le \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_aI_b^{(\alpha)} f(x) \le \frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)} \big(f(a) + f(b)\big).$$

Proof. Taking advantage of $f(tx) \leq t^{\alpha} f(x)$ and the inequalities (3.4), we acquire that

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{1}{2}\right)^{\alpha} f(a+b),$$

and

$$\begin{aligned}
&\left(\frac{1}{b-a}\right)^{\alpha} \frac{1}{\Gamma(1+\alpha)} \int_0^1 f(x)(\mathrm{d}x)^{\alpha} \\
&\leq \frac{1}{\Gamma(1+\alpha)} \int_0^1 f(ta)(\mathrm{d}t)^{\alpha} + \frac{1}{\Gamma(1+\alpha)} \int_0^1 f(tb)(\mathrm{d}t)^{\alpha} \\
&\leq \frac{1}{\Gamma(1+\alpha)} \int_0^1 t^{\alpha}(\mathrm{d}t)^{\alpha} [f(a)+f(b)] = \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} [f(a)+f(b)].
\end{aligned}$$

As a consequence, we derive that

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{1}{2}\right)^{\alpha} f(a+b) \leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} {}_aI_b^{(\alpha)} f(x) \leq \frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)} \big(f(a)+f(b)\big).$$

$\square$

Next, we propose an another form of the generalized Hermite–Hadamard-type inequality for the generalized subadditive functions in the frame of fractal space.

Theorem 2. *Suppose that the hypotheses mentioned in Theorem 1 are met. Then we have the following inequalities for the generalized subadditivity:*

$$\begin{aligned}
&\left(\frac{1}{4}\right)^{\alpha} \left[f\left(\frac{3a+b}{2}\right) + f\left(\frac{a+3b}{2}\right)\right] \leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} {}_aI_b^{(\alpha)} f(x) \\
&\leq \left(\frac{1}{2}\right)^{\alpha} \left[\frac{\Gamma(1+\alpha)}{a^{\alpha}} {}_0I_a^{(\alpha)} f(x) + \frac{\Gamma(1+\alpha)}{b^{\alpha}} {}_0I_b^{(\alpha)} f(x)\right] + \frac{2^{\alpha}\Gamma(1+\alpha)}{(a+b)^{\alpha}} {}_0I_{\frac{a+b}{2}}^{(\alpha)} f(x).
\end{aligned} \tag{3.5}$$

Proof. Taking advantage of Theorem 1 over the intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$, we deduce that

$$\begin{aligned}
\left(\frac{1}{4}\right)^{\alpha} f\left(\frac{3a+b}{2}\right) &\leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} {}_aI_{\frac{a+b}{2}}^{(\alpha)} f(x) \\
&\leq \left(\frac{1}{2}\right)^{\alpha} \frac{\Gamma(1+\alpha)}{a^{\alpha}} {}_0I_a^{(\alpha)} f(x) + \frac{\Gamma(1+\alpha)}{(a+b)^{\alpha}} {}_0I_{\frac{a+b}{2}}^{(\alpha)} f(x)
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
\left(\frac{1}{4}\right)^{\alpha} f\left(\frac{a+3b}{2}\right) &\leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} {}_{\frac{a+b}{2}}I_b^{(\alpha)} f(x) \\
&\leq \left(\frac{1}{2}\right)^{\alpha} \frac{\Gamma(1+\alpha)}{b^{\alpha}} {}_0I_b^{(\alpha)} f(x) + \frac{\Gamma(1+\alpha)}{(a+b)^{\alpha}} {}_0I_{\frac{a+b}{2}}^{(\alpha)} f(x).
\end{aligned} \tag{3.7}$$

Adding the inequalities (3.6) and (3.7), we derive that

$$\left(\frac{1}{4}\right)^{\alpha}\left[f\left(\frac{3a+b}{2}\right)+f\left(\frac{a+3b}{2}\right)\right] \leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_{a}I_{b}^{(\alpha)}f(x)$$
$$\leq \left(\frac{1}{2}\right)^{\alpha}\left[\frac{\Gamma(1+\alpha)}{a^{\alpha}}{}_{0}I_{a}^{(\alpha)}f(x)+\frac{\Gamma(1+\alpha)}{b^{\alpha}}{}_{0}I_{b}^{(\alpha)}f(x)\right]+\frac{2^{\alpha}\Gamma(1+\alpha)}{(a+b)^{\alpha}}{}_{0}I_{\frac{a+b}{2}}^{(\alpha)}f(x).$$

This closures the proof. □

On the basis of Theorem 2, if the function f is generalized starshaped, then we can acquire a refined version of the generalized Hermite–Hadamard-type inequalities.

Theorem 3. *Suppose that the hypotheses mentioned in Theorem 2 are met. If we attempt to take $f(tx) \leq t^{\alpha} f(x)$, then we have the following inequalities for the generalized subadditivity:*

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\leq \left(\frac{1}{2}\right)^{\alpha}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\\ &\leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_{a}I_{b}^{(\alpha)}f(x)\\ &\leq \frac{\Gamma^{2}(1+\alpha)}{\Gamma(1+2\alpha)}\left[\frac{f(a)+f(b)}{2^{\alpha}}+f\left(\frac{a+b}{2}\right)\right]\\ &\leq \frac{\Gamma^{2}(1+\alpha)}{\Gamma(1+2\alpha)}\left[f(a)+f(b)\right].\end{aligned} \tag{3.8}$$

Proof. Taking advantage of the generalized subadditivity of the function f, we acquire that

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\leq \left(\frac{1}{2}\right)^{\alpha} f(a+b)\\ &= \left(\frac{1}{2}\right)^{\alpha} f\left(\frac{3a+b}{4}+\frac{a+3b}{4}\right)\\ &\leq \left(\frac{1}{2}\right)^{\alpha}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right].\end{aligned} \tag{3.9}$$

Using $f(tx) \leq t^{\alpha} f(x)$, it yields that

$$\left(\frac{1}{2}\right)^{\alpha}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \leq \left(\frac{1}{4}\right)^{\alpha}\left[f\left(\frac{3a+b}{2}\right)+f\left(\frac{a+3b}{2}\right)\right]. \tag{3.10}$$

In accordance with Theorem 2, we know that

$$\left(\frac{1}{4}\right)^{\alpha}\left[f\left(\frac{3a+b}{2}\right)+f\left(\frac{a+3b}{2}\right)\right] \leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_{a}I_{b}^{(\alpha)}f(x)$$

$$\leq \left(\frac{1}{2}\right)^{\alpha}\left[\frac{\Gamma(1+\alpha)}{a^{\alpha}}{}_{0}I_{a}^{(\alpha)}f(x)+\frac{\Gamma(1+\alpha)}{b^{\alpha}}{}_{0}I_{b}^{(\alpha)}f(x)\right]+\frac{2^{\alpha}\Gamma(1+\alpha)}{(a+b)^{\alpha}}{}_{0}I_{\frac{a+b}{2}}^{(\alpha)}f(x). \tag{3.11}$$

As a consequence, in view of the inequalities (3.6) and (3.7), and in terms of $f(tx)\leq t^{\alpha}f(x)$ again, we deduce that

$$\begin{aligned}
&\left(\frac{1}{2}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{a^{\alpha}}{}_{0}I_{a}^{(\alpha)}f(x)+\frac{\Gamma(1+\alpha)}{(a+b)^{\alpha}}{}_{0}I_{\frac{a+b}{2}}^{(\alpha)}f(x)\\
&=\frac{\Gamma(1+\alpha)}{2^{\alpha}}\left[\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}f(at)(\mathrm{d}t)^{\alpha}+\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}f\left(\frac{a+b}{2}t\right)(\mathrm{d}t)^{\alpha}\right]\\
&\leq\frac{\Gamma(1+\alpha)}{2^{\alpha}}\left[f(a)+f\left(\frac{a+b}{2}\right)\right]\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}t^{\alpha}(\mathrm{d}t)^{\alpha}\\
&=\left(\frac{1}{2}\right)^{\alpha}\frac{\Gamma^{2}(1+\alpha)}{\Gamma(1+2\alpha)}\left[f(a)+f\left(\frac{a+b}{2}\right)\right]
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
&\left(\frac{1}{2}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{b^{\alpha}}{}_{0}I_{b}^{(\alpha)}f(x)+\frac{\Gamma(1+\alpha)}{(a+b)^{\alpha}}{}_{0}I_{\frac{a+b}{2}}^{(\alpha)}f(x)\\
&=\frac{\Gamma(1+\alpha)}{2^{\alpha}}\left[\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}f(bt)(\mathrm{d}t)^{\alpha}+\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}f\left(\frac{a+b}{2}t\right)(\mathrm{d}t)^{\alpha}\right]\\
&\leq\left(\frac{1}{2}\right)^{\alpha}\frac{\Gamma^{2}(1+\alpha)}{\Gamma(1+2\alpha)}\left[f\left(\frac{a+b}{2}\right)+f(b)\right].
\end{aligned} \tag{3.13}$$

Adding the inequalities (3.12) and (3.13), we derive that

$$\begin{aligned}
&\left(\frac{1}{2}\right)^{\alpha}\left[\frac{\Gamma(1+\alpha)}{a^{\alpha}}{}_{0}I_{a}^{(\alpha)}f(x)+\frac{\Gamma(1+\alpha)}{b^{\alpha}}{}_{0}I_{b}^{(\alpha)}f(x)\right]+\frac{2^{\alpha}\Gamma(1+\alpha)}{(a+b)^{\alpha}}{}_{0}I_{\frac{a+b}{2}}^{(\alpha)}f(x)\\
&\leq\frac{\Gamma^{2}(1+\alpha)}{\Gamma(1+2\alpha)}\left[\frac{f(a)+f(b)}{2^{\alpha}}+f\left(\frac{a+b}{2}\right)\right].
\end{aligned} \tag{3.14}$$

By virtue of the generalized subadditivity as well as $f(tx)\leq t^{\alpha}f(x)$, we acquire that

$$\begin{aligned}
\frac{\Gamma^{2}(1+\alpha)}{\Gamma(1+2\alpha)}\left[\frac{f(a)+f(b)}{2^{\alpha}}+f\left(\frac{a+b}{2}\right)\right]&\leq\frac{\Gamma^{2}(1+\alpha)}{\Gamma(1+2\alpha)}\left[\frac{f(a)+f(b)}{2^{\alpha}}+\frac{f(a)+f(b)}{2^{\alpha}}\right]\\
&=\frac{\Gamma^{2}(1+\alpha)}{\Gamma(1+2\alpha)}\left[f(a)+f(b)\right].
\end{aligned} \tag{3.15}$$

Combining the inequalities (3.9), (3.10), (3.11), (3.14) and (3.15), we obtain the desired outcomes (3.8). This fulfills the proof. □

Corollary 2. *If one attempts to take $\alpha = 1$ in Theorem 3, then one acquires the following Hermite–Hadamard-type inequality for convex functions:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &\le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \\ &\le \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right] \le \frac{f(a)+f(b)}{2}, \end{aligned}$$

which is proposed by Tseng et al. in [19].

Finally, we present the Pachpatte-type integral inequalities in the settings of fractal sets.

Theorem 4. *Suppose that $f,g\colon I=[0,\infty)\to\mathbb{R}_+^{\alpha}$ are both nonnegative generalized subadditive functions. If $a<b$ $(a,b\in I^{\circ})$ and $f\in C_{\alpha}[0,\infty)$, then the following inequalities in the frame of fractal space hold*

$$\begin{aligned} &\left(\frac{2}{b-a}\right)^{\alpha} {}_aI_b^{(\alpha)} f(x)g(x) \\ &\quad\le \left(\frac{2}{a}\right)^{\alpha} {}_0I_a^{(\alpha)} f(x)g(x) + \left(\frac{2}{b}\right)^{\alpha} {}_0I_b^{(\alpha)} f(x)g(x) \\ &\qquad + \frac{2^{\alpha}}{\Gamma(1+\alpha)}\int_0^1 f(ta)g((1-t)b)(\mathrm{d}t)^{\alpha} + \frac{2^{\alpha}}{\Gamma(1+\alpha)}\int_0^1 f((1-t)a)g(tb)(\mathrm{d}t)^{\alpha} \\ &\quad\le \left(\frac{1}{a}\right)^{\alpha} {}_0I_a^{(\alpha)}\left[f^2(x)+g^2(x)\right] + \left(\frac{1}{b}\right)^{\alpha} {}_0I_b^{(\alpha)}\left[f^2(x)+g^2(x)\right] \\ &\qquad + \frac{2^{\alpha}}{\Gamma(1+\alpha)}\int_0^1 f(ta)f((1-t)b)(\mathrm{d}t)^{\alpha} + \frac{2^{\alpha}}{\Gamma(1+\alpha)}\int_0^1 g(ta)g((1-t)b)(\mathrm{d}t)^{\alpha}. \end{aligned}$$

Proof. Taking advantage of the generalized subadditivity of the functions f and g, we can figure out that

$$f(ta+(1-t)b) \le f(ta)+f((1-t)b) \tag{3.16}$$

and

$$g(ta+(1-t)b) \le g(ta)+g((1-t)b). \tag{3.17}$$

Multiplying the inequalities (3.16) and (3.17), as well as noticing that all these terms are nonnegative, we obtain that

$$\begin{aligned} &f(ta+(1-t)a)g(ta+(1-t)a) \\ &\quad\le f(ta)g(ta)+f(ta)g((1-t)b)+f((1-t)b)g(ta)+f((1-t)b)g((1-t)b) \\ &\quad\le \left(\frac{1}{2}\right)^{\alpha}\left[\big(f(ta)+f((1-t)b)\big)^2+\big(g(ta)+g((1-t)b)\big)^2\right]. \end{aligned} \tag{3.18}$$

Integrating the resulting inequalities (3.18) regarding the variate t over the interval $[0,1]$ in the fractal sense, it yields that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)(\mathrm{d}t)^\alpha \\
&\le \frac{1}{\Gamma(1+\alpha)}\int_0^1 f(ta)g(ta)(\mathrm{d}t)^\alpha+\frac{1}{\Gamma(1+\alpha)}\int_0^1 f(ta)g((1-t)b)(\mathrm{d}t)^\alpha \\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 f((1-t)b)g(ta)(\mathrm{d}t)^\alpha+\frac{1}{\Gamma(1+\alpha)}\int_0^1 f((1-t)b)g((1-t)b)(\mathrm{d}t)^\alpha \\
&\le \left(\frac{1}{2}\right)^\alpha\frac{1}{\Gamma(1+\alpha)}\int_0^1\left[f^2(ta)+g^2(ta)\right](\mathrm{d}t)^\alpha \\
&\quad+\left(\frac{1}{2}\right)^\alpha\frac{1}{\Gamma(1+\alpha)}\int_0^1\left[f^2((1-t)b)+g^2((1-t)b)\right](\mathrm{d}t)^\alpha \\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 f(ta)f((1-t)b)(\mathrm{d}t)^\alpha+\frac{1}{\Gamma(1+\alpha)}\int_0^1 g(ta)g((1-t)b)(\mathrm{d}t)^\alpha.
\end{aligned} \tag{3.19}$$

As a consequence, by making appropriate substitutions for the inequalities (3.19), we deduce the desired results. This fulfills the proof. □

Corollary 3. *If one attempts to take $f(tx)\le t^\alpha f(x)$ in Theorem 4, then one acquires the following inequalities for the generalized subadditive functions:*

$$\begin{aligned}
\left(\frac{1}{b-a}\right)^\alpha {}_aI_b^{(\alpha)}f(x)g(x) &\le \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}R(a,b)+\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]S(a,b) \\
&\le \left(\frac{1}{2}\right)^\alpha\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left[\left(f(a)-f(b)\right)^2+\left(g(a)-g(b)\right)^2\right] \\
&\quad+\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[f(a)f(b)+g(a)g(b)\right],
\end{aligned}$$

where

$$R(a,b)=f(a)g(a)+f(b)g(b)$$

and

$$S(a,b)=f(a)g(b)+f(b)g(a).$$

Proof. Taking advantage of $f(tx)\le t^\alpha f(x)$ and the inequalities (3.19), it yields that

$$\begin{aligned}
&\left(\frac{1}{b-a}\right)^\alpha {}_aI_b^{(\alpha)}f(x)g(x) \\
&\le f(a)g(a)\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{2\alpha}(\mathrm{d}t)^\alpha+f(a)g(b)\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^\alpha(1-t)^\alpha(\mathrm{d}t)^\alpha
\end{aligned}$$

$$+f(b)g(a)\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{\alpha}(1-t)^{\alpha}(\mathrm{d}t)^{\alpha}+f(b)g(b)\frac{1}{\Gamma(1+\alpha)}\int_0^1(1-t)^{2\alpha}(\mathrm{d}t)^{\alpha}$$
$$\leq\left(\frac{1}{2}\right)^{\alpha}\left[\left(f^2(a)+g^2(a)\right)\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{2\alpha}(\mathrm{d}t)^{\alpha}\right.$$
$$\left.+\left(f^2(b)+g^2(b)\right)\frac{1}{\Gamma(1+\alpha)}\int_0^1(1-t)^{2\alpha}(\mathrm{d}t)^{\alpha}\right]$$
$$+\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{\alpha}(1-t)^{\alpha}(\mathrm{d}t)^{\alpha}\left[f(a)f(b)+g(a)g(b)\right].$$

Calculating the following integrals, we derive that

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{2\alpha}(\mathrm{d}t)^{\alpha}=\frac{1}{\Gamma(1+\alpha)}\int_0^1(1-t)^{2\alpha}(\mathrm{d}t)^{\alpha}=\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)},$$

and

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{\alpha}(1-t)^{\alpha}(\mathrm{d}t)^{\alpha}=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}.$$

This terminates the proof. □

Theorem 5. *Suppose that $f,g\colon I=[0,\infty)\to\mathbb{R}_+^{\alpha}$ are both nonnegative generalized subadditive functions. If $a<b$ $(a,b\in I^{\circ})$ and $f\in C_{\alpha}[0,\infty)$, then the following inequalities in the fractal sense hold*

$$\left(\frac{1}{2}\right)^{\alpha}\frac{1^{\alpha}}{\Gamma(1+\alpha)}f(a+b)g(a+b)$$
$$\leq\left(\frac{1}{b-a}\right)^{\alpha}{}_aI_b^{(\alpha)}f(x)g(x)$$
$$+\frac{1}{\Gamma(1+\alpha)}\int_0^1\left[f(ta)g((1-t)a)+f(tb)g((1-t)b)\right](\mathrm{d}t)^{\alpha}$$
$$+\frac{1}{\Gamma(1+\alpha)}\int_0^1\left[f(ta)g(tb)+f(tb)g(ta)\right](\mathrm{d}t)^{\alpha} \tag{3.20}$$

and

$$\left(\frac{1}{b-a}\right)^{\alpha}{}_aI_b^{(\alpha)}f(x)g(x)$$
$$\leq\left(\frac{1}{a}\right)^{\alpha}{}_0I_a^{(\alpha)}f(x)g(x)+\left(\frac{1}{b}\right)^{\alpha}{}_0I_b^{(\alpha)}f(x)g(x)$$
$$+\frac{1}{\Gamma(1+\alpha)}\int_0^1 f(ta)g((1-t)b)(\mathrm{d}t)^{\alpha}$$
$$+\frac{1}{\Gamma(1+\alpha)}\int_0^1 f(tb)g((1-t)a)(\mathrm{d}t)^{\alpha}. \tag{3.21}$$

Proof. Taking advantage of the generalized subadditivity of the functions f and g, we can figure out that

$$f(a+b) \leq f(ta+(1-t)b)+f((1-t)a+tb),$$

and

$$g(a+b) \leq g(ta+(1-t)b)+g((1-t)a+tb).$$

Multiplying the above inequalities, as well as noticing that all these terms are non-negative, we attain that

$$\begin{aligned}
&f(a+b)g(a+b)\\
&\leq f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)\\
&\quad+[f(ta)+f((1-t)b)]\,[g((1-t)a)+g(tb)]\\
&\quad+[f((1-t)a)+f(tb)]\,[g(ta)+g((1-t)b)]\\
&= f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)\\
&\quad+f(ta)g((1-t)a)+f(ta)g(tb)+f((1-t)b)g((1-t)a)+f((1-t)b)g(tb)\\
&\quad+f((1-t)a)g(ta)+f((1-t)a)g((1-t)b)+f(tb)g(ta)+f(tb)g((1-t)b).
\end{aligned} \tag{3.22}$$

Integrating the resulting inequality (3.22) regarding the variate t over the interval $[0,1]$ in the fractal sense, it yields that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^1 f(a+b)g(a+b)(\mathrm{d}t)^\alpha\\
&\leq \frac{1}{\Gamma(1+\alpha)}\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)(\mathrm{d}t)^\alpha\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 f((1-t)a+tb)g((1-t)a+tb)(\mathrm{d}t)^\alpha\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 \big[f(ta)g((1-t)a)+f(tb)g((1-t)b)\big](\mathrm{d}t)^\alpha\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 \big[f((1-t)a)g(ta)+f((1-t)b)g(tb)\big](\mathrm{d}t)^\alpha\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 \big[f(ta)g(tb)+f(tb)g(ta)\big](\mathrm{d}t)^\alpha\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 \big[f((1-t)a)g((1-t)b)+f((1-t)b)g((1-t)a)\big](\mathrm{d}t)^\alpha.
\end{aligned} \tag{3.23}$$

By virtue of appropriate substitutions for the inequality (3.23), we can deduce inequality (3.20).

On the other hand, taking advantage of the subadditivity of functions f and g again, it follows that

$$\begin{aligned}
&f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)\\
&\leq f(ta)g(ta)+f(ta)g((1-t)b)+f((1-t)b)g(ta)+f((1-t)b)g((1-t)b)\\
&\quad+f((1-t)a)g((1-t)a)+f((1-t)a)g(tb)+f(tb)g((1-t)a)+f(tb)g(tb).
\end{aligned} \tag{3.24}$$

Similarly, integrating the inequalities (3.24) with regard to the variate t over the interval $[0,1]$ in the settings of fractal sets, as well as making appropriate substitutions, we can acquire the desired inequality (3.21). This fulfills the proof. □

Corollary 4. *If one attempts to take $f(tx)\leq t^{\alpha}f(x)$ in Theorem 5, then one acquires the following inequalities for the generalized subadditive functions:*

$$\begin{aligned}
&\frac{2^{\alpha}}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\leq\left(\frac{1}{2}\right)^{\alpha}\frac{1^{\alpha}}{\Gamma(1+\alpha)}f(a+b)g(a+b)\\
&\leq\left(\frac{1}{b-a}\right)^{\alpha}{}_aI_b^{(\alpha)}f(x)g(x)+\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]R(a,b)+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}S(a,b)
\end{aligned}$$

and

$$\left(\frac{1}{b-a}\right)^{\alpha}{}_aI_b^{(\alpha)}f(x)g(x)\leq\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}R(a,b)+\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]S(a,b),$$

where $R(a,b)$ and $S(a,b)$ are same defined as Corollary 3.

Proof. Taking advantage of the inequality (3.23) and $f(tx)\leq t^{\alpha}f(x)$, we can figure out that

$$\begin{aligned}
&\left(\frac{1}{2}\right)^{\alpha}\frac{1^{\alpha}}{\Gamma(1+\alpha)}f(a+b)g(a+b)\\
&\leq\left(\frac{1}{b-a}\right)^{\alpha}{}_aI_b^{(\alpha)}f(x)g(x)\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{\alpha}(1-t)^{\alpha}(\mathrm{d}t)^{\alpha}\left[f(a)g(a)+f(b)g(b)\right]\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{2\alpha}(\mathrm{d}t)^{\alpha}\left[f(a)g(b)+f(b)g(a)\right]\\
&=\left(\frac{1}{b-a}\right)^{\alpha}{}_aI_b^{(\alpha)}f(x)g(x)\\
&\quad+\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]R(a,b)+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}S(a,b).
\end{aligned}$$

As a consequence, we can acquire the first desired outcome. Similarly, we can deduce another inequality by taking advantage of the same procedure. □

4. Conclusions

A class of generalized subadditive functions is first presented in the current article. With the aid of the proposed functions and generalized starshaped functions, several Hermite–Hadamard- and Pachpatte-type fractal integral inequalities with different bounds are derived as well, respectively. We believe that the present article can provide ideas for interested researchers to develop the fractal inequalities similarly using the fractal-fractional integral operators, see for example the published article [22].

Acknowledgments

The authors would like to thank the reviewer for his/her valuable comments and suggestions.

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