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ON (M,P)-FUNCTIONS WITH SOME FEATURES AND NEW INEQUALITIES

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Abstract. In this study, we introduce a generalization of P-function, called (M,P)-functions, via weighted mean functions given by İşcan. Then, we prove some new inequalities for (M,P)-functions. Also, we give new properties for (M,P)-functions and present some results for the special cases of M.

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1. INTRODUCTION

Historically, pedagogically and logically, the study of convex functions begins in the context of real-valued functions of real variable. Convex functions have important applications and at same time they give rise to a variety of generalizations. The geometric definition of a convex function specifies the following. A real-valued function is said to be convex if the line segment connecting two points of its graph lies above the graph. Equivalently, a real-valued function is convex if its epigraph (the set of points on or above its graph) is convex. A convex function $f\colon [a,b]\subset\mathbb{R}\to\mathbb{R}$ is bounded and its restriction to (a,b) is continuous. Simple examples of convex functions are $f(x)=x^2$ on $(-\infty,\infty)$, $g(x)=\sin x$ on $[-\pi,0]$, $k(x)=|x|$ on $(-\infty,\infty)$. The analytic definition of a convex function is as follows.

Definition 1. The function $f\colon [a,b]\subset\mathbb{R}\to\mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y) \tag{1.1}$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

Definition 2 ([5])**.** A function $f\colon I\subseteq\mathbb{R}\to\mathbb{R}$ is P function or that f belongs to the class of $P(I)$, if it is nonnegative and, for all $x,y\in I$ and $\lambda\in[0,1]$, satisfies the following inequality;

$$f\big(\lambda x+(1-\lambda)y\big)\leq f(x)+f(y). \tag{1.2}$$

Convex functions play an important role in many areas of mathematics. They are especially important in the study of optimization problems where they are distinguished by a number of convenient properties. The generalized condition of convexity, i.e. MN-convexity with respect to arbitrary means M and N, was proposed in 1933 by Aumann [4]. Recently many authors have dealt with these generalizations. In particular, Niculescu [14] compared MN-convexity with relative convexity. In [3], Anderson et al. studied certain generalizations of these notions for a positive-valued function of a positive variable as follows:

Definition 3. A function $M\colon (0,\infty)\times(0,\infty)\to(0,\infty)$ is called a mean function if the following conditions are satisfied.

(M1) $M(u,v)=M(v,u)$,
(M2) $M(u,u)=u$,
(M3) $u<M(u,v)<v$ whenever $u<v$,
(M4) $M(\lambda u,\lambda v)=\lambda M(u,v)$ for all $\lambda>0$.

Example 1. For $u,v\in(0,\infty)$

$$M(u,v)=A(u,v)=A=\frac{u+v}{2}$$

is the Arithmetic Mean,

$$M(u,v)=G(u,v)=G=\sqrt{uv}$$

is the Geometric Mean,

$$M(u,v)=H(u,v)=H=A^{-1}(u^{-1},v^{-1})=\frac{2uv}{u+v}$$

is the Harmonic Mean,

$$M(u,v)=L(u,v)=L=\begin{cases}\frac{u-v}{\ln u-\ln v} & u\neq v\\ u & u=v\end{cases}$$

is the Logarithmic Mean,

$$M(u,v)=I(u,v)=I=\begin{cases}\frac{1}{e}\left(\frac{u^u}{v^v}\right)^{\frac{1}{u-v}} & u\neq v\\ u & u=v\end{cases}$$

is the Identric Mean,

$$M(u,v)=M_p(u,v)=M_p=\begin{cases}A^{1/p}(u^p,v^p)=\left(\frac{u^p+v^p}{2}\right)^{1/p} & p\in\mathbb{R}\setminus\{0\}\\ G(u,v)=\sqrt{uv} & p=0\end{cases}$$

is the p-Power Mean, In particular, we have the following inequality

$$M_{-1}=H\leq M_0=G\leq L\leq I\leq A=M_1.$$

In [3], Anderson et al. gave a new definition of MN-convex functions called MN-midpoint convex with the help of M and N weighted mean as follows.

Definition 4. Let M and N be two means defined on the intervals $I \subset (0,\infty)$ and $J \subset (0,\infty)$ respectively, a function $f: I \to J$ is called MN-midpoint convex if it satisfies

$$f(M(u,v)) \leq N(f(u),f(v))$$

for all $u,v \in I$.

In [9], İşcan gave a new definition of function called weighted mean function as follows.

Definition 5. A function $M: (0,\infty)\times(0,\infty)\times[0,1] \to (0,\infty)$ is called a weighted mean function if

(WM1) $M(u,v,\lambda) = M(v,u,1-\lambda)$.

(WM2) $M(u,u,\lambda) = u$.

(WM3) $u < M(u,v,\lambda) < v$ whenever $u < v$ and $\lambda \in (0,1)$. Also $\{M(u,v,0), M(u,v,1)\} = \{u,v\}$.

(WM4) $M(\alpha u, \alpha v, \lambda) = \alpha M(u,v,\lambda)$ for all $\alpha > 0$.

(WM5) Let $\lambda \in [0,1]$ be fixed. Then $M(u,v,\lambda) \leq M(w,v,\lambda)$ whenever $u \leq w$ and $M(u,v,\lambda) \leq M(u,\omega,\lambda)$ whenever $v \leq \omega$.

(WM6) Let $u,v \in (0,\infty)$ be fixed and $u \neq v$. Then $M(u,v,.)$ is a strictly monotone and continuous function on $[0,1]$.

(WM7) $M(M(u,v,\lambda), M(z,w,\lambda), s) = M(M(u,z,s), M(v,w,s), \lambda)$ for all $u,v,z,w \in (0,\infty)$ and $s,\lambda \in [0,1]$.

(WM8) $M(u,v,s\lambda_1 + (1-s)\lambda_2) = M(M(u,v,\lambda_1), M(u,v,\lambda_2), s)$ for all $u,v \in (0,\infty)$ and $s,\lambda_1,\lambda_2 \in [0,1]$.

Remark 1 ([9]). According to the above definition every weighted mean function is a mean function with $\lambda = 1/2$. Also, By (WM6) we can say that for each $x \in [u,v] \subseteq (0,\infty)$ there exists a $\lambda \in [0,1]$ such that $x = M(u,v,\lambda)$. Morever;

i) If $M(u,v,.)$ is a strictly increasing, then $M(u,v,0) = u$ and $M(u,v,1) = v$ whenever $u < v$ (i.e. $M(u,v,\lambda)$ is in the positive direction)

ii) If $M(u,v,.)$ is a strictly deccreasing, then $M(u,v,0) = v$ and $M(u,v,1) = u$ whenever $u < v$ (i.e. $M(u,v,\lambda)$ is in the negative direction) and $M(u,v,.)([0,1]) = [\min\{u,v\}, \max\{u,v\}]$.

Throughout this paper, we will assume that different weighted means have the same direction unless otherwise stated.

Example 2 ([9]).

$$M(u,v,\lambda) = A(u,v,\lambda) = A_\lambda = (1-\lambda)u + \lambda v$$

is the Weighted Arithmetic Mean,

$$M(u,v,\lambda) = G(u,v,\lambda) = G_\lambda = u^{1-\lambda}v^\lambda$$

is the Weighted Geometric Mean,

$$M(u,v,\lambda) = H(u,v,\lambda) = H_\lambda = A^{-1}(u^{-1},v^{-1},\lambda) = \frac{uv}{\lambda u + (1-\lambda)v}$$

is the Weighted Harmonic Mean,

$$M(u,v,\lambda) = M_p(u,v,\lambda) = M_{p,\lambda} = \begin{cases} A^{1/p}(u^p,v^p,\lambda) = ((1-\lambda)x^p + \lambda y^p)^{1/p} & p \in \mathbb{R}\setminus\{0\} \\ G(u,v,\lambda) = u^{1-\lambda}v^\lambda & p = 0 \end{cases}$$

is the p-Power Mean. In particular, we have the following inequality

$$M_{-1,\lambda} = H_\lambda \leq M_{0,\lambda} = G_\lambda \leq M_{1,\lambda} = A_\lambda \leq M_{p,\lambda}$$

for all $x,y \in (0,\infty), t \in [0,1]$ and $p \geq 1$.

İşcan [9] proved the equalities in the following proposition.

Proposition 1. *If* $M\colon (0,\infty)\times(0,\infty)\times[0,1] \to (0,\infty)$ *is a weighted mean function, then the following identities hold:*

$$M\left(M\left(a,M(a,b,s),\lambda\right),M\left(b,M(a,b,s),\lambda\right),s\right) = M(a,b,s), \tag{1.3}$$

$$M\left(M(a,b,\lambda),M(b,a,\lambda),1/2\right) = M(a,b,1/2). \tag{1.4}$$

Many different definitions of convexity have been made by mathematicians until now. One of these definition was given by İşcan as follows.

Definition 6 ([9])**.** Let M and N be two weighted means defined on the intervals $I \subseteq (0,\infty)$ and $J \subseteq (0,\infty)$ respectively, a function $f\colon I \to J$ is called MN-convex (concave) if it satisfies

$$f\left(M(u,v,\lambda)\right) \leq (\geq) N\left(f(u),f(v),\lambda\right)$$

for all $u,v \in I$ and $\lambda \in [0,1]$.

We note that by considering the special cases of M and N, we obtain several different results. For some recent results related to convex functions, MN-convexity and some kinds of convexity obtained by using weighted means, see [1,4,6,7,11–14,16].

Definition 7 ([9])**.** Let M and N be two weighted means defined on the intervals $[u,v] \subseteq (0,\infty)$ and $J \subseteq (0,\infty)$ respectively and $f\colon [u,v] \to J$ be a function. We say that f is symmetric with respect to $M(u,v,1/2)$, if it satisfies

$$f\left(M(u,v,\lambda)\right) = f\left(M(u,v,1-\lambda)\right)$$

for all $\lambda \in [0,1]$.

Definition 8 ([10]). A function f defined on $[a,b]$ is said to be of bounded variation on $[a,b]$ if its total variation $Var(f)$ on $[a,b]$ is finite, where

$$Var(f) = \sup \sum_{j=1}^{n} \left| f(t_j) - f(t_{j-1}) \right|, \tag{1.5}$$

the supremum being taken over all partitions

$$a = t_0 < t_1 < \ldots < t_n = b \tag{1.6}$$

of the interval $[a,b]$; here, $n \in \mathbb{N}$ is arbitrary and so is the choice of values $t_1,\ldots,t_{n-1}$ in $[a,b]$ which, however, must satisfy (1.6).

Obviously, all functions of bounded variation on $[a,b]$ form a vector space. A norm on this space is given by

$$\|f\| = |f(a)| + Var(f). \tag{1.7}$$

The normed space thus defined is denoted by $BV[a,b]$, where BV suggest "bounded variation".

In 1905, E. Almansi [2] proved the following theorem.

Theorem 1. *Let f and f' be continuous functions on the interval (a,b) and let $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. Then*

$$\int_a^b [f(x)]^2 dx \leq \left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 dx. \tag{1.8}$$

The aim of this paper is to give a new definition called (M,P)-function of P-functions that belongs to the class of $P(I)$ via the weighted means, obtain new inequalities using (M,P)-functions and present some properties of (M,P)-functions.

2. Main Results

Definition 9. Let $I \subset (0,\infty)$ be an interval, let $M\colon I \times I \times [0,1] \to (0,\infty)$ be a weighted mean function and let $f\colon I \to \mathbb{R}$ be a function. Then f is said to be an (M,P)-function if the inequality

$$f\big(M(x,y,t)\big) \leq f(x) + f(y)$$

holds for all $x,y \in I$ and $t \in [0,1]$.

Remark 2. If we choose M as the weighted arithmetic mean in Definition 9, we obtain the class of P-function.

Remark 3. If $f\colon I \to \mathbb{R}$ is an (M,P)-function on I,

$$f(x) \geq 0 \quad \forall x \in I.$$

Theorem 2. *Every MN-convex function is an (M,P)-function.*

Proof. Let $M\colon I\times I\times[0,1]\to(0,\infty)$, $N : J\times J\times[0,1]\to(0,\infty)$ be two weighted mean functions on intervals $J\subset(0,\infty)$, $I\subset(0,\infty)$ respectively and $f\colon I\to J$ be a MN-convex function. Then we can write

$$f\big(M(x,y,t)\big)\le N\left(f(x),f(y),t\right) \tag{2.1}$$

for all $x,y\in I$ and all $t\in[0,1]$.

On the other hand, we can write

$$f(x)\le f(x)+f(y)$$

and

$$f(y)\le f(x)+f(y).$$

Then, we obtain,

$$N\big(f(x),f(y),t\big)\le N\big(f(x)+f(y),f(x)+f(y),t\big)=f(x)+f(y). \tag{2.2}$$

So, using (2.1) and (2.2), the proof is completed. □

Theorem 3 (Hermite-Hadamard's inequalities for (M,P)-functions)**.** *Let M be a weighted mean function defined on the interval $I\subset(0,\infty)$ and $f\colon I\to J$ is an (M,P)-function. If the following integral exists, then we have the following inequalities for (M,P)-functions*

$$f\big(M(x,y,1/2)\big)\le 2\int_0^1 f\big(M(x,y,t)\big)dt\le 2[f(x)+f(y)] \tag{2.3}$$

for all $x,y\in I$ with $x<y$.

Proof. Since f is (M,P)-function, using (WM1) and (1.4) equality, we have

$$\begin{aligned} f\big(M(x,y,1/2)\big) &= f\Big(M\big(M(x,y,t),M(x,y,1-t),1/2\big)\Big) \\ &\le f\big(M(x,y,t)\big)+f\big(M(x,y,1-t)\big) \end{aligned} \tag{2.4}$$

for all $t\in[0,1]$. Integrating both sides of (2.4) inequality respect to t over [0,1], we obtain

$$\begin{aligned} \int_0^1 f\big(M(x,y,1/2)\big)dt &= f\big(M(x,y,1/2)\big) \\ &\le \int_0^1 f\big(M(x,y,t)\big)dt+\int_0^1 f\big(M(x,y,1-t)\big)dt \\ &= 2\int_0^1 f\big(M(x,y,t)\big)dt. \end{aligned} \tag{2.5}$$

Otherwise, we can write

$$f\big(M(x,y,t)\big)\le f(x)+f(y) \tag{2.6}$$

for all $t \in [0,1]$. Integrating both sides of (2.6) inequality respect to t over [0,1], we obtain

$$\int_0^1 f\big(M(x,y,t)\big)dt \leq f(x)+f(y). \tag{2.7}$$

Then, using (2.5) and (2.7) inequalities, we get the desired result. □

Remark 4. Let $I \subset (0,\infty)$ and $f\colon I \to \mathbb{R}$. If f is an (M,P)-function and $M = A$ (A is the weighted arithmetic mean), then using (2.3), we have the following Hermite-Hadamard's inequalities (see [5], Theorem 3.1).

$$\begin{aligned} f\big(A(x,y,1/2)\big) = f\left(\frac{x+y}{2}\right) &\leq 2\int_0^1 f\big(A(x,y,t)\big)dt \\ &= 2\int_0^1 f\big((1-t)x+ty\big)dt \\ &= \frac{2}{y-x}\int_x^y f(u)du \\ &\leq 2[f(x)+f(y)]. \end{aligned}$$

Remark 5. Let $I \subset (0,\infty)$ and $f\colon I \to \mathbb{R}$. If f is an (M,P)-function and $M = G$ (G is the weighted geometric mean), then using (2.3), we have the following Hermite-Hadamard's inequalities (see [15], Theorem 2.2, Corollary 2.2, for $h(t)$=1).

$$\begin{aligned} f\big(G(x,y,1/2)\big) = f\left(\sqrt{xy}\right) &\leq 2\int_0^1 f\big(G(x,y,t)\big)dt \\ &= 2\int_0^1 f\big(x^{1-t}y^t\big)dt \\ &= \frac{2}{\ln y - \ln x}\int_x^y \frac{f(u)}{u}du \\ &\leq 2[f(x)+f(y)]. \end{aligned}$$

Remark 6. Let $I \subset (0,\infty)$ and $f\colon I \to \mathbb{R}$. If f is an (M,P)-function and $M = H$ (H is the weighted harmonic mean), then using (2.3), we have the following Hermite-Hadamard's inequalities (see [8], Theorem 4).

$$\begin{aligned} f\big(H(x,y,1/2)\big) = f\left(\frac{2xy}{x+y}\right) &\leq 2\int_0^1 f\big(H(x,y,t)\big)dt \\ &= 2\int_0^1 f\left(\frac{xy}{tx+(1-t)y}\right)dt \\ &= \frac{2xy}{y-x}\int_x^y \frac{f(u)}{u^2}du \\ &\leq 2[f(x)+f(y)]. \end{aligned}$$

Theorem 4. *If $f\colon [a,b] \subset (0,\infty) \to \mathbb{R}$ is an (M,P)-function, f is bounded on $[a,b]$.*

Proof. Since f is an (M,P)-function, $f(x) \geq 0$ respect to Remark 3 for all $x \in [a,b]$. Then, f is a function bounded below. Also, we can write $x = M(a,b,t)$ for $\forall x \in [a,b]$ and $\exists t \in [0,1]$. Then, we get

$$f(x) = f\big(M(a,b,t)\big) \leq f(a) + f(b) = k$$

and f is a function bounded above. Consequently, f is a bounded function. □

Theorem 5. *Let M be weighted mean defined on the interval $I \subset (0,\infty)$. If $f\colon I \to \mathbb{R}$ is an (M,P)-function and $\alpha > 0$, then αf is an (M,P)-function.*

Proof. Since f is an (M,P)-function, we have

$$\begin{aligned} \alpha f\big(M(x,y,t)\big) &\leq \alpha\big(f(x) + f(y)\big) \\ &= \alpha f(x) + \alpha f(y). \end{aligned}$$

This shows that αf is an (M,P)-function. So, the proof of theorem is completed. □

Theorem 6. *Let M be weighted mean function defined on the interval $I \subset (0,\infty)$. If $f_\alpha : I \to \mathbb{R}$ be an arbitrary family of (M,P)-functions and let $f(x) = \sup_\alpha f_\alpha(x)$. If $K = \{u \in I : f(u) < \infty\}$ is nonempty, then K is an interval and f is an (M,P)-function on K.*

Proof. Let $t \in [0,1]$ and $x, y \in K$ be arbitrary. Also, since f_α is an (M,P)-function, f_α is bounded. Then

$$\begin{aligned} f\big(M(x,y,t)\big) &= \sup_\alpha f_\alpha\big(M(x,y,t)\big) \\ &\leq \sup_\alpha \left(f_\alpha(x) + f_\alpha(y)\right) \\ &\leq \sup_\alpha f_\alpha(x) + \sup_\alpha f_\alpha(y) \\ &= f(x) + f(y) \\ &< \infty. \end{aligned}$$

This shows simultaneously that K is an interval, since it contains every point between any two of its points and that f is an (M,P)-function on K. The proof of the theorem is completed. □

Theorem 7. *Let M be weighted mean function defined on the interval $[x,y] \subseteq (0,\infty)$. If function $f\colon [x,y] \to \mathbb{R}$ is an (M,P)-function and symmetric with respect to $M(x,y,1/2)$, then we have*

$$f\big(M(x,y,1/2)\big) \leq 2f(u) \leq 2\big[f(x) + f(y)\big] \tag{2.8}$$

for all $u \in [x,y]$.

Proof. Let $u \in [x,y]$ be arbitrary point. Then there exist a $t \in [0,1]$ such that $u = M(x,y,t)$. Since $f\colon [x,y] \to J$ is an (M,P)-function and symmetric with respect to $M(x,y,1/2)$, by using equality (1.4) we have

$$f\big(M(x,y,1/2)\big) = f\Big(M\big(M(x,y,t), M(x,y,1-t), 1/2\big)\Big)$$

$$
\begin{aligned}
&\leq f\big(M(x,y,t)\big)+f\big(M(x,y,1-t)\big)\\
&= f\big(M(x,y,t)\big)+f\big(M(x,y,t)\big)\\
&= 2f(u).
\end{aligned}
$$

Thus, we obtain the left-hand side of inequality (2.8). Secondly, since f is an (M,P)-function and $(WM5)$ with (1.4), we get

$$
\begin{aligned}
2f(u) &= f\big(M(x,y,t)\big)+f\big(M(x,y,t)\big)\\
&\leq f(x)+f(y)+f(x)+f(y)\\
&= 2f(x)+2f(y)\\
&= 2\left[f(x)+f(y)\right].
\end{aligned}
$$

So, the proof of the theorem is completed. □

Theorem 8. *Let M be weighted mean function defined on the interval $I\subset(0,\infty)$. If the functions $f,g\colon \;\to\mathbb{R}$ are (M,P)-functions, then $f+g$ is also an (M,P)-function.*

Proof. Since f and g are (M,P)-functions, we have

$$f\big(M(x,y,t)\big)\leq f(x)+f(y)$$

and

$$g\big(M(x,y,t)\big)\leq g(x)+g(y)$$

for all $x,y\in I$ and $t\in[0,1]$. Then we can write

$$
\begin{aligned}
(f+g)\big(M(x,y,t)\big) &= f\big(M(x,y,t)\big)+g\big(M(x,y,t)\big)\\
&\leq f(x)+f(y)+g(x)+g(y)\\
&= f(x)+g(x)+f(y)+g(y)\\
&= (f+g)(x)+(f+g)(y).
\end{aligned}
$$

So, this completes the proof. □

Theorem 9. *Let $0<a<b$ and $M\colon [a,b]\times[a,b]\times[0,1]\to(0,\infty)$ be a weighted mean function defined on $[a,b]$, $f\colon [a,b]\to(0,\infty)$, f and f' be continuous functions on (a,b) with $f(a)=f(b)$ and $\int_0^1 f\big(M(a,b,t)\big)dt=0$. If $|f'|$ is an (M,P)-function on $[a,b]$, then the following inequality holds*

$$\int_0^1 f^2\big(M(a,b,t)\big)dt\leq\frac{\left[|f'(a)|+|f'(b)|\right]^2}{4\pi^2}\int_0^1(\varphi'(t))^2dt,$$

where $\varphi(t)=M(a,b,t)$, $\forall t\in[0,1]$.

Proof. Let $\varphi(t)=M(a,b,t)$ and $\hbar(t)=f\circ\varphi(t)$. Since φ is strictly monotone, $\varphi\in\mathrm{BV}[0,1]$, then $\varphi'\in L[0,1]$. Also, we can write $\varphi(0)=a$, $\varphi(1)=b$ and therefore

$\hbar(0) = f\big(M(a,b,0)\big) = f(a) = f(b) = f\big(M(a,b,1)\big) = \hbar(1)$. Also, since $\int_0^1 \hbar(t)dt = \int_0^1 f\big(M(a,b,t)\big)dt = 0$, φ satisfies the hypothesis of Theorem 1. So that, we can write

$$\int_0^1 \hbar^2(t)dt \le \frac{1}{4\pi^2}\int_0^1 \left(\hbar'(t)\right)^2 dt. \tag{2.9}$$

Then, we have,

$$\begin{aligned}\frac{1}{4\pi^2}\int_0^1 \left(\hbar'(t)\right)^2 dt &= \frac{1}{4\pi^2}\int_0^1 \left[f'\big(\varphi(t)\big)\varphi'(t)\right]^2 dt \\ &= \frac{1}{4\pi^2}\int_0^1 |f'\big(\varphi(t)\big)|^2 \left(\varphi'(t)\right)^2 dt.\end{aligned}$$

Since $|f'|$ is (M,P)-function on $[a,b]$, we get

$$\begin{aligned}\frac{1}{4\pi^2}\int_0^1 \left(\hbar'(t)\right)^2 dt &\le \frac{1}{4\pi^2}\int_0^1 \left[|f'(a)|+|f'(b)|\right]^2 \left(\varphi'(t)\right)^2 dt \\ &= \frac{\left[|f'(a)|+|f'(b)|\right]^2}{4\pi^2}\int_0^1 (\varphi'(t))^2 dt.\end{aligned} \tag{2.10}$$

Using (2.9) and (2.10), we get the desired result. □

Corollary 1. *If we take $M = A$ (A is the weighted arithmetic mean) in Theorem 9, we get*

$$\int_a^b f^2(x)dx \le \frac{(b-a)^3}{4\pi^2}\left[|f'(a)|+|f'(b)|\right]^2.$$

Corollary 2. *If we take $M = G$ (G is the weighted geometric mean) in Theorem 9, we get*

$$\int_a^b \frac{f^2(x)}{x}dx \le \frac{[\ln b - \ln a]^2(b^2-a^2)}{8\pi^2}\left[|f'(a)|+|f'(b)|\right]^2.$$

Corollary 3. *If we take $M = H$ (H is the weighted harmonic mean) in Theorem 9, we get*

$$\int_a^b \frac{f^2(x)}{x^2}dx \le \frac{(b^3-a^3)(b-a)^2}{12(ab)^2\pi^2}\left[|f'(a)|+|f'(b)|\right]^2.$$

Theorem 10. *Let $0 < a < b$ and $M\colon [a,b]\times[a,b]\times[0,1] \to (0,\infty)$ be a weighted mean function defined on $[a,b]$, $f\colon [a,b] \to (0,\infty)$, f and f' be continuous functions on (a,b) with $f(a) = f(b)$ and $\int_0^1 f\big(M(a,b,t)\big)dt = 0$. If $|f'|^q$ is an (M,P)-function on $[a,b]$, then the following inequality holds*

$$\int_0^1 f^2\big(M(a,b,t)\big)dt \le \frac{\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{2}{q}}}{4\pi^2}\left(\int_0^1 |\varphi'(t)|^{2p}dt\right)^{\frac{1}{p}},$$

where $\frac{1}{p}+\frac{1}{q}, q > 1$, $\varphi(t) = M(a,b,t)$, $\forall t \in [0,1]$.

Proof. Let $\varphi(t) = M(a,b,t)$ and $\hbar(t) = f \circ \varphi(t)$. Since φ is strictly monotone, $\varphi \in \mathrm{BV}[0,1]$, then $\varphi' \in L[0,1]$. Also, we can write $\varphi(0) = a$, $\varphi(1) = b$ and therefore $\hbar(0) = f\big(M(a,b,0)\big) = f(a) = f(b) = f\big(M(a,b,1)\big) = \hbar(1)$. Also, since $\int_0^1 \hbar(t)dt = \int_0^1 f\big(M(a,b,t)\big)dt = 0$, φ is satisfies the hypothesis of Theorem 1. So that, we can write

$$\int_0^1 \hbar^2(t)dt \le \frac{1}{4\pi^2}\int_0^1 \big(\hbar'(t)\big)^2 dt. \tag{2.11}$$

Using Hölder inequality, we have

$$\begin{aligned}
\frac{1}{4\pi^2}\int_0^1 \big(\hbar'(t)\big)^2 dt &= \frac{1}{4\pi^2}\int_0^1 \big[f'\big(\varphi(t)\big)\varphi'(t)\big]^2 dt \\
&= \frac{1}{4\pi^2}\int_0^1 |f'\big(\varphi(t)\big)|^2 \left|\varphi'(t)\right|^2 dt \\
&\le \frac{1}{4\pi^2}\left(\int_0^1 \left(\left|f'\big(\varphi(t)\big)\right|^2\right)^q dt\right)^{\frac{1}{q}} \left(\int_0^1 \left(\left|\varphi'(t)\right|^2\right)^p dt\right)^{\frac{1}{p}} \\
&= \frac{1}{4\pi^2}\left(\int_0^1 \left(\left|f'\big(\varphi(t)\big)\right|^q\right)^2 dt\right)^{\frac{1}{q}} \left(\int_0^1 \left|\varphi'(t)\right|^{2p} dt\right)^{\frac{1}{p}}.
\end{aligned}$$

Since $|f'|^q$ is (M,P)-function on $[a,b]$, we get

$$\begin{aligned}
\frac{1}{4\pi^2}\int_0^1 \big(\hbar'(t)\big)^2 dt &\le \frac{1}{4\pi^2}\left(\int_0^1 \left(|f'(a)|^q + |f'(b)|^q\right)^2 dt\right)^{\frac{1}{q}} \left(\int_0^1 \left|\varphi'(t)\right|^{2p} dt\right)^{\frac{1}{p}} \\
&= \frac{\left[|f'(a)|^q + |f'(b)|^q\right]^{\frac{2}{q}}}{4\pi^2}\left(\int_0^1 \left|\varphi'(t)\right|^{2p} dt\right)^{\frac{1}{p}}.
\end{aligned} \tag{2.12}$$

Using (2.11) and (2.12), we get the desired result. □

Corollary 4. *If we take $M = A$ (A is the weighted arithmetic mean) in Theorem 10, we get*

$$\int_a^b f^2(x)dx \le \frac{(b-a)^3}{4\pi^2}\left[|f'(a)|^q + |f'(b)|^q\right]^{\frac{2}{q}}.$$

Corollary 5. *If we take $M = G$ (G is the weighted geometric mean) in Theorem 10, we get*

$$\int_a^b \frac{f^2(x)}{x}dx \le \frac{\left[\ln b - \ln a\right]^{3-\frac{1}{p}}\left(b^{2p} - a^{2p}\right)^{\frac{1}{p}}}{4\pi^2(2p)^{\frac{1}{p}}}\left[|f'(a)|^q + |f'(b)|^q\right]^{\frac{2}{q}}.$$

Corollary 6. *If we take $M = H$ (H is the weighted harmonic mean) in Theorem 10, we get*

$$\int_a^b \frac{f^2(x)}{x^2}dx \leq \frac{(b-a)^{3-\frac{1}{p}}(ab)(b^{1-4p}-a^{1-4p})^{\frac{1}{p}}}{4\pi^2(1-4p)^{\frac{1}{p}}}\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{2}{q}}.$$

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