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NEW EXTENSIONS VERSION OF HERMITE–HADAMARD TYPE INEQUALITIES BY MEANS OF CONFORMABLE FRACTIONAL INTEGRALS

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Abstract. In the current investigation, we acquire the upper and lower bounds for inequalities of midpoint-type and trapezoid-type involving conformable fractional integral operators with the help of the mappings whose second derivatives are bounded. We support the established inequalities with examples. Moreover, we use graphs to demonstrate the correctness of the given examples. What's more, we prove the Hermite-Hadamard inequality, which includes conformable fractional integrals, with the aid of condition $f'(a+b-t)-f'(t)\geq 0$, $t\in\left[a,\frac{a+b}{2}\right]$ rather than the convexity of function.

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1. INTRODUCTION

The theory of inequalities in literature is widely used in engineering, physics, convexity theory, optimization theory, time scale, and quantum analysis. Many mathematicians have obtained new inequalities in these subjects. The new inequalities acquired and the new bounds of the mean value of the mappings find use in many other fields of mathematics. In this respect, inequalities are closely related to other branches of mathematics. The most famous of these inequalities are Hermite-Hadamard type, Trapezoid type, Midpoint type, Simpson type, Newton type, and Bullen type inequalities.

The famous Hermite-Hadamard inequality, which was presented separately by Charles Hermite and Jacques Hadamard in the literature, has attracted the attention of many researchers. Studies on this inequality have been the subject of much scientific research. In particular, studies on the Trapezoid inequality and the Midpoint inequality, that is, the right and left sides of the Hermite–Hadamard inequality, constitute the majority of the studies on this subject. Dragomir and Agarwal first presented trapezoid inequalities via convex mappings in [6], whereas Kirmacı first, acquired midpoint inequalities with the help of the convex mappings in [13]. Moreover, Dragomir et al. also proved new trapezoid and midpoint type extensions by utilizing the bounds of the twice differentiable instead of the condition of convexity in [8] and [7], respectively. For research with extensions on the Hermite-Hadamard inequality, midpoint type inequality, and trapezoid type inequality, refer to the following references [2, 4, 17].

Fractional calculus has taken its place in the literature to clarify the problems that classical analysis cannot answer. In particular, the function "Is it possible to take fractional derivatives and fractional integrals of a function?" answers the question. Many studies have been carried out that have contributed to the literature by using the convexity of the function in some studies and by using the bounds of the second derivative in some studies. In addition, new inequalities are obtained by using the condition $f'(a+b-x) \geq f'(x), x \in \left[a, \frac{a+b}{2}\right]$ instead of the convexity condition. There are two basic approaches to fractional calculations. One of these basic approaches is the Riemann-Liouville approach and the other is the Grünwald-Letnikov approach. Among these two approaches, research based on the Riemann-Liouville approach has been popular in recent years. In examples of these studies, Chen acquired extensions of the Hermite–Hadamard type inequality via convex mappings including Riemann-Liouville fractional integrals in [5]. Budak et al. presented the left and right-hand sides of fractional Hermite–Hadamard with the help of bounds in [3]. Sarikaya et al. gave Hermite–Hadamard type based on Riemann-Liouville fractional integrals in [15]. The conformable approach has emerged to eliminate the shortcomings of the Grünwald-Letnikov and Riemann-Liouville approaches. The conformable approach, based on the basic definition of the derivative, was developed by Khalil in [11]. Abdelhakim et al. presented that the conformable approach in [11] cannot yield good results when compared to the Caputo definition via specific mappings in [1]. This flaw in the conformable fractional approach has been fixed with some extensions in [9, 18]. Inspired by all these studies, Jarad et al. defined conformable fractional integrals. These definitions accelerated the research in fractional calculus.

One of the advantages of the conformable fractional integral is that it can be used to define a new class of functions that have the properties of both smooth and non-smooth functions. These functions are called conformable smooth functions, and they have been shown to have several interesting properties, such as the ability to capture sharp changes in a function without introducing discontinuities. Conformable fractional integrals have been used in several applications, such as signal processing, image processing, and control theory. They have been shown to be useful in the analysis of fractional-order systems, such as fractional-order differential equations, and in the design of fractional-order controllers.

This study consists of four sections including the introduction. In Section 2, we will give the necessary definitions and theorems to construct the main results.

In Section 3, we will present some Midpoint type, Trapezoid type, and Hermite-Hadamard type inequalities by means of conformable fractional integrals. In some of these obtained inequalities, we will use the bounds of the second derivative of the function instead of the convexity of the function. We will use the condition $f'(a+b-t)-f'(t)\geq 0$, $t\in\left[a,\frac{a+b}{2}\right]$ instead of the convexity when obtaining some inequalities. In Section 4, suggestions are made to researchers for future studies.

2. Preliminaries

This section presents the necessary definitions and theorems to form our principal outcomes.

Dragomir et al. obtained new bounds for classical midpoint type and trapezoid type inequalities.

Theorem 1 ([7]). *Assume that $f\colon [a,b]\to\mathbb{R}$ be a twice differentiable mapping such that there exists real constants m and M so that $m\leq f''\leq M$. Then, the following inequalities hold:*

$$m\frac{(b-a)^2}{24}\leq\frac{1}{b-a}\int\limits_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\leq M\frac{(b-a)^2}{24} \tag{2.1}$$

and

$$m\frac{(b-a)^2}{24}\leq\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int\limits_a^b f(x)dx\leq M\frac{(b-a)^2}{24}. \tag{2.2}$$

Let's give the definitions of Riemann-Liouville fractional integrals in the literature.

Definition 1. The Euler Gamma function and Euler Beta function are defined

$$\Gamma(x):=\int\limits_0^\infty t^{x-1}e^{-t}dt,$$

$$\mathcal{B}(x,y):=\int\limits_0^1 t^{x-1}(1-t)^{y-1}dt,$$

Here, $0<x,y<\infty$ and $x,y\in\mathbb{R}$.

Let's give the definitions of Riemann-Liouville fractional integrals in the literature.

Definition 2 ([12, 14]). Let $f\in L_1[a,b]$. The Riemann-Liouville fractional integrals of order $\beta>0$ are described by

$$J_{a^+}^{\beta}f(x)=\frac{1}{\Gamma(\beta)}\int_a^x(x-t)^{\beta-1}f(t)dt,\ \ x>a$$

and

$$J_{b-}^{\beta}f(x)=\frac{1}{\Gamma(\beta)}\int_x^b (t-x)^{\beta-1}f(t)dt,\ \ x<b.$$

Here, Γ is the Euler Gamma function and $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$.

Sarikaya first presented the inequality of Hermite–Hadamard-type involving Riemann-Liouville integrals as follows in [15].

Theorem 2. *Let consider $f\colon\ [a,b]\to\mathbb{R}$ be a positive mapping with $f\in L_1[a,b]$. If f is a convex mapping on $[a,b]$, then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\beta+1)}{2(b-a)^{\beta}}\left[J_{a+}^{\beta}f(b)+J_{b-}^{\beta}f(a)\right]\leq\frac{f(a)+f(b)}{2} \tag{2.3}$$

with $\beta>0$.

In [5], Chen acquired extensions of the trapezoid type inequality and midpoint type inequality via convex mappings utilizing Riemann-Liouville fractional integrals.

Theorem 3 ([5])**.** *Consider $f\colon\ [a,b]\longrightarrow\mathbb{R}$ is a positive, twice differentiable function and $f\in L_1[a,b]$. If f'' is bounded $[a,b]$, then we derive*

$$\begin{aligned}m\frac{(b-a)^2(\beta^2-\beta+2)}{8(\beta+1)(\beta+2)}&\leq\frac{\Gamma(\beta+1)}{2(b-a)^{\beta}}\left[J_{a+}^{\beta}f(b)+J_{b-}^{\beta}f(a)\right]-f\left(\frac{a+b}{2}\right)\\&\leq\frac{(b-a)^2(\beta^2-\beta+2)}{8(\beta+1)(\beta+2)}M\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}m\frac{(b-a)^2\beta}{2(\beta+1)(\beta+2)}&\leq\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{2(b-a)^{\beta}}\left[J_{a+}^{\beta}f(b)+J_{b-}^{\beta}f(a)\right]\\&\leq\frac{(b-a)^2\beta}{2(\beta+1)(\beta+2)}M.\end{aligned} \tag{2.5}$$

for $\beta>0$. *Here,* $m=\inf_{t\in[a,b]}f''(t)$, $M=\sup_{t\in[a,b]}f''(t)$.

Chen also established the inequality obtained in Theorem 2 by utilizing the condition $f'(a+b-x)\geq f'(x)$ instead of the convexity property of f in [5].

In 2017, Jarad et al. [10] defined the following fractional conformable integral operators. They also presented certain characteristics and relationships between these operators and several other fractional operators in the literature. The conformable fractional integral operators are described as follows.

Definition 3 ([10]). For $f \in L^1[a,b]$, the conformable fractional integral operators of order $\beta \in C$, $Re(\beta) > 0$ and $\alpha \in (0,1]$ are respectively given by

$$
{}^{\beta}_{a^+}\Upsilon^{\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left(\frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}} dt,
$$
$$
{}^{\beta}\Upsilon^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b \left(\frac{(b-x)^{\alpha} - (b-t)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{f(t)}{(b-t)^{1-\alpha}} dt.
$$

Set et al. achieved a new Hermite–Hadamard inequality with the help of the conformable fractional integral operators in [16].

Theorem 4 ([16]). *Note that f is a convex function on $[a,b]$. Then the following inequality is satisfied.*

$$
f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}} \left[{}^{\beta}_{a^+}\Upsilon^{\alpha} f(b) + {}^{\beta}\Upsilon^{\alpha}_{b^-} f(a) \right] \le \frac{f(a)+f(b)}{2}. \tag{2.6}
$$

Here, $\beta > 0$, $\alpha \in (0,1]$ and Γ is the Euler Gamma function.

3. Principal Outcomes

In this section, some new midpoint-type, trapezoid-type, and Hermite-Hadamard-type inequalities are given.

Theorem 5. *Let $f\colon [a,b] \to \mathbb{R}$ be a twice differentiable function such that there exists real constants m and M so that $m \le f'' \le M$ and $\beta > 0$, $\alpha \in (0,1]$. The following inequality is*

$$
\begin{aligned}
& m\frac{\beta(b-a)^2}{2}\left[\frac{1}{4\beta} + \mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right) - \mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)\right] \qquad (3.1)\\
& \le \frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}} \left[{}^{\beta}_{a^+}\Upsilon^{\alpha} f(b) + {}^{\beta}\Upsilon^{\alpha}_{b^-} f(a) \right] - f\left(\frac{a+b}{2}\right)\\
& \le M\frac{\beta(b-a)^2}{2}\left[\frac{1}{4\beta} + \mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right) - \mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)\right]
\end{aligned}
$$

valid. Here, $\mathcal{B}$ is Euler Beta function.

Proof. With the help of Definition 3, we obtain

$$
\begin{aligned}
& \frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}} \left[{}^{\beta}_{a^+}\Upsilon^{\alpha} f(b) + {}^{\beta}\Upsilon^{\alpha}_{b^-} f(a) \right]\\
& = \frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}} \left[\frac{1}{\Gamma(\beta)} \int_a^b \left(\frac{(b-a)^{\alpha} - (t-a)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}} dt \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\beta)}\int_a^b\left(\frac{(b-a)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)}{(b-t)^{1-\alpha}}dt\Bigg]\\
&=\frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\int_a^b\left[\left(\frac{(b-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}\right.\\
&\left.+\left(\frac{(b-a)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{1}{(b-t)^{1-\alpha}}\right]f(t)\,dt\\
&=\frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[\int_a^{\frac{a+b}{2}}\left(\frac{(b-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}\right.\\
&+\left(\frac{(b-a)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{1}{(b-t)^{1-\alpha}}\\
&+\int_{\frac{a+b}{2}}^b\left(\frac{(b-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}\\
&\left.+\left(\frac{(b-a)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{1}{(b-t)^{1-\alpha}}\right]f(t)\,dt.
\end{aligned}
$$

With the aid of the change of variables, we get

$$
\begin{aligned}
&\frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{b^-}f(a)\right] \qquad (3.2)\\
&=\frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[\int_a^{\frac{a+b}{2}}\left(\left(\frac{(b-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}\right.\right.\\
&\left.+\left(\frac{(b-a)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{1}{(b-t)^{1-\alpha}}\right)f(t)dt\\
&+\int_a^{\frac{a+b}{2}}\left(\left(\frac{(b-a)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{1}{(b-t)^{1-\alpha}}\right.\\
&\left.\left.+\left(\frac{(b-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}\right)f(a+b-t)dt\right]\\
&=\frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left[\left(\frac{(b-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}\right.\\
&\left.+\left(\frac{(b-a)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{1}{(b-t)^{1-\alpha}}\right](f(t)+f(a+b-t))\,dt
\end{aligned}
$$

$$= \frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}} \int_{a}^{\frac{a+b}{2}} U_{\alpha}^{\beta}(t)\left[f(t)+f(a+b-t)\right]dt,$$

where

$$U_{\alpha}^{\beta}(t) = \left(\frac{(b-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}} + \left(\frac{(b-a)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{1}{(b-t)^{1-\alpha}}.$$

From equality (3.2), we can write

$$\begin{aligned} &\frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}_{a^+}^{\beta}\Upsilon^{\alpha} f(b) + {}^{\beta}\Upsilon_{b^-}^{\alpha} f(a)\right] - f\left(\frac{a+b}{2}\right) \\ &= \frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}} \int_{a}^{\frac{a+b}{2}} U_{\alpha}^{\beta}(t)\left[f(t)+f(a+b-t)-2f\left(\frac{a+b}{2}\right)\right]dt. \end{aligned} \tag{3.3}$$

If we take advantage of the facts that

$$f(t)-f\left(\frac{a+b}{2}\right) = \int_{\frac{a+b}{2}}^{t} f'(s)ds$$

and

$$f(a+b-t)-f\left(\frac{a+b}{2}\right) = \int_{\frac{a+b}{2}}^{a+b-t} f'(s)ds,$$

then we obtain

$$\begin{aligned} f(t)+f(a+b-t)-2f(\frac{a+b}{2}) &= \int_{\frac{a+b}{2}}^{t} f'(s)ds + \int_{\frac{a+b}{2}}^{a+b-t} f'(s)ds \\ &= \int_{\frac{a+b}{2}}^{a+b-t} f'(s)ds - \int_{\frac{a+b}{2}}^{a+b-t} f'(a+b-s)ds \\ &= \int_{\frac{a+b}{2}}^{a+b-t} \left[f'(s)-f'(a+b-s)\right]ds. \end{aligned} \tag{3.4}$$

We also obtain

$$f'(s)-f'(a+b-s) = \int_{a+b-s}^{s} f''(y)dy. \tag{3.5}$$

By using the condition $m \le f''(y) \le M$ for all $y \in [a,b]$ and with the aid of equality (3.5), we get

$$\int_{a+b-s}^{s} mdy \le \int_{a+b-s}^{s} f''(y)dy \le \int_{a+b-s}^{s} Mdy$$

which gives

$$m(2s-a-b) \le f'(s)-f'(a+b-s) \le M(2s-a-b).$$

From equality (3.4), we establish

$$m\int_{\frac{a+b}{2}}^{a+b-t}(2s-a-b) \le \int_{\frac{a+b}{2}}^{a+b-t}\left[f'(s)-f'(a+b-s)\right]ds \le M\int_{\frac{a+b}{2}}^{a+b-t}(2s-a-b).$$

This implies that

$$m\left(\frac{a+b}{2}-t\right)^2 \leq f(t)+f(a+b-t)-2f\left(\frac{a+b}{2}\right) \leq M\left(\frac{a+b}{2}-t\right)^2. \qquad (3.6)$$

Multiplying inequality (3.6) by $\frac{\beta\alpha^{\beta}U_{\alpha}^{\beta}(t)}{2(b-a)^{\alpha\beta}}$ and then integrating with respect to t on the interval $\left[a,\frac{a+b}{2}\right]$, we have

$$\begin{aligned}
& m\frac{\beta(b-a)^2}{2}\left[\frac{1}{4\beta}+\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)\right] \\
& \leq \frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}} U_{\alpha}^{\beta}(t)\left[f(t)+f(a+b-t)-2f\left(\frac{a+b}{2}\right)\right]dt \\
& \leq M\frac{\beta(b-a)^2}{2}\left[\frac{1}{4\beta}+\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)\right].
\end{aligned}$$

By using equality (3.3), we establish the required result. □

Remark 1. If we take $\alpha=1$ in Theorem 5, then inequalities (3.1) reduce to inequalities (2.4).

Remark 2. If we choose $\alpha=1$ and $\beta=1$ in Theorem 5, then inequalities (3.1) turns into inequalities (2.1).

Example 1. If the function $f\colon [a,b]=[0,1]\to\mathbb{R}$ is described as $f(t)=t^4+t^2$ such that $2\leq f''(t)\leq 14$ for $t\in[0,1]$. Under these assumptions, the mid-term of inequality (3.1) becomes to

$$\begin{aligned}
& \frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}_{a^+}^{\beta}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon_{b^-}^{\alpha}f(a)\right]-f\left(\frac{a+b}{2}\right) \\
& =\frac{\Gamma(\beta+1)\alpha^{\beta}}{2}\left[{}_{0^+}^{\beta}\Upsilon^{\alpha}f(1)+{}^{\beta}\Upsilon_{1^-}^{\alpha}f(0)\right]-f\left(\frac{1}{2}\right) \\
& =\frac{\Gamma(\beta+1)\alpha^{\beta}}{2}\left[\frac{1}{\Gamma(\beta)}\int_0^1\left(\frac{1^{\alpha}-t^{\alpha}}{\alpha}\right)^{\beta-1}\frac{(t^4+t^2)}{t^{1-\alpha}}dt\right. \\
& \quad\left.+\frac{1}{\Gamma(\beta)}\int_0^1\left(\frac{1^{\alpha}-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{(t^4+t^2)}{(1-t)^{1-\alpha}}dt\right]-\left(\frac{1}{2^4}+\frac{1}{2^2}\right) \\
& =\frac{\beta\alpha}{2}\left[\int_0^1(1^{\alpha}-t^{\alpha})^{\beta-1}(t^{\alpha+3}+t^{\alpha+2})\,dt+\right. \\
& \quad\left.+\int_0^1(1^{\alpha}-(1-t)^{\alpha})^{\beta-1}(t^4+t^2)(1-t)^{\alpha-1}dt\right]-\frac{5}{16} \\
& =\frac{\beta}{2}\left[2\mathcal{B}\left(\frac{4}{\alpha}+1,\beta\right)-4\mathcal{B}\left(\frac{3}{\alpha}+1,\beta\right)+8\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right.
\end{aligned}$$

$$-6\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)+\frac{2}{\beta}\bigg]-\frac{5}{16}.$$

Consequently, inequalities (3.1) can be written as follows

$$\begin{aligned}
&\beta\left[\frac{1}{4\beta}+\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)\right] \\
&\leq \frac{\beta}{2}\left[2\mathcal{B}\left(\frac{4}{\alpha}+1,\beta\right)-4\mathcal{B}\left(\frac{3}{\alpha}+1,\beta\right)\right. \\
&\quad \left. +8\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)-6\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)+\frac{2}{\beta}\right]-\frac{5}{16} \\
&\leq 7\beta\left[\frac{1}{4\beta}+\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)\right].
\end{aligned} \tag{3.7}$$

To illustrate the correctness of (3.7), one can refer to Figure 1.

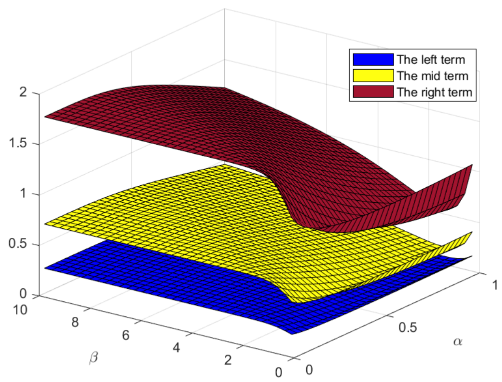

FIGURE 1. Graph for the result of Example 1 computed and plotted in MATLAB.

Theorem 6. *Let $f\colon [a,b]\to\mathbb{R}$ be a twice differentiable mapping such that there exists real constants m and M so that $m\leq f''\leq M$. Then, we establish*

$$\begin{aligned}
&m\frac{\beta(b-a)^2}{2}\left[\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right] \\
&\leq \frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha} f(b)+{}^{\beta}\Upsilon^{\alpha}_{b^-} f(a)\right] \\
&\leq M\frac{\beta(b-a)^2}{2}\left[\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right]
\end{aligned} \tag{3.8}$$

where $\mathcal{B}$ is Euler Beta function.

Proof. By equality (3.2), we have

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}_{a^{+}}^{\beta}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon_{b^{-}}^{\alpha}f(a)\right] \tag{3.9}$$
$$=\frac{f(a)+f(b)}{2}-\frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\int_{a}^{\frac{a+b}{2}}U_{\alpha}^{\beta}(t)\left[f(t)+f(a+b-t)\right]dt$$
$$=\frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\int_{a}^{\frac{a+b}{2}}U_{\alpha}^{\beta}(t)\left[f(a)+f(b)-(f(t)+f(a+b-t))\right]dt.$$

With the aid of the equalities

$$f(a)-f(t)=-\int_{a}^{t}f'(s)ds$$

and

$$f(b)-f(a+b-t)=\int_{a+b-t}^{b}f'(s)ds,$$

we acquire

$$f(a)+f(b)-(f(t)+f(a+b-t))=\int_{a+b-t}^{b}f'(s)ds-\int_{a}^{t}f'(s)ds \tag{3.10}$$
$$=\int_{a}^{t}f'(a+b-s)ds-\int_{a}^{t}f'(s)ds$$
$$=\int_{a}^{t}\left[f'(a+b-s)-f'(s)\right]ds.$$

We also acquire

$$f'(a+b-s)-f'(s)=\int_{s}^{a+b-s}f''(y)dy. \tag{3.11}$$

By equality (3.11) and the condition $m\leq f''\leq M$, we have

$$m(a+b-2s)\leq f'(s)-f'(a+b-s)\leq M(a+b-2s). \tag{3.12}$$

By utilizing equality (3.10) and inequality (3.12), we get

$$\int_{a}^{t}m(a+b-2s)ds\leq\int_{a}^{t}\left[f'(s)-f'(a+b-s)\right]ds\leq\int_{a}^{t}M(a+b-2s)ds$$

i.e.

$$m(b-t)(t-a)\leq f(a)+f(b)-(f(t)+f(a+b-t))\leq M(b-t)(t-a). \tag{3.13}$$

Multiplying inequality (3.13) by $\frac{\beta\alpha^{\beta}U_{\alpha}^{\beta}(t)}{2(b-a)^{\alpha\beta}}$ and then integrating with respect to t on the interval $\left[a,\frac{a+b}{2}\right]$, we derive

$$m\frac{\beta(b-a)^{2}}{2}\left[\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right] \tag{3.14}$$

$$\leq \frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha} f(b) + {}^{\beta}\Upsilon^{\alpha}_{b^-} f(a)\right]$$
$$\leq M\frac{\beta(b-a)^2}{2}\left[\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right) - \mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right].$$

So, the proof is accomplished. □

Remark 3. If we take $\alpha = 1$ in Theorem 6, then inequalities (3.8) reduce to inequalities (2.5).

Remark 4. If we allow $\alpha = \beta = 1$ in Theorem 5, then inequalities (3.8) and inequalities (2.2) identical.

Example 2. Let us consider the function $f: [a,b] = [-2,1] \to \mathbb{R}$ is described as $f(t) = t^3$ such that $-12 \leq f''(t) \leq 6$ for $t \in [-2,1]$. By these assumptions, the mid-term of (3.8) becomes as follows

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha} f(b) + {}^{\beta}\Upsilon^{\alpha}_{b^-} f(a)\right]$$
$$= \frac{f(-2)+f(1)}{2} - \frac{\Gamma(\beta+1)\alpha^{\beta}}{2\cdot 3^{\alpha\beta}}\left[{}^{\beta}_{-2^+}\Upsilon^{\alpha} f(1) + {}^{\beta}\Upsilon^{\alpha}_{1^-} f(-2)\right.$$
$$\left. + \frac{1}{\Gamma(\beta)}\int_{-2}^{1}\left(\frac{3^{\alpha}-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{t^3}{(1-t)^{1-\alpha}}dt\right]$$
$$= -\frac{7}{2} - \frac{\beta}{2}\left[\left(27\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right) - 27\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right) - \frac{7}{\beta}\right)\right].$$

Finally, inequality (3.8) would be as follows

$$-54\beta\left[\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right) - \mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right] \tag{3.15}$$
$$\leq -\frac{7}{2} - \frac{\beta}{2}\left[\left(27\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right) - 27\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right) - \frac{7}{\beta}\right)\right]$$
$$\leq 27\beta\left[\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right) - \mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right].$$

To demonstrate the accuracy of (2), one can refer to Figure 2.

Theorem 7. *Let $f: [a,b] \to \mathbb{R}$ be a differentiable function and $f \in L_1[a,b]$. If $f'(a+b-x) \geq f'(x)$ for all $x \in \left[a, \frac{a+b}{2}\right]$, then the following Hemite-Hadamard inequality for conformable fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha} f(b) + {}^{\beta}\Upsilon^{\alpha}_{b^-} f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{3.16}$$

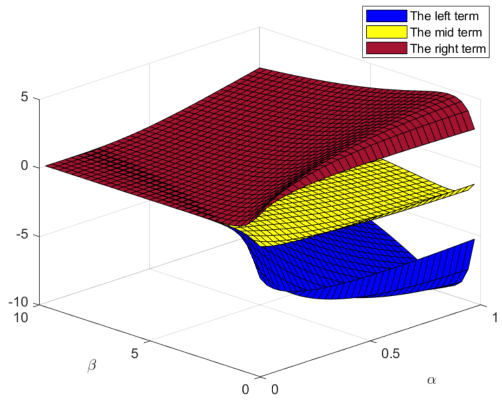

FIGURE 2. Graph for the result of Example 2 computed and plotted in MATLAB.

Proof. Using equalities (3.3) and (3.4), we can write

$$\begin{aligned}
&\frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{b^-}f(a)\right]-f\left(\frac{a+b}{2}\right)\\
&=\frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}U_{\alpha}^{\beta}(t)\left[f(t)+f(a+b-t)-2f(\frac{a+b}{2})\right]\\
&=\frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}U_{\alpha}^{\beta}(t)\left[\int_{\frac{a+b}{2}}^{a+b-t}\left[f'(s)-f'(a+b-s)\right]ds\right]dt\\
&=\frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}U_{\alpha}^{\beta}(t)\left[\int_t^{\frac{a+b}{2}}\left[f'(a+b-u)-f'(u)\right]ds\right]dt\\
&\geq 0
\end{aligned}$$

which presents the first inequality in (3.16).

Likewise, by equalities (3.9) and (3.10), we establish

$$\begin{aligned}
&\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{b^-}f(a)\right]\\
&=\frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}U_{\alpha}^{\beta}(t)\left[f(a)+f(b)-(f(t)+f(a+b-t))\right]dt\\
&=\frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}U_{\alpha}^{\beta}(t)\left[\int_a^{t}\left[f'(a+b-s)-f'(s)\right]ds\right]dt\\
&\geq 0.
\end{aligned}$$

Thus, the proof of Theorem 7 is completed. □

4. Conclusion

In this article, we derived some new midpoint-type and trapezoid-type inequalities for conformable fractional integrals. In deriving these inequalities, we took advantage of the bounds of the second derivative of the function. We also used a different technique for the conformable fractional Hermite Hadamard inequality instead of the convexity of the function. As can be seen from the paper, interested readers can use the condition $m \leq f''(t) \leq M$ for all $t \in [a,b]$ instead of convexity, and they can try to find better bounds using the condition $f'(a+b-t) - f'(t) \geq 0$, $t \in \left[a, \frac{a+b}{2}\right]$. More precisely, researchers can examine the other kinds of fractional integrals of the obtained these inequalities.

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