Miskolc Mathematical Notes
Vol. 25 (2024), No. 1, pp. 35–48

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4496

A LEGENDRE WAVELET COLLOCATION METHOD FOR SOLVING NEUTRAL DELAY DIFFERENTIAL EQUATIONS

UZAIR AHMED, MO FAHEEM, ARSHAD KHAN, AND GEETAN MANCHANDA

Received 21 November, 2022

Abstract. The Legendre wavelet based method has been employed in this paper to investigate neutral delay differential equations. The highest order derivative is approximated by Legendre wavelet using the integral operator technique. Then integrations of Legendre wavelet are used to approximate the lower order derivatives and unknown function. To get an algebraic system of linear or nonlinear equations, approximated values of unknown function and its derivatives are substituted in neutral delay differential equations. On solving the developed system, we get unknown wavelet coefficients and subsequently the approximate solution. To analyze the theoretical usability of the approach, the upper bound of error norm is established. Moreover, the theoretical results are confirmed through few numerical experiments. A comparison of the results of presented method with method available in literature is given to conclude the superiority of the proposed method.

2010 *Mathematics Subject Classification:* 65L05; 65T60

Keywords: neutral delay differential equations, Legendre wavelet, convergence, error norm, MATLAB

This paper was presented in 5th International Conference on Mathematical Advances and Applications (ICOMAA-2022) which was held at Yıldız Teknik Üniversitesi, Istanbul, Turkey.

1. INTRODUCTION

The term “delay” in differential equations refers to the temporal latencies between observation and control action in the mathematical formulation of natural and technological problems. So we can say that a differential equation in which delay exist in unknown variable and/or its derivatives, is known as delay differential equation. It occurs in many real-life problems, such as SARS-CoV-2, where there is a temporal lag between the period of an infected person transmits the virus, and the time susceptible person get infected. The main reason for which, neutral delay differential equations (NDDEs) is getting attention is that it plays vital role in many areas of sciences. NDDEs have wide range of applications in applied mathematics, physics, ecology, engineering, etc. For example, in biological science, delay differential equation exhibit better picture of population fluctuation than ordinary differential equation [16]. It has many applications in the dynamical system too. As a result, it can be claimed that most of the mathematical problems have a time delay.

We shall investigate the following type of neutral delay differential equation (NDDE):

$$\begin{aligned} w''(t) =& f(t, w(t), w(t-\pi_1(t,w(t))), w'(t), w'(t-\pi_2(t,w(t))), \\ & w''(t), w''(t-\pi_3(t,w(t)))), \qquad t \in [\alpha, \beta], \end{aligned} \tag{1.1}$$

with initial and delay conditions

$$w(t) = \phi(t), \quad t \leq \alpha, \tag{1.2}$$

and boundary conditions (BCs):

$$y(\alpha) = \xi, \quad \text{and} \quad y(\beta) = \eta, \tag{1.3}$$

where $f : [\alpha, \beta] \times \mathbb{R}^5 \longrightarrow \mathbb{R}$ is a differentiable function, $\pi_1(t,w(t))$, $\pi_2(t,w(t))$, and $\pi_3(t,w(t))$ are continuous functions on $[\alpha,\beta] \times \mathbb{R}$ such that $t-\pi_1(t,w(t))$, $t-\pi_2(t, w(t))$, $t-\pi_3(t,w(t)) < \beta$. Also, $\phi(t)$ represents the initial function which is given in Vanani and Aminataei [23].

In the late 1970s, technique for solving second-order delay differential equation having constant delay with boundary conditions was given by Nevers and Schmitt [11], where they solved the problem by using Euler's method and shooting technique.

In literature review, we found various methods which have been used to solve DDEs, like finite difference, Richardson method and collocation method (see [10], [6],[15], [19], [14], [20]). For more details about the collocation methods for a variety of differential equations, please refer to ([4],[1],[2],[3], [7], [17], [18]). However very few people have obtained numerical solution of second order DDEs for non constant delay with BCs. For example, convergent method of order three was given by Backke and Jackiewicz [6] where they applied Richardson's extrapolation technique, and Cahlon and Nachman [8] solved time dependent delay problems using Atkinson's product integration technique.

In recent decades, methods based on wavelet basis function has grown manifolds and the main reason of its attraction is that, they are more efficient and gives more accurate numerical results as compared to other well known methods (see [5],[14],[21], [13]). Wavelets are a powerful and efficient mathematical tool that divides a data function into multiple frequency constituents and investigates each constituent with a resolution that corresponds to its scale [22].

This paper aims to propose integral operator approach in which the highest order derivative is approximated in terms of Legendre wavelet for solving second order NDDEs with boundary conditions. The paper's outline is presented as follows. In Section 2, we present definition of multiresolution analysis, basic definition of wavelet, Legendre wavelet and function approximation by Legendre wavelet. In Section 3, we discussed our method for solving NDDEs. We carried out the Legendre wavelet's convergence analysis in Section 4. Section 5, consist of two linear and one nonlinear problem respectively, to demonstrate the proposed method's validation using maximum absolute errors. Moreover, we made a comparison between exact solution and existing method such as direct block method [15].

2. Preliminary considerations

This section gives an insight into the basic definitions that is used in the rest part of the paper.

Definition 2.1. A multiresolution analysis (MRA), which is also referred as the wavelet's 'heart', was first introduced in the year 1989. It plays a vital role in writing the wavelet in a broad sense. It gives the ability to write any arbitrary function $w \in \mathscr{L}^2(\mathbb{R})$ over the multiresolution approximation space. MRA's goal is to break down the entire function spaces into spaces, $\mathcal{V}^k$ and $\mathcal{W}^k$, namely wavelet subspace and scaling function subspace, respectively. An arbitrary function $w \in \mathscr{L}^2(\mathbb{R})$ is projectable on $\mathcal{V}^k$, if $\mathcal{V}^k$ satisfies the following conditions:

(1) $\mathcal{V}^k \subset \mathcal{V}^{k+1}$,
(2) $\overline{\bigcup_{k\in\mathbb{Z}} \mathcal{V}^k} = \mathscr{L}^2(\mathbb{R})$, i.e., $\{\mathcal{V}^k\}$'s are dense in $\mathscr{L}^2(\mathbb{R})$,
(3) The collection $\{\phi(t-n), n \in \mathbb{Z}\}$ forms an orthonomal basis for $\mathcal{V}^0$,
(4) $w(.) \in \mathcal{V}^k \Longleftrightarrow w(2.) \in \mathcal{V}^{k+1}$, for all $k \in \mathbb{N}$,
(5) $\bigcap_{k\in\mathbb{Z}} \mathcal{V}^k = \{0\}$, i.e., there is nothing common in all the subspaces.

The wavelet subspace is defined in the following manner:

$$\mathcal{W}^k = \left\{\psi_k^\lambda;\ k, \lambda \in \mathbb{Z}\right\},$$

where $\mathcal{W}^k$ is orthogonal complement of $\mathcal{V}^k$ in $\mathcal{V}^{k+1}$ such that

$$\mathcal{V}^{k+1} = \mathcal{V}^k \oplus \mathcal{W}^k. \tag{2.1}$$

On repeating the above steps, we get

$$\mathcal{V}^K = \mathcal{V}^{K_0} \oplus \bigoplus_{k=K_0}^{K-1} \mathcal{W}^k, \quad K > K_0. \tag{2.2}$$

If $P_{\mathcal{V}^K}$ project any arbitrary function $w(t) \in \mathscr{L}^2(\mathbb{R})$ on $\mathcal{V}^k$, we can conclude from dense criteria of MRA that

$$P_{\mathcal{V}^K} w(t) \longrightarrow w(t), \quad \text{as} \quad K \longrightarrow \infty. \tag{2.3}$$

From equations (2.2) and (2.3), we can define scaling function projection and wavelet projection in the following way:

$$P_{\mathcal{V}^K} w(t) \approx \sum_{\lambda} \tilde{\mu}_k^\lambda \phi_k^\lambda(t), \tag{2.4}$$

$$P_{\mathcal{V}^K} w(t) \approx \sum_{\lambda} \tilde{\mu}^{\lambda}_{K_0} \phi^{\lambda}_{K_0}(t) + \sum_{\lambda} \sum_{k=K_0}^{K-1} \mu^{\lambda}_k \psi^{\lambda}_k(t), \tag{2.5}$$

where the coefficients μ^{λ}_k and $\tilde{\mu}^{\lambda}_{K_0}$ can be evaluated by applying the orthogonal property of the wavelet $\psi(t)$ and scaling function $\phi(t)$ as

$$\tilde{\mu}^{\lambda}_{K_0} = \int_{-\infty}^{\infty} w(t)\phi^{\lambda}_{K_0}(t)dt, \quad \mu^{\lambda}_k = \int_{-\infty}^{\infty} w(t)\psi^{\lambda}_k(t)dt. \tag{2.6}$$

2.1. *Wavelet and Legendre Wavelet*

Definition 2.2. Any orthogonal system which comes from MRA is called mother wavelet, if its total integration is exactly zero. i.e.

$$\int_{-\infty}^{\infty} \psi(t)dt = 0.$$

The dilation and translation of mother wavelet gives birth to a group of functions which is referred as wavelet, and defined as:

$$\psi^{\mathcal{T}}_{\mathcal{D}}(t) = \frac{1}{\sqrt{\mathcal{D}}}\psi\left(\frac{t-\mathcal{T}}{\mathcal{D}}\right), \quad \mathcal{D} \neq 0, \mathcal{T} \in \mathbb{R}, \tag{2.7}$$

where $\mathcal{D}$ and $\mathcal{T}$ are representing dilation and translation parameters, respectively ([9], [12]). On restricting these two parameters upto the discrete values $\mathcal{D} = \mathcal{D}_0^{-k}$, $\mathcal{T} = \lambda \mathcal{T}_0 \mathcal{D}_0^{-k}$, where $\mathcal{D}_0 > 1$ and $\mathcal{T}_0 > 1$, we get the following discrete wavelet:

$$\psi^{\lambda}_k = \left(\sqrt{\mathcal{D}_0}\right)^k (\mathcal{D}_0^k t - \lambda \mathcal{T}_0). \tag{2.8}$$

Definition 2.3. The n^{th}-order Legendre polynomials having weight function $w(t) = 1$, denoted by $\mathcal{P}_n(t)$ are orthogonal system on the interval $[-1,1]$, it can be determined by the following recurrence relation:

$$\mathcal{P}_{m+1}(t) = \left(\frac{2m+1}{m+1}\right) t\mathcal{P}_m(t) - \left(\frac{m}{m+1}\right)\mathcal{P}_{m-1}(t), \qquad m = 1,2,3\dots$$
$$\mathcal{P}_0(t) = 1, \ \mathcal{P}_1(t) = t, \dots.$$

Now, we define Legendre wavelet with four arguments in such a way that $\mathcal{L}^{\lambda}_k(t) = \mathcal{L}(k,\lambda,m,t)$ over the interval $[0,1)$ [24],

$$\mathcal{L}^{\lambda}_k(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\, 2^{\frac{k}{2}} \mathcal{P}_m(2^k t - \lambda), & t \in \left[\frac{\lambda-1}{2^k}, \frac{\lambda+1}{2^k}\right) \\ 0, & \text{elsewhere} \end{cases} \tag{2.9}$$

where 'm' denotes the order of Legendre polynomial varying from 0 to a fixed positive value $M-1$, $\lambda = 1,2,3.....2^k - 1$. The family of Legendre wavelet produces an orthonormal basis of $\mathscr{L}^2(\mathbb{R})$. For each pair of m and λ, we can re-write the Legendre wavelet in single index in the form:

$$\mathcal{L}_{\imath}(t) = \mathcal{L}^{\lambda}_k(t), \tag{2.10}$$

where the wavelet number $\imath$ satisfies, $\imath = \lambda + 2^{k-1}m$.

2.1.1. *Approximation of function by Legendre wavelet*

An arbitrary function $w \in \mathscr{L}^2[0,1)$ is capable of being expanded into series of Legendre wavelet as [25], from equation (2.5), we get

$$w(t) = \sum_{\lambda} \tilde{\mu}_{K_0}^{\lambda} \mathcal{L}_{K_0}^{\lambda}(t) + \sum_{\lambda} \sum_{k=K_0}^{K-1} \mu_k^{\lambda} \mathcal{L}_k^{\lambda}(t) = \sum_{\imath=1}^{\infty} \mu_\imath \mathcal{L}_\imath(t). \tag{2.11}$$

To get the best approximation, we truncate the infinite series given in equation (2.11) in the manner shown below for a fixed natural number N:

$$w(t) \approx \sum_{\imath=1}^{N} \mu_\imath \mathcal{L}_\imath(t) = \mu^T \mathcal{L}(t), \tag{2.12}$$

where

$$\mu^T = [\mu_1, \mu_2, \ldots. \mu_N],$$
$$\mathcal{L}(t) = [\mathcal{L}_1(t), \mathcal{L}_2(t) \ldots. \mathcal{L}_N(t)]^T,$$

where $N = 2^{k-1}M$ and collocation points are determined by $t(p) = \frac{p-0.5}{N}$, where $1 \le p \le N$, $N = 2^K$, $K \in \mathbb{N}$.

2.1.2. *Integration of Legendre Wavelet*

Let us denote the first and second integrations of Legendre wavelet by $\mathscr{I}_1$ and $\mathscr{I}_2$. These integrations can be easily calculated using equation (2.9) and can be expressed as follows:

$$\mathscr{I}_\imath^1(t) = \int_0^t \mathcal{L}_\imath(\tilde{t}) d\tilde{t},$$
$$\mathscr{I}_\imath^2(t) = \int_0^t \int_0^t \mathcal{L}_\imath(\tilde{t}) d\tilde{t} d\tilde{t}.$$

The above integrations can be obtained as :

$$\mathscr{I}_\imath^1(t) = \begin{cases} \left(\frac{1}{\sqrt{2}}\right)^k \rho(\frac{1}{2m+1})\{\mathcal{P}_{m+1}(t) - \mathcal{P}_{m-1}(t) - \mathcal{P}_{m+1}(-1) \\ \quad +\mathcal{P}_{m-1}(-1)\}, & t \in [\nu_1,\ \nu_2) \\ \left(\frac{1}{\sqrt{2}}\right)^k \rho(\frac{1}{2m+1})\{\mathcal{P}_{m+1}(1) - \mathcal{P}_{m-1}(1) - \mathcal{P}_{m+1}(-1) \\ \quad +\mathcal{P}_{m-1}(-1)\}, & t \in [\nu_2,\ 1) \end{cases}$$

$$\mathscr{I}_\iota^2(t)=\begin{cases}\left(\frac{1}{\sqrt{2}}\right)^{3k}\rho(\frac{1}{2m+1})\{(\frac{1}{2m+2})\{\mathcal{P}_{m+2}(t)-\mathcal{P}_m(t)-\mathcal{P}_{m+2}(-1)+\mathcal{P}_m(-1)\}\\ \quad -\left(\frac{1}{2m-2}\right)\{\mathcal{P}_m(t)-\mathcal{P}_{m-2}(t)-\mathcal{P}_m(-1)+\mathcal{P}_{m-2}(-1)\}\\ \quad -2(t+1)\{\mathcal{P}_{m+1}(-1)+\mathcal{P}_{m-1}(-1)\}\}, & t\in[\nu_1,\ \nu_2)\\ \left(\frac{1}{\sqrt{2}}\right)^{3k}\rho(\frac{1}{2m+1})\{(\frac{1}{2m+2})\{\mathcal{P}_{m+1}(1)-\mathcal{P}_{m-1}(1)-\mathcal{P}_{m+1}(-1)\\ \quad +\mathcal{P}_{m-1}(-1)\}-(\frac{1}{2m-2})\{\mathcal{P}_m(1)-\mathcal{P}_{m-2}(1)-\mathcal{P}_m(-1)\\ \quad +\mathcal{P}_{m-2}(-1)\}-2\{\mathcal{P}_{m+1}(-1)+\mathcal{P}_{m-1}(-1)\}+(t-1)\\ \quad \times\{\mathcal{P}_{m+1}(1)-\mathcal{P}_{m-1}(1)-\mathcal{P}_{m+1}(-1)+\mathcal{P}_{m-1}(-1)\}\}, & t\in[\nu_2,\ 1)\end{cases}$$

where $\nu_1=\frac{\lambda-1}{2^k}$, $\nu_2=\frac{\lambda+1}{2^k}$ and $\rho=\sqrt{\frac{2m+1}{2}}$.

3. Method for solution of NDDE

For the sake of convenience, we use '$\sum$' instead of $\sum_{\iota=1}^{n}$ throughout the paper. Now, approximate the higher order derivative in the form of Legendre wavelet

$$w''(t)\approx\sum d_\iota\mathcal{L}_\iota(t). \tag{3.1}$$

Now, integrate equation (3.1) twice from 0 to t, we get

$$w'(t)\approx\sum d_\iota\mathscr{I}_\iota^1(t)+w'(0), \tag{3.2}$$
$$w(t)\approx\sum d_\iota\mathscr{I}_\iota^2(t)+tw'(0)+w(0). \tag{3.3}$$

On putting $t=1$ in equation (3.3), we get

$$w(1)\approx w(0)+w'(0)+\sum d_\iota\mathscr{I}_\iota^2(1),$$
$$w'(0)\approx w(1)-w(0)-\sum d_\iota\mathscr{I}_\iota^2(1). \tag{3.4}$$

On substituting $w'(0)$ in equations (3.2) and (3.3), we get the following equations:

$$w'(t)\approx\sum d_\iota\mathscr{I}_\iota^1(t)+\Lambda-\sum d_\iota\mathscr{I}_\iota^2(1), \tag{3.5}$$
$$w(t)\approx\sum d_\iota\mathscr{I}_\iota^2(t)+t(\Lambda-\sum d_\iota\mathscr{I}_\iota^2(1))+w(0), \tag{3.6}$$

where $\Lambda=w(1)-w(0)$.

Replace t by $(t-\pi_3(t,w(t)))$, $(t-\pi_2(t,w(t)))$ and $(t-\pi_1(t,w(t)))$ in equations (3.1), (3.5) and (3.6) respectively, we get

$$w''(t-\pi_3(t,w(t)))\approx\sum d_\iota\mathcal{L}_\iota(t-\pi_3(t,w(t))), \tag{3.7}$$
$$w'(t-\pi_2(t,w(t)))\approx\sum d_\iota\mathscr{I}_\iota^1(t-\pi_2(t,w(t)))+w'(0), \tag{3.8}$$
$$w(t-\pi_1(t,w(t)))\approx\sum d_\iota\mathscr{I}_\iota^2(t-\pi_1(t,w(t)))+(t-\pi_1(t,w(t)))$$
$$\times(\Lambda-\sum d_\iota\mathscr{I}_\iota^2(1))+w(0). \tag{3.9}$$

On substituting the values from equations (3.1) and ((3.5)-(3.9)), in equation (1.1) we get the following system of equation:

$$\begin{aligned}\sum d_\imath \mathcal{L}_\imath \approx & f(t,\sum d_\imath \mathscr{I}_\imath^2(t)+t(\Lambda-\sum d_\imath \mathscr{I}_\imath^2(1))+w(0),\sum d_\imath \mathscr{I}_\imath^2(t-\pi_1(t,w(t)))\\ &+(t-\pi_1(t,w(t)))\times(\Lambda-\sum d_\imath \mathscr{I}_\imath^2(1))+w(0),\sum d_\imath \mathscr{I}_\imath^1(t)\\ &+\Lambda-\sum d_\imath \mathscr{I}_\imath^2(1),\sum d_\imath \mathscr{I}_\imath^1(t-\pi_2(t,w(t)))\\ &+\Lambda-\sum d_\imath \mathscr{I}_\imath^2(1),\sum d_\imath \mathcal{L}_\imath(t),\sum d_\imath \mathcal{L}_\imath(t-\pi_3(t,w(t)))). \end{aligned} \tag{3.10}$$

We determine the Legendre wavelet coefficients by solving the above system of equations. Then, in order to obtain the approximate solution, we put the values of these coefficients into equation (3.6). While dealing with nonlinear NDDE, we employ Newton's method to solve the resulting system.

4. Convergence Analysis

The convergence analysis of Legendre wavelet basis is covered in this section.

To demonstrate the convergence analysis of the proposed method we use the analytic version of equation (3.6) as

$$w(t)=\sum_{\imath=1}^{\infty} d_\imath \mathscr{I}_\imath^2(t)+t(\Lambda-\sum_{\imath=1}^{\infty} d_\imath \mathscr{I}_\imath^2(1))+w(0). \tag{4.1}$$

Theorem 1. *Let $w(t)\in\mathscr{L}^2[0,1]$ be such that $\left|w''(t)\right|\le\alpha_0,\forall\ t\in(0,1)$ and $\alpha_0>0$. If $w''(t)=\sum_i^\infty d_\imath \mathcal{L}_\imath(t)$, then we have the following inequality:*

$$\left|d_\imath\right|\le 2^{-\frac{k}{2}}\sigma\alpha_0\wp, \tag{4.2}$$

where $\sigma=\sqrt{m+\frac{1}{2}}$ and $\wp$ is constant defined in the proof below.

Proof. We have

$$w''(t)=\sum_{\imath=1}^{\infty} d_\imath \mathcal{L}_\imath(t), \tag{4.3}$$

$$\left|d_\imath\right|=\left|\int_0^1 w''(t)\mathcal{L}_\imath(t)dt\right|\le\sup_{t\in[0,1]}\left|w''(t)\right|\int_0^t\left|\mathcal{L}_\imath(t)\right|dt\le\alpha_0 2^{-\frac{k}{2}}\sigma\wp.$$

Taking inner product of equation (4.3) and applying orthonormality condition of $\mathcal{L}_\imath(t)$, we get equation (4.2). We have applied mean value theorem for integral and $\wp=\frac{\int_{-1}^1|\mathscr{P}'_{m+1}(t)-\mathscr{P}'_{m-1}(t)|dt}{\sqrt{2m+1}}$. Therefore, we have

$$|d_\imath|\le 2^{-\frac{k}{2}}\sigma\alpha_0\wp. \tag{4.4}$$

□

Theorem 2. *Let the analytic and approximate solution of equation (1.1) is denoted by $w(t)$ and $P_{\mathcal{V}^K}w(t)$ respectively and $w(t) \in \mathscr{L}^2[0,1], |w''(t)| \leq \alpha_0, \forall\ t \in (0,1)$ with $\alpha_0 > 0$. If ε_K is the error of the approximation then we have following inequality:*

$$\left\|\varepsilon_K\right\|_2 \leq \frac{1}{3}\rho^2\alpha_0\Im^2(\mathfrak{F}-\lambda)2^{-K+1}.$$

Proof. Suppose $w(t) \in \mathscr{L}^2[0,1]$ be the analytical solution of (1.1) and $P_{\mathcal{V}^K}w(t)$ is the projection of $w(t)$ on the multiresolution space $\mathcal{V}^K$. To obtain the error norm, we calculate the $\mathscr{L}^2$ norm of the difference of exact solution $w(t)$ and projection $P_{\mathcal{V}^K}w(t)$ of it. Detailed description of determining the error norm's upper bound is given below: We have

$$\begin{aligned}\left\|\varepsilon_K\right\|_2 &= \left\|w(t) - P_{\nu^K}w(t)\right\|_2\\ &= \Big\|\sum_{k=K+1}^{\infty}\sum_{\iota=2^K}^{2^{K+1}-1} d_\iota \mathscr{I}_\iota^2(t) - \sum_{k=K+1}^{\infty}\sum_{\iota=2^K}^{2^{K+1}-1} d_\iota \mathscr{I}_\iota^2(1)\Big\|_2.\end{aligned}$$

Using Minskowski inequality, we get

$$\begin{aligned}\left\|\varepsilon_K\right\|_2 &\leq \Big\|\sum_{k=K+1}^{\infty}\sum_{\iota=2^K}^{2^{K+1}-1} d_\iota \mathscr{I}_\iota^2(t)\Big\|_2 + \Big\|\sum_{k=K+1}^{\infty}\sum_{\iota=2^K}^{2^{K+1}-1} d_\iota \mathscr{I}_\iota^2(1)\Big\|_2\\ &\leq \Big\|\sum_{k=K+1}^{\infty}\sum_{\iota=2^K}^{2^{K+1}-1} d_\iota \mathscr{I}_\iota^2(t)\Big\|_2 + \Big\|\sum_{k=K+1}^{\infty}\sum_{\iota=2^K}^{2^{K+1}-1} d_\iota \mathscr{I}_\iota^2(1)\Big\|_2 \qquad (4.5)\\ &\leq \sum_{k=K+1}^{\infty}\sum_{\iota=2^K}^{2^{K+1}-1} |d_\iota| \left(\int_0^1 |\mathscr{I}_\iota^2(t)|^2 dt\right)^{\frac{1}{2}} + \sum_{k=K+1}^{\infty}\sum_{\iota=2^K}^{2^{K+1}-1} |d_\iota|\\ &\left(\int_0^1 |\mathscr{I}_\iota^2(1)|^2 dt\right)^{\frac{1}{2}}. \qquad (4.6)\end{aligned}$$

Now, consider the Cauchy's formula for repeating integration as

$$I^n \mathcal{L}_\iota(t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}\mathcal{L}_\iota(s)ds. \qquad (4.7)$$

For $n = 2$, we get

$$I^2 \mathcal{L}_\iota(t) = \mathscr{I}_\iota^2(t) = \int_0^t (t-s)^2 \mathcal{L}_\iota(s)ds.$$

Now,

$$\begin{aligned}I^2\mathcal{L}_\iota(t) &= \int_{\nu_1}^{\nu_2}(t-s)\rho 2^{\frac{k}{2}}\mathcal{P}_m(2^k s-\lambda)ds = \int_{-1}^{1}(t-\frac{\varsigma+\lambda}{2^k})\rho 2^{\frac{k}{2}}2^{-k}\mathcal{P}_m(\varsigma)d\varsigma\\ &= \int_{-1}^{1}\frac{(2^k t-\varsigma-\lambda)}{2^k}\rho 2^{-\frac{k}{2}}\mathcal{P}_m(\varsigma)d\varsigma\end{aligned}$$

$$\leq 2^{\frac{-3k}{2}}\rho \frac{\max\limits_{-1\leq t\leq 1}|\mathcal{P}'_{m+1}(t)-\mathcal{P}'_{m-1}(t)|}{2m+1}\int_{-1}^{1}(2^k t-\varsigma-\lambda)d\varsigma$$

$$|I^2\mathcal{L}_\imath(t)| = |\mathscr{I}_\imath^2(t)| \leq 2^{\frac{-3k+2}{2}}\rho\mathfrak{I}\mathfrak{F}, \tag{4.8}$$

where $\varsigma = 2^k s-\lambda$, $\mathfrak{I} = \frac{\max\limits_{-1\leq t\leq 1}|\mathcal{P}'_{m+1}(t)-\mathcal{P}'_{m-1}(t)|}{2m+1}$ and $\mathfrak{F} = \max\limits_{t\in[0,1]}|(2^k t-\lambda)|$.

In similar fashion, we can get

$$|I^2\mathcal{L}_\imath(1)| = |\mathscr{I}_\imath^2(1)| \leq 2^{\frac{-3k+2}{2}}(2^k-\lambda)\rho\mathfrak{I}, \tag{4.9}$$

where $\mathfrak{I} = \frac{\max\limits_{-1\leq t\leq 1}|\mathcal{P}'_{m+1}(t)-\mathcal{P}'_{m-1}(t)|}{2m+1}$. On substituting equations (4.2), (4.8) and (4.9) in equation (4.6) and after simplification, we get

$$\left\|\varepsilon_K\right\|_2 \leq \sum_{k=K+1}^{\infty}\sum_{\imath=2^K}^{2^{K+1}-1} 2^{-2k+1}\rho^2\alpha_0\mathfrak{I}^2\left(\mathfrak{F}+2^k-\lambda\right)$$

$$\leq \sum_{k=K+1}^{\infty} 2^{-2k+1}\left\{\rho^2\alpha_0\mathfrak{I}^2\left(\mathfrak{F}+2^k-\lambda\right)\right\}(2^{K+1}-2^K)$$

$$\leq 2^K\sum_{k=K+1}^{\infty} 2^{-2k+1}\rho^2\alpha_0\mathfrak{I}^2\left(\mathfrak{F}+2^k-\lambda\right)$$

$$\left\|\varepsilon_K\right\|_2 \leq \frac{1}{3}\rho^2\alpha_0\mathfrak{I}^2(\mathfrak{F}-\lambda)2^{-K+1}. \tag{4.10}$$

□

From equation (4.10) we can say that the error and the resolution level K are related to each other inversely, which implies that as $K\to\infty$, then $\left\|\varepsilon_K\right\|\to 0$.

5. Numerical Examples

Problem 1. *Let*

$$w''(t)+w'(t)+\sqrt{\cos t}\,w'(\sqrt{t})+(\sin(\sqrt{t})+e^t)w(\sin t) = f(t), \tag{5.1}$$

which satisfies the BCs:

$$w(0)=1, \qquad w(1)=e.$$

The exact solution is

$$w(t)=e^t. \tag{5.2}$$

The source function $f(t)$ can be calculated with the help of exact solution.

We have applied Legendre wavelet series method (LWSM) for solving equation (5.1). The obtained maximum absolute errors (MAE) and CPU time for various convergence parameters $(M,\ K)$ are reported in Table 1. The table shows clearly that as we increase the values of the convergence parameters $(M,\ K)$, the MAE decreases gradually and CPU time increases at a low rate. A comparison of exact solution with LWSM solution is displayed in Figure 1.

TABLE 1. MAE with CPU time of Problem 1

(M,K)	Exact solution	MAE	CPU time (seconds)
$(8,1)$	2.5536	$2.2662e-12$	0.1180
$(8,2)$	2.6346	$3.6326e-13$	0.2442
$(8,3)$	2.6761	$1.7764e-15$	0.7261
$(8,4)$	2.6971	$8.8818e-16$	2.5288

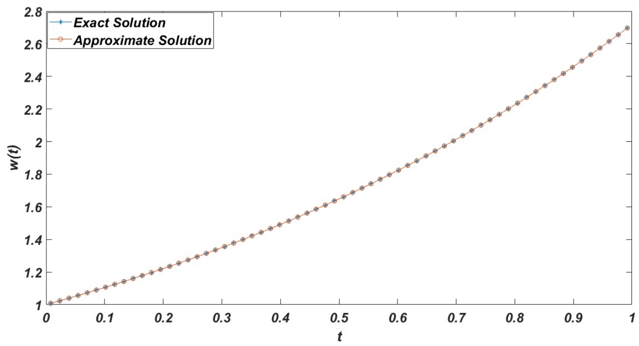

FIGURE 1. Graph of exact and LWSM solutions of Problem 1 for $M=8$ and $K=4$

Problem 2. *Let*

$$w''(t) = -2e^{-t} + \frac{w(t)}{2} + e^{\frac{-t}{2}} w\left(\frac{t}{2}\right), \qquad t \in [0,1], \tag{5.3}$$

which satisfies the BCs:

$$w(0) = 0, \qquad w(1) = e^{-1}.$$

The exact solution is

$$w(t) = te^{-t}. \tag{5.4}$$

The source function $f(t)$ can be calculated with the help of the exact solution.

We have applied LWSM for solving equation (5.3). The obtained MAE and CPU time for various convergence parameters $(M,\ K)$ are reported in Table 2. The table shows clearly that as we increase the values of the convergence parameters $(M,\ K)$,the MAE decreases gradually and CPU time increases at a low rate. A comparison of exact solution with LWSM solution is displayed in Figure 2.

TABLE 2. MAE with CPU time of Problem 2

(M,K)	Exact solution	MAE	CPU time (seconds)
$(8,1)$	0.3671	$7.2649e-12$	0.1072
$(8,2)$	0.3677	$1.0288e-12$	0.1696
$(8,3)$	0.3678	$4.0523e-15$	0.4781
$(8,4)$	0.3679	$1.6653e-16$	1.6053

From Table 2, we can see that the MAE of the developed method is $1.0288e-012$ where as the MAE in [15] is $2.3421e-08$, so we can say that the developed method is more accurate.

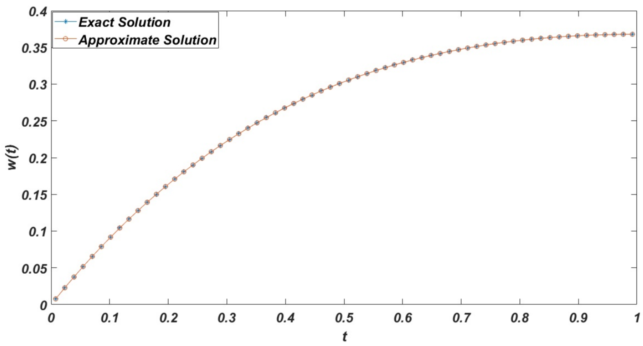

FIGURE 2. Graph of exact and LWSM solutions of Problem 2 for $M=8$ and $K=4$

Problem 3. *Let*

$$w(t)w''(t)+w'(t)+\sqrt{\cos t}w'(t)(\sqrt{t})+\sin(\sqrt{t})+e^t w(\sin t)=f(t), \tag{5.5}$$

which satisfies the BCs:

$$w(0)=1, \qquad w(1)=e.$$

The exact solution is

$$w(t)=e^t. \tag{5.6}$$

The source function $f(t)$ can be calculated with the help of exact solution.

We have applied LWSM for solving equation (5.5). The obtained MAE and CPU time for various convergence parameters $(M,\ K)$ are reported in Table 3. The table shows clearly that as we increase the values of the convergence parameters $(M,\ K)$,the MAE decreases gradually and CPU time increases at a low rate. A comparison of exact solution with LWSM solution is displayed in Figure 3.

TABLE 3. MAE with CPU time of Problem 3

(M,K)	Exact solution	MAE	CPU time (seconds)
$(8,1)$	2.5536	$2.4698e-12$	0.2279
$(8,2)$	2.6346	$4.0767e-13$	0.3362
$(8,3)$	2.6761	$2.2204e-15$	0.7627
$(8,4)$	2.6971	$8.8818e-16$	2.4628

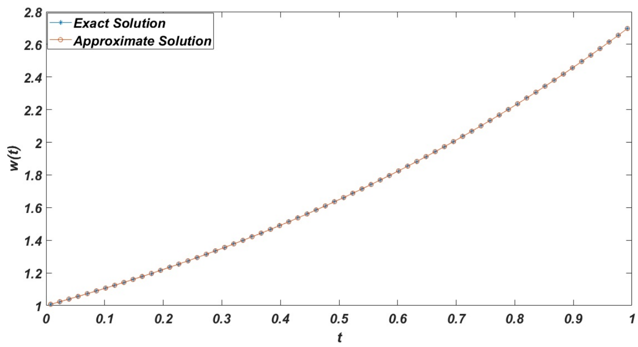

FIGURE 3. Graph of exact and LWSM solutions of Problem 3 for $M = 8$ and $K = 4$

CONCLUSION

In this paper, we have used LWSM to get the numerical solution of second order NDDEs. We have employed integral operator technique, i.e., the highest order derivative is approximated in terms of Legendre wavelet basis and then the integrations of Legendre wavelet are used to approximate the unknown variable and its lower order derivatives. The advantage of this technique is that we don't need to deal with the BCs separately. These conditions are automatically taken into consideration. We approximate delay term directly by using Legendre wavelet, and not by any series expansion such as Taylor series. In addition, the proposed method converges significantly fast as compared to previous method such as direct block method [15] and is easy to execute. For all computational work we have used MATLAB 2021, intel i5 and windows 10.

ACKNOWLEDGEMENT

The authors are thankful to anonymous reviewer(s) for their fruitful suggestions, which greatly improved the quality and presentation of the paper. The first author is also thankful to the Council of Scientific and Industrial Research (CSIR), Govt. of India for providing Junior Research Fellowship with grant number 09/0466 (12384)/2021-EMR-I.

REFERENCES

[1] M. P. Alam, T. Begum, and A. Khan, "A new spline algorithm for solving non-isothermal reaction diffusion model equations in a spherical catalyst and spherical biocatalyst," *Chemical Physics Letters*, vol. 754, p. 137651, 2020, doi: 10.1016/j.cplett.2020.137651.

[2] M. P. Alam, T. Begum, and A. Khan, "A high-order numerical algorithm for solving Lane–Emden equations with various types of boundary conditions," *Computational and Applied Mathematics*, vol. 40, no. 6, pp. 1–28, 2021, doi: 10.1007/s40314-021-01591-7.

[3] M. P. Alam, A. Khan, and D. Baleanu, "A high-order unconditionally stable numerical method for a class of multi-term time-fractional diffusion equation arising in the solute transport models," *International Journal of Computer Mathematics*, pp. 1–28, 2022, doi: 10.1080/00207160.2022.2082248.

[4] M. P. Alam, D. Kumar, and A. Khan, "Trigonometric quintic B-spline collocation method for singularly perturbed turning point boundary value problems," *International Journal of Computer Mathematics*, vol. 98, no. 5, pp. 1029–1048, 2021, doi: 10.1080/00207160.2020.1802016.

[5] E. Babolian and F. Fattahzadeh, "Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration," *Applied Mathematics and Computation*, vol. 188, no. 1, pp. 417–426, 2007, doi: 10.1016/j.amc.2006.10.008.

[6] V. L. Bakke and Z. Jackiewicz, "The numerical solution of boundary-value problems for differential equations with state dependent deviating arguments," *Aplikace matematiky*, vol. 34, no. 1, pp. 1–17, 1989, doi: 10.21136/am.1989.104330.

[7] A. Boutiara, M. M. Matar, M. K. Kaabar, F. Martínez, S. Etemad, and S. Rezapour, "Some qualitative analyses of neutral functional delay differential equation with generalized caputo operator," *Journal of Function Spaces*, vol. 2021, 2021, doi: 10.1155/2021/9993177.

[8] B. Cahlon and L. J. Nachman, "Numerical methods for discontinuous linear boundary value problems with deviation arguments," *Journal of Mathematical Analysis and Applications*, vol. 154, no. 2, pp. 529–542, 1991, doi: 10.1016/0022-247X(91)90056-6.

[9] C. K. Chui, *An introduction to wavelets*. Texas: Academic press, 1992, vol. 1.

[10] C. W. Cryer, "The numerical solution of boundary value problems for second order functional differential equations by finite differences," *Numerische Mathematik*, vol. 20, no. 4, pp. 288–299, 1972, doi: 10.1007/BF01407371.

[11] K. de Nevers and K. Schmitt, "An application of the shooting method to boundary value problems for second order delay equations," *Journal of Mathematical Analysis and Applications*, vol. 36, no. 3, pp. 588–597, 1971, doi: 10.1016/0022-247X(71)90041-2.

[12] L. Debnath and F. A. Shah, *Wavelet transforms and their applications*. Springer, 2002.

[13] M. Faheem, A. Khan, and A. Raza, "A high resolution hermite wavelet technique for solving space–time-fractional partial differential equations," *Mathematics and Computers in Simulation*, vol. 194, pp. 588–609, 2022, doi: 10.1016/j.matcom.2021.12.012.

[14] M. Faheem, A. Raza, and A. Khan, "Wavelet collocation methods for solving neutral delay differential equations," *International Journal of Nonlinear Sciences and Numerical Simulation*, 2021, doi: 10.1515/ijnsns-2020-0103.

[15] N. T. Jaaffar, Z. Abdul Majid, and N. Senu, "Numerical approach for solving delay differential equations with boundary conditions," *Mathematics*, vol. 8, no. 7, p. 1073, 2020, doi: 10.3390/math8071073.

[16] S. Palaniswami, "Existence of asymptotic solutions of second order neutral differential equation with multiple delays," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 8, no. 1, pp. 77–83, 1995, doi: 10.1155/S1048953395000074.

[17] E. Piskin and H. Yüksekkaya, "Nonexistence of global solutions of a delayed wave equation with variable- exponents," *Miskolc Mathematical Notes*, vol. 22, no. 2, pp. 841–859, 2021, doi: 10.18514/MMN.2021.3487.

[18] K. Ramkumar, K. Ravikumar, A. Anguraj, and H. M. Ahmed, "Well posedness results for higher-order neutral stochastic differential equations driven by poisson jumps and rosenblatt process," *Filomat*, vol. 35, no. 2, pp. 353–365, 2021, doi: 10.2298/FIL2102353R.

[19] A. Raza and A. Khan, "Haar wavelet series solution for solving neutral delay differential equations," *Journal of King Saud University-Science*, vol. 31, no. 4, pp. 1070–1076, 2019, doi: 10.1016/j.jksus.2018.09.013.

[20] A. Raza and A. Khan, "Treatment of singularly perturbed differential equations with delay and shift using haar wavelet collocation method," *Tamkang Journal of Mathematics*, 2021, doi: 10.5556/j.tkjm.53.2022.3250.

[21] U. Saeed and M. u. Rehman, "Hermite wavelet method for fractional delay differential equations," *Journal of Difference Equations*, vol. 2014, p. 359093, 2014, doi: 10.1155/2014/359093.

[22] I. Singh and S. Kumar, "Wavelet methods for solving three-dimensional partial differential equations," *Mathematical Sciences*, vol. 11, no. 2, pp. 145–154, 2017, doi: 10.1007/s40096-017-0220-6.

[23] S. K. Vanani and A. Aminataei, "On the numerical solution of neutral delay differential equations using multiquadric approximation scheme," *Bulletin of the Korean Mathematical Society*, vol. 45, no. 4, pp. 663–670, 2008, doi: 10.4134/BKMS.2008.45.4.663.

[24] S. Venkatesh, S. Ayyaswamy, and S. R. Balachandar, "The Legendre wavelet method for solving initial value problems of Bratu-type," *Computers & Mathematics with Applications*, vol. 63, no. 8, pp. 1287–1295, 2012, doi: 10.1016/j.camwa.2011.12.069.

[25] S. A. Yousefi, "Legendre wavelets method for solving differential equations of Lane–Emden type," *Applied Mathematics and Computation*, vol. 181, no. 2, pp. 1417–1422, 2006, doi: 10.1016/j.amc.2006.02.031.

Authors' addresses

Uzair Ahmed
(**Corresponding author**) Jamia Millia Islamia, Department of Mathematics, Jamia Nagar, 110025 Delhi, India
E-mail address: `ahmeduzair571@gmail.com`

Mo Faheem
Jamia Millia Islamia, Department of Mathematics, Jamia Nagar, 110025 Delhi, India

Arshad Khan
Jamia Millia Islamia, Department of Mathematics, Jamia Nagar, 110025 Delhi, India

Geetan Manchanda
Maitreyi College, Department of Mathematics, 110021 Delhi, India