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NOTE ON BLOW UP SOLUTIONS FOR A GENERAL CLASS OF SEMILINEAR PARABOLIC EQUATIONS INVOLVING SECOND ORDER OPERATOR

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Abstract. This work deals with an initial-boundary value problem of Laplacian parabolic equation

$$\begin{aligned} (h(u))_t + \triangle_p u = f(u(x,t)), &\quad \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, &\quad \text{on } \partial\Omega \times (0,\infty), \\ u(x_0) = u_0 \geq 0, &\quad x \in \bar{\Omega}, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$. Our contribution is to give a new condition on nonlinearity to obtain the blow-up solutions of the above equations.

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1. INTRODUCTION

Over the past few years, a blow-up phenomenon of global solutions for non linear reaction diffusion equations and systems has been extensively investigated by many authors. These authors studied the questions of global existence, blow-up at some finite time, blow-up rate, blow-up set, asymptotic behavior for solutions and so on as well as a variety of methods used to research these questions. Particularly, the problems of the blow-up and global solutions for non linear reaction diffusion equations under Dirichlet boundary conditions have been considered in [1, 4, 5, 7–9, 12, 13]. However, due to the explosive nature of the solutions, it is very important in applications to determine lower bounds on the blow-up time. Presently, the research on the lower bound of the blow-up time for the non local problems with Dirichlet or Neumann boundary condition had some new progress. We provide the reader to the literature [3, 6] and [11], for some recent interesting research on the local reaction-diffusion equation with non local boundary conditions we can refer to [9].

Inspired by the above mentioned papers, especially from [3] and [13], we discuss the blow-up solutions of the following initial-boundary value problem of p-Laplacian parabolic equation:

$$\begin{aligned} (h(u))_t + \triangle_p u &= f(u(x,t)), && \text{in } \Omega\times(0,\infty),\\ u(x,t) &= 0, && \text{on } \partial\Omega\times(0,\infty),\\ u(x_0) &= u_0 \geq 0, && x\in\bar{\Omega}, \end{aligned} \tag{1.1}$$

where $h(\tau), f(\tau) \in C^\infty(\mathbb{R}^+), h(\tau) > 0, h^{'}(\tau) \geq 1$ and $f(\tau) > 0$ for $\tau > 0$.

We borrow some ideas from the work [3] and we extend them to more general parabolic problem which involves the $p-$Laplacian operator.

The papers [4, 8], and [9] studied the special cases of problem (1.1). In [9], the following problem was considered by Payne and Schaefe:

$$\begin{aligned} u_t &= \Delta_p u + f(u(x,t)), && \text{in } \Omega\times(0,\infty),\\ u(x,t) &= 0, && \text{on } \partial\Omega\times(0,\infty),\\ u(x_0) &= u_0 \geq 0, && x\in\bar{\Omega}, \end{aligned}$$

where Ω is a bounded domain on $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. They used a differential inequality technique and a comparison principle to obtain a lower bound on blow-up time when blow-up occurs.

Little later, Payne, Philippin and Schaefer investigated the problem in [8]:

$$\begin{aligned} u_t &= \nabla.(\rho(|\nabla u|^2)\nabla u)u + f(u), && \text{in } \Omega\times(0,\infty),\\ u(x,t) &= 0, && \text{on } \partial\Omega\times(0,\infty),\\ u(x_0) &= u_0 \geq 0, && x\in\bar{\Omega}, \end{aligned}$$

where Ω is a bounded domain on $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Under appropriate assumptions on the functions f, ρ and u_0, a lower bound on blow-up time was showed by applying a differential technique when blow-up does occur. Moreover, a criterion for blow-up and conditions which ensure that blow-up cannot occur were also obtained. Finally, the following problem was studied by Payne and Philippin in [4]:

$$\begin{aligned} u_t &= \triangle u + K(t) f(u(x,t)), && \text{in } \Omega\times(0,\infty),\\ u(x,t) &= 0, && \text{on } \partial\Omega\times(0,\infty),\\ u(x_0) &= u_0 \geq 0, && x\in\bar{\Omega}, \end{aligned}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $(N \geq 2)$ with smooth boundary $\partial\Omega$.

A first-order differential inequality technique and Sobolev inequality were used to give the sufficient conditions which guarantee the blow-up or the global existence of the solution. In addition, lower and upper bounds on blow-up time were also derived.

We proceed as follows. In Section 2 we establish a sufficient condition on the data of problem (1.1) to guarantee the blow-up of the solution and obtain an upper bound on blow-up time by a new condition noted (C_p).

We introduce a condition sufficient to ensure the solution blows up at some finite time and obtain an upper bound on blow-up time.

$$(C_p) \quad \lambda F(u) \le uf(u) + \mu u^p + \delta, \quad u > 0,$$

where $0 < \mu \le \frac{\lambda - 2}{2}\lambda_{1,p}$, and $\lambda_{1,p}$, is the first eigenvalue of the p-Laplacian $\triangle_p$. Blow-up phenomena for this kind of problems in bounded domains have been extensively studied. Concavity method has been used so far to derive the blow-up solutions for some variants of the equations (1.1), see [1, 6, 8, 13]. By using the concavity method, Philippin and Proytcheva, in [10], obtained the blow-up solutions for problem (1.1) (with $h = id$) under the conditions:

$$(C^*) \quad (2+\varepsilon)F(u) \le uf(u), \quad u > 0, \varepsilon > 0.$$

We point out that the used condition (C_p) is more weaker than (C^*), so it is more interesting. Meanwhile, local existence of classical solutions (or weak solutions) to such problems have been also established by many works. Since that not all solutions of these equations exist for all time, so many authors deal with the sufficient conditions for the local existence of solutions see for example [2–4] .

2. Main result and proofs

Let consider the functional H such that:

$$H(t) = \int_0^t h^{'}(\tau)\tau d\tau.$$

Through this paper, we suppose that:

$$(H_0) \quad H(t) \ge h^{'}(t)t, \text{ for } t > 0.$$

In this section, our main result can be read as follows:

Theorem 1. *Under the hypothesis (H_0), let a function f satisfy the condition (C_p) and $p > 2$. If the initial data $u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ such that*

$$-\frac{1}{p}\int_\Omega |\bigtriangledown u_0(x)|^p dx + \int_\Omega [F(u_0(x)) - \delta]\, dx \ge 0, \tag{2.1}$$

then the nonnegative weak solution to the problem (1.1) blows up at finite time T^*, i.e

$$\lim_{t \mapsto T^*} \int_0^t \int_\Omega u^2(x,s)dx = +\infty,$$

where $s \ge 2$ and δ is the constant in the condition (C_p).

Consider the Sobolev space $W_0^{1,p}(\Omega)$, which is the closure of C^∞ functions compactly supported in Ω for the norm:

$$||u||_{1,p}^p = \int_\Omega |u(x)|^p dx + \int_\Omega |\bigtriangledown u(x)|^p dx.$$

The p-Laplacian operator is defined by

$$\begin{aligned} \Delta_p : W_0^{1,p}(\Omega) &\longrightarrow W^{-1,q}(\Omega) \\ u &\longmapsto \Delta_p(u) = div(|\nabla u|^{p-2}\nabla u), \end{aligned}$$

where $W^{-1,q}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$ and we have

$$1 < p,\ q < \infty,\ \frac{1}{p} + \frac{1}{q} = 1.$$

Let u be a function of $W_0^{1,p}(\Omega)$, not identically 0. Through the above definitions, the function u is called an eigenfunction if

$$\int_\Omega |\nabla u(x)|^{p-2} \nabla u \nabla \phi dx = \lambda \int_\Omega |u(x)|^{p-2} u \phi dx \ \text{ for all } \ \phi \in C_0^\infty(\Omega).$$

The corresponding real number λ is called an eigenvalue.

Lemma 1 ([4]). *For $1 < P < \infty$, there exist $\lambda_{1,p} > 0$ and an eigenfunction $e_{1,p} \in W_0^{1,p}(\Omega)$ such that*

$$\begin{aligned} \Delta_p e_{1,p} &= \lambda_{1,p} e_{1,p} \\ e_{1,p}(x) &= 0, \quad x \in \partial\Omega. \end{aligned}$$

Moreover, $\lambda_{1,p}$ is given by:

$$\lambda_{1,p} = \inf_{W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla v|^p}{\int_\Omega |v|^p} > 0.$$

Proof of Theorem 1.

$$\begin{aligned} \int_0^t \int_\Omega (u_t^2(x,s))_t dx ds \leq -\frac{1}{p} \int_\Omega [|\nabla u(x,t)|^p - |\bigtriangledown u_0(x)|^p] dx \\ + \int_\Omega [F(u(x,t)) - F(u_0(x))] dx. \end{aligned} \tag{2.2}$$

We define a function J_p by:

$$J_p(t) = -\frac{1}{p} \int_\Omega |\bigtriangledown u(x,t)|^p dx + \int_\Omega [F(u(x,t)) - \delta] dx.$$

Then it follows from (2.2) that

$$J_p(t) = -\frac{1}{p} \int_\Omega |\nabla u(x,t)|^p dx + \int_\Omega [F(u(x,t)) - \delta] dx$$

$$\geq -\frac{1}{p}\int_{\Omega}|\nabla u_0|^p dx+\int_{\Omega}[F(u_0)-\delta]dx+\int_0^t\int_{\Omega}h^{'}(u)u_t^2(x,s)dxds$$
$$=J_p(0)+\int_0^t\int_{\Omega}h^{'}(u)u_t^2(x,s)dxds.$$

On the other hand, we define a function by

$$\psi_p(t)=\int_0^t\int_{\Omega}H(u(x,s))dxds+M,\ \ t\geq 0,$$

where $M>0$ is a constant to be determined later. Accordingly, by virtue the definition of the functional H and the fact that $du=\frac{\partial u}{\partial t}dt$, it yields

$$\psi_p^{'}(t)=\int_{\Omega}H(u(x,s))dx=\int_{\Omega}\int_0^t h^{'}(u)u_t u ds dx+\int_{\Omega}H(u_0)dx$$
$$=\int_0^t\int_{\Omega}(H(u(x,t)))_t dx. \tag{2.3}$$

Then

$$\psi_p^{''}(t)=\int_{\Omega}H(u(x,s))_t dx$$
$$=\int_{\Omega}|\Delta_p u|u-f(u)udx$$
$$=\int_{\Omega}-|\nabla u|^p dx+\int_{\partial\Omega}\frac{\partial u}{\partial \nu}udS+\int_{\Omega}f(u(x,s))u(x,s)dx$$
$$=\int_{\Omega}-|\nabla u|^p dx+\int_{\Omega}f(u(x,s))u(x,s)dx.$$

By using the condition (C_p), Lemma 1, and (2.3) in turn, we obtain that

$$\psi^{''}(t)\geq -\int_{\Omega}|\nabla u|^p dx+\int_{\Omega}[\lambda F(u)-\beta u^p-\lambda\delta]dx$$
$$=\lambda J_p(t)+\frac{\lambda-p}{p}[-\int_{\Omega}|\nabla u|^p dx-\beta\int_{\Omega}u^p(x,t)pdx$$
$$\geq \lambda J_p(t)+\left[\frac{(\lambda-p)\lambda_{1,p}}{p}-\beta\right]\int_{\Omega}u^p dx$$
$$\geq \lambda J_p(t)$$
$$\geq \lambda\left[J_p(0)+\int_0^t\int_{\Omega}h^{'}(u)u_t^2(x,s)dxds\right]. \tag{2.4}$$

Applying the Schwartz inequality, we obtain that

$$(\psi')^2(t)\leq 4(1+\delta)\left(\int_{\Omega}\int_0^t h^{'}(u)uu_t(x,s)dsdx)^2+(1+\frac{1}{\delta})(H(u_0(x))dx\right)^2$$
$$\leq 4(1+\delta)\left[\int_{\Omega}(\int_0^t(h^{'}(u)u)^2ds)^{\frac{1}{2}}(\int_0^t u_t^2(x,s)ds)^{\frac{1}{2}}\right]^2+(1+\frac{1}{\delta})\left[H(u_0(x))dx\right]^2$$

$$\leq 4(1+\delta)\left(\int_{\Omega}\int_{0}^{t}(h^{'}(u)u)^{2}dsdx\right)\left(\int_{\Omega}\int_{0}^{t}u_{t}^{2}(x,s)dsdx\right)$$
$$+(1+\frac{1}{\delta})\left[H(u_{0}(x))dx\right]^{2}, \tag{2.5}$$

where $\delta > 0$ is arbitrary. Combining the above estimates (2.4) and (2.5), we obtain that

$$\psi_{p}^{''}(t)\psi_{p}(t)-(1+\sigma)\psi_{p}^{'}(t)^{2} > 0$$

by choosing $\sigma = \delta = \sqrt{\frac{\lambda}{2}} - 1 > 0$ and M is large enough. This means that the solutions u blow up in finite time T^*. □

Remark 1. By the same argument with slight changes, we may prove that our result is still available for the following problem

$$\begin{aligned} (h(u))_t + \triangle u &= f(u(x,t)), && \text{in } \Omega\times(0,\infty), \\ u(x,t) &= 0, && \text{on } \partial\Omega\times(0,\infty), \\ u(x_0) &= u_0 \geq 0, && x\in\bar{\Omega}, \end{aligned}$$

In fact, the above theorem can work for the case $p = 2$.

Example 1. For $p \geq 2$, where the functional $h = id$ and $f(u(x,t)) = u^p$, the weak solution to the problem (1.1) blows up at finite time T^* provided

$$-\int_{\Omega}|\bigtriangledown u_0(x)|^p dx + \int_{\Omega}\left[|u_0(x)|^p - \delta\right]dx \geq 0.$$

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REFERENCES

[1] A. M. Alghamd, S. Gala, and M. A. Ragusa , "Global regularity for the 3D micropolar fluid flows." *Filomat*, vol. 36, no. 6, pp. 1967–1970, 2022, doi: 10.2298/FIL2206967A.

[2] J. M. Ball, "Remarks on blow up and nonexistence theorems for nonlinear evolution equations." *The Quarterly Journal of Mathematics*, vol. 28, no. 4, pp. 473–486, 1977, doi: 10.1093/qmath/28.4.473.

[3] S.-Y. Chung and M.-J. Choi, "A new condition for the concavity method of blow-up solutions to $p-$Laplacian parabolic equations." *J. Differential Equations.*, vol. 265, no. 12, pp. 6384–6399, 2018, doi: 10.1016/j.jde.2018.07.032.

[4] S.-Y. Chung, M.-J. Choi, and J. Hwang, "A condition for blow-up solutions to discrete p-Laplacian parabolic equations under the mixed boundary conditions on networks." *Boundary Value Problems.*, 2019, article number: 180, doi: 10.1186/s13661-019-01294-3.

[5] J. Ding, "Blow-up solutions for a class of nonlinear parabolic equations with Dirichlet boundary conditions." *Nonlinear Analysis*, vol. 52, no. 6, pp. 1645–1654, 2003, doi: 10.1016/S0362-546X(02)00277-8.

[6] J. Ding, "Global existence and blow-up for a class of nonlinear reaction diffusion problems." *Boundary Value Problems*, 2014, article number: 168, doi: 10.1186/s13661-014-0168-5.

[7] L. Payne and J. chul Song, "Lower bounds for blow-up time in a nonlinear parabolic problem." *Journal of Mathematical Analysis and Applications*, vol. 354, no. 1, pp. 394–396, 2009, doi: 10.1016/j.jmaa.2009.01.010.

[8] L. Payne, G. A. Philippin, and P. W. Schaefer, "Blow-up phenomena for some nonlinear parabolic problems." *Nonlinear Analysis*, vol. 69, no. 10, pp. 3495–3502, 2008, doi: 10.1016/j.na.2007.09.035.

[9] L. Payne and P. W. Schaefer, " Lower bounds for blow-up time in parabolic problems under Dirichlet conditions." *Journal of Mathematical Analysis and Applications*, vol. 328, no. 2, pp. 1196–1205, 2007, doi: 10.1016/j.jmaa.2006.06.015.

[10] G. A. Philippin and V. Proytcheva, "Some remarks on the asymptotic behaviour of the solutions of a class of parabolic problems." *Mathematical Methods in the Applied Sciences*, vol. 29, no. 3, pp. 297–307, 2006, doi: 10.1002/mma.679.

[11] M. Protter and H. Weinberger, *Maximum Principles in Differential Equations*. Springer, 1967.

[12] M. A. Ragusa, "Dirichlet problem associated to divergence form parabolic equations." *Communications in Contemporary Mathematics*, vol. 06, no. 03, pp. 377–393, 2004, doi: 10.1142/S0219199704001392.

[13] X. Zhang and Z. Cui, "Blow-up results for a class of quasilinear parabolic equation with power nonlinearity and nonlocal source." *Journal of Function Spaces*, vol. 2021, 2021, article ID 2208818, doi: 10.1155/2021/2208818.

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