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TWO-POINT BOUNDARY VALUE PROBLEMS FOR 4TH ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The new optimal efficient sufficient conditions are established for solvability and uniqueness of a solution of the linear and nonlinear fourth order ordinary differential equations

$$u^{(4)}(t) = p(t)u(t) + q(t) \quad \text{for} \quad t \in [a, b],$$

$$u^{(4)}(t) = p(t)u(t) + f(t, u(t)) \quad \text{for} \quad t \in [a, b],$$

under the following two-point boundary conditions

$$u^{(i)}(a) = 0, \quad u^{(i)}(b) = 0 \ \ (i = 0, 1),$$

and

$$u^{(i)}(a) = 0 \ \ (i = 0, 1, 2), \quad u(b) = 0,$$

where $p \in L([a, b]; \mathbb{R})$ is a nonconstant sign function and $f \in K([a, b] \times \mathbb{R}; \mathbb{R})$.

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1. Introduction

Consider on the interval $I := [a, b]$ the fourth order ordinary differential equations

$$u^{(4)}(t) = p(t)u(t) + q(t), \tag{1.1}$$

and

$$u^{(4)}(t) = p(t)u(t) + f(t, u(t)), \tag{1.2}$$

under the boundary conditions

$$u^{(j)}(a) = 0, \qquad u^{(j)}(b) = 0 \ (j = 0, 1), \tag{1.3$_1$}$$

$$u^{(j)}(a) = 0 \ (j = 0, 1, 2), \qquad u(b) = 0, \tag{1.3$_2$}$$

where $p \in L(I; \mathbb{R}), f \in K(I \times \mathbb{R}; \mathbb{R})$.

By a solution of problem (1.2), (1.3_1), (1.3_2) we understand a function $u \in \widetilde{C}^3(I;\mathbb{R})$, which satisfies equation (1.2) a. e. on I, and conditions (1.3_1), (1.3_2).

The following notations are used throughout the paper:

- $\mathbb{N}$ is the set of all natural numbers;
- $\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_0^+ = [0, +\infty[$, $\mathbb{R}^+ =]0, +\infty[$, $\mathbb{R}_0^- = \mathbb{R} \setminus \mathbb{R}^+$;
- $C(I;\mathbb{R})$ is the Banach space of continuous functions $u : I \to \mathbb{R}$ with the norm $\|u\|_C = \max\{|u(t)| : t \in I\}$;
- $\widetilde{C}^{(3)}(I;\mathbb{R})$ is the set of functions $u : I \to \mathbb{R}$ which are absolutely continuous together with their third derivatives;
- $L(I;\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : I \to \mathbb{R}$ with the norm $\|p\|_L = \int_a^b |p(s)|ds$;
- $K(I \times \mathbb{R};\mathbb{R})$ is the set of functions $f : I \times \mathbb{R} \to \mathbb{R}$ satisfying the Carathéodory conditions, i.e., $f(\cdot, x) : I \to \mathbb{R}$ is a measurable function for all $x \in \mathbb{R}$, $f(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is a continuous function for almost all $t \in I$, and for arbitrary $r > 0$ the inclusion
$$\sup\{|f(t,x)| : |x| \le r\} \in L(I;\mathbb{R}_+)$$
holds;
- For arbitrary $x, y \in L(I;R)$, the notation
$$x(t) \preccurlyeq y(t) \ (x(t) \succcurlyeq y(t)) \quad \text{for} \quad t \in I,$$
means that $x \le y$ $(x \ge y)$ and $x \ne y$;
- We also use the notation $[x]_\pm = (|x| \pm x)/2$.

The fourth order equations appear as model equations for a large class of higher order differential equations, arising for example, in hydrodynamics, suspension bridge models, etc. and have become the subject of many fundamental works. These studies particulary focus on the questions of oscillatory properties and Green's function sign for some two-point boundary value problems (see, e.g., [2–5],[11–13]) which give the basis for the study of questions of solvability and unique solvability of the fourth order differential equations under different boundary conditions (see, e.g., [6]-[7], [10], [14], [16], [17]).

The aim of our paper is to study the solvability of the above mentioned problems. We have proved the unimprovable sufficient conditions of the unique solvability for the linear problem, which show that the solvability of problem (1.1), (1.3_1), ((1.1), (1.3_2)) depends only on the nonnegative (nonpositive) part of the coefficient p if this nonnegative (nonpositive) part is small enough. Based on these results, for the nonlinear problems, sufficient conditions of solvability have been proved, which in some sense improves previously known results.

Below we present some definitions and results from the works [1], [15], on which our paper is based.

Definition 1. Equation

$$u^{(4)}(t) = p(t)u(t) \quad \text{for} \quad t \in I \tag{1.3}$$

is said to be disconjugate (non-oscillatory) on I if every nontrivial solution u has less then four zeros on I, the multiple zeros being counted according to their multiplicity.

Definition 2. We will say that $p \in D_+(I)$ if $p \in L(I;\mathbb{R}_0^+)$, and problem (1.3), ($1.3_1$) has a solution u such that

$$u(t) > 0 \qquad \text{for} \qquad t \in]a,b[. \tag{1.4}$$

Definition 3. We will say that $p \in D_-(I)$ if $p \in L(I;\mathbb{R}_0^-)$, and problem (1.3), ($1.3_2$) has a solution u such that inequality (1.4) holds.

Remark 1. Let $p \in L(I;\mathbb{R}_0^+)$ $(p \in L(I;\mathbb{R}_0^-))$, and consider the equation

$$u^{(4)}(t) = \lambda^4 p(t)u(t) \qquad \text{for} \qquad t \in I. \tag{1.5}$$

In [15] it is shown that the set $D_+(I)$ $(D_-(I))$ can be interpreted as a set of the functions for which $\lambda = 1$ is the first eigenvalue of problem (1.5), (1.3_1) ((1.5), (1.3_2)). Also the fact that $\lambda > 0$ is the first eigenvalue of problem (1.5), (1.3_1) ((1.5), (1.3_2)) is equivalent to the inclusion $\lambda^4 p \in D_+(I)$ $(\lambda^4 p \in D_-(I))$.

The importance of the classes D_+ and D_- for our investigation follows from the following propositions.

Proposition 1. *([15], Theorem 2) Let $p \in L(I;\mathbb{R}_0^+)$. Then equation (1.3) is disconjugate on I if and only if there exists $p^* \in D_+(I)$ such that $p(t) \preccurlyeq p^*(t)$ on I.*

Proposition 2. *([15], Theorem 4) Let $p \in L(I;\mathbb{R}_0^-)$. Then equation (1.3) is disconjugate on I if and only if there exists $p_* \in D_-(I)$ such that $p(t) \succcurlyeq p_*(t)$ on I.*

Also we need the following propositions:

Proposition 3. *([1], Theorem 5) Let $p \in L(I,\mathbb{R})$, and Green's function G of the equation*

$$u^{(4)}(t) = [p(t)]_+ u(t), \tag{1.6_1}$$

under boundary conditions (1.3_1) be totally positive kernel. Then problem (1.1), (1.3_1) is uniquely solvable for an arbitrary $q \in L(I,\mathbb{R})$.

Proposition 4. *([1], Theorem 4) Let $p \in L(I;\mathbb{R})$, and Green's function G of the equation*

$$u^{(4)}(t) = -[p(t)]_- u(t), \tag{1.6_2}$$

under boundary conditions (1.3_2) be such that $-G$ is totally positive kernel. Then problem (1.1), (1.3_2) is uniquely solvable for an arbitrary $q \in L(I,\mathbb{R})$.

Proposition 5. *(Gantmacher-Krein, see [13]) Let $i \in \{1,2\}, p \in L(I,\mathbb{R})$ be such that equation (1.3) is disconjugate and G is Green's function of problem (1.3), (1.3_1), ((1.3), (1.3_2)). Then $(-1)^{i-1}G$ is the totally positive kernel.*

2. Main Results

2.1. *Linear Problem*

Theorem 1. *Let $i \in \{1,2\}$ and the function $p_0 \in L(I;\mathbb{R})$ be such that the equation*

$$u^{(4)}(t) = [p_0(t)]_+ u(t) \qquad \text{if} \quad i=1,$$
$$u^{(4)}(t) = -[p_0(t)]_- u(t) \qquad \text{if} \quad i=2,$$

is diconjugate on I. Then if the inequality

$$(-1)^{i-1}[p(t)-p_0(t)] \le 0 \quad \text{for} \quad t \in I \tag{2.1$_i$}$$

holds, problem (1.1), (1.3$_1$) ((1.3$_2$)) is uniquely solvable.

From the last theorem with $p_0 = [p]_+$, by Proposition 1 immediately follows the corollary.

Corollary 1. *Let there exist $p^* \in D_+(I)$ such that the inequality*

$$[p(t)]_+ \preccurlyeq p^*(t) \quad \text{for} \quad t \in I \tag{2.2$_1$}$$

holds. Then problem (1.1), (1.3$_1$) is uniquely solvable.

Analogously, from the last theorem with $p_0 = -[p]_-$, by Proposition 2 we get

Corollary 2. *Let there exist $p_* \in D_-(I)$ such that the inequality*

$$-[p(t)]_- \succcurlyeq p_*(t) \quad \text{for} \quad t \in I \tag{2.2$_2$}$$

holds. Then problem (1.1), (1.3$_2$) is uniquely solvable.

Remark 2. Condition (2.2$_1$) ((2.2$_2$)) in Corollary 1 (2) is optimal in the sense that the inequality $\preccurlyeq$ ($\succcurlyeq$) can not be replaced by the inequality $\le$ ($\ge$).

Corollary 3. *Let there exist $M \in \mathbb{R}_0^+$ such that*

$$M\frac{b-a}{2} + \int_a^b [[p(s)]_+ - M]_+ ds \le \frac{192}{(b-a)^3}. \tag{2.3$_1$}$$

Then problem (1.1), (1.3$_1$) is uniquely solvable.

Corollary 4. *Let $p \in L(I;\mathbb{R})$ and there exist $M \in \mathbb{R}_0^+$ such that*

$$M\frac{495}{1024}(b-a) + \int_a^b [[p(s)]_- - M]_+ ds \le \frac{110}{(b-a)^3}. \tag{2.3$_2$}$$

Then problem (1.1), (1.3$_2$) is uniquely solvable.

The validity of Corollary 3 (4) follows from Theorem 1 and the fact that condition (2.3$_1$) ((2.3$_2$)) guarantees the disconjugacy of equation (1.6$_1$) ((1.6$_2$)) (see [15], Theorems 3,5).

2.2. *Nonlinear Problem*

Theorem 2. *Let $i \in \{1,2\}$ and there exist $r \in \mathbb{R}^+$ and $g \in L(I;\mathbb{R}_0^+)$ such that a. e. on I the inequality*

$$-g(t)|x| \le (-1)^{i-1} f(t,x)\operatorname{sgn} x \le \delta(t,|x|) \qquad \text{for} \qquad |x| > r \tag{2.4$_i$}$$

holds, where the function $\delta \in K(I \times \mathbb{R}_0^+;\mathbb{R}_0^+)$ is nondecreasing in the second argument and

$$\lim_{\rho\to+\infty} \frac{1}{\rho}\int_a^b \delta(s,\rho)ds = 0. \tag{2.5}$$

Then if equation (1.6$_1$) ((1.6$_2$)) is disconjugate, problem (1.2), (2.3$_1$) ((2.3$_2$)) has at least one solution.

Corollary 5. *Let $i \in \{1,2\}$ and conditions (2.4$_i$), (2.5) of Theorem 2 hold. Then if condition (2.3$_1$) ((2.3$_2$)) is fulfilled, problem (1.2), (2.3$_1$) ((2.3$_2$)) has at least one solution.*

Theorem 3. *Let $p^* \in D_+(I)$ and a. e. on I the inequality*

$$[f(t,x_1) - f(t,x_2)]\operatorname{sgn}(x_1 - x_2) < [p^*(t) - p(t)]|x_1 - x_2| \tag{2.6$_1$}$$

holds for $x_1, x_2 \in R$, $x_1 \ne x_2$. Then problem (1.2), (1.3$_1$) has at most one solution.

Theorem 4. *Let $p_* \in D_-(I)$ and a. e. on I the inequality*

$$[f(t,x_1) - f(t,x_2)]\operatorname{sgn}(x_1 - x_2) > [p_*(t) - p(t)]|x_1 - x_2| \tag{2.6$_2$}$$

holds for $x_1, x_2 \in R$, $x_1 \ne x_2$. Then problem (1.2), (1.3$_2$) has at most one solution.

Example 1. Consider on I the equations

$$u^{(4)}(t) = p(t)u(t) + f_0(t)|u(t)|^{\alpha}\operatorname{sgn} u(t) + h(t), \tag{2.7}$$

$$u^{(4)}(t) = p(t)u(t) + f_0(t)\frac{u(t)}{(1+|u(t)|)^{\beta}} + h(t), \tag{2.8}$$

where p, $f_0 \in L(I;\mathbb{R})$, $\alpha \in [0,1[$, and $\beta \in]0,1[$.
Then if

$$||[p]_+||_L \le \frac{192}{(b-a)^3} \quad \text{for} \quad i=1, \qquad ||[p]_-||_L \le \frac{110}{(b-a)^3} \quad \text{for} \quad i=2,$$

from Corollary 5 it follows that for an arbitrary $h \in L(I;\mathbb{R})$ problem (2.7) (1.3$_1$) ((1.3$_2$)) has at least one solution.
Also, if a.e. on I the inequalities

$$f_0(t) < p^*(t) - p(t) \quad \text{for} \quad i=1, \qquad f_0(t) > p^*(t) - p(t) \quad \text{for} \quad i=2,$$

hold, from Corollary 1 if $i=1$, and Corollary 2 if $i=2$, it follows that for an arbitrary $h \in L(I;\mathbb{R})$ problem (2.8) (1.3$_1$) ((1.3$_2$)) has at most one solution.

3. Auxiliary Propositions

Lemma 1. *Let $i \in \{1,2\}, p \in L(I;\mathbb{R}), g \in L(I;\mathbb{R}_0^+)$ be the functions defined in Theorem 2, and let the function $\widetilde{p} \in L(I;R)$ admit the inequalities*

$$
\begin{aligned}
p(t)-g(t) \le \widetilde{p}(t) \le p(t) \quad & \textit{for} \quad t \in I, \quad i=1, \\
p(t) \le \widetilde{p}(t) \le p(t)+g(t) \quad & \textit{for} \quad t \in I, \quad i=2,
\end{aligned} \tag{3.1}
$$

Then there exists a number $\rho_0 > 0$ such that an arbitrary solution u of the equation

$$u^{(4)}(t) = \widetilde{p}(t)u(t) + q(t), \tag{3.2}$$

under boundary condition (1.3_1) ((1.3_2)) admits the estimate

$$||u||_C \le \rho_0 ||q||_L. \tag{3.3}$$

To prove this proposition, we need Lemma 1.1 from [9].

Lemma 2. *Let $\widetilde{p}, p_k \in L(I;\mathbb{R}),\ v_0, v_k \in C(I;\mathbb{R})\ (k \in \mathbb{N})$,*

$$\lim_{k\to+\infty} ||v_k - v_0||_C = 0, \ \lim\sup_{k\to+\infty} ||p_k||_L < +\infty,$$

and

$$\lim_{k\to+\infty} \int_a^t p_k(s)ds = \int_a^t \widetilde{p}(s)ds \ \textit{ uniformly on } \ I.$$

Then

$$\lim_{k\to+\infty} \int_a^t p_k(s)v_k(s)ds = \int_a^t \widetilde{p}(s)v_0(s)ds \quad \textit{uniformly on } \ I. \tag{3.4}$$

Proof. We will prove our lemma only for $i=1$, for $i=2$ the proof is similar. Assume that Lemma 1 is not true. Then for all $k \in \mathbb{N}$ there exist functions $p_k, q_k \in L(I;\mathbb{R})$ such that

$$h_0(t) \le p_k(t) \le p(t) \quad \text{for} \quad t \in I, \tag{3.5}$$

where $h_0 := p - g$, and the problem

$$u_k^{(4)}(t) = p_k(t)u_k(t) + q_k(t), \qquad u_k^{(j)}(a) = 0, \qquad u_k^{(j)}(b) = 0 \ (j = 0,1),$$

has a solution u_k such that $||u_k||_C \ge k||q_k||_L$. Then if we suppose that

$$v_k(t) = u_k(t)/||u_k||_C, \widetilde{q}_k(t) = q_k(t)/||u_k||_C,$$

we obtain

$$||v_k||_C = 1, \quad ||\widetilde{q}_k||_L \le \frac{1}{k}, \tag{3.6}$$

and

$$
\begin{aligned}
& v_k^{(4)}(t) = p_k(t)v_k(t) + \widetilde{q}_k(t) \quad \text{for} \quad t \in I, \\
& v_k^{(j)}(a) = 0, \qquad v_k^{(j)}(b) = 0 \ (j = 0,1),
\end{aligned} \tag{3.7}
$$

from which due to (3.5) and (3.6) we have

$$||v_k^{(4)}||_L \le ||h||_L + \frac{b-a}{k} \quad \text{for} \quad t \in I, \tag{3.8}$$

where $h(t) = \max(|h_0(t)|, |p(t)|)$.

In view of the equality $||v_k||_C = 1$, (3.8), and the boundary conditions (3.7), it is clear that sequences $\{v_k^{(j)}\}_{k=1}^{+\infty}$ $(j = \overline{0,3})$ are uniformly bounded and equicontinuous on I. By the Arzela-Ascoli lemma, without loss of generality it can be assumed that these sequences are uniformly convergent on I. Therefore there exists a function $v_0 \in \widetilde{C}^3(I;R)$ such that

$$v_0^{(j)}(t) = \lim_{k\to+\infty} v_k^{(j)}(t)\,(j = \overline{0,3}) \quad \text{uniformly on} \quad I, \tag{3.9}$$

and due to (3.6) we have

$$||v_0||_C = 1. \tag{3.10}$$

Set $P_k(t) = \int_a^t p_k(s)ds$. Then from (3.5) we get

$$P_k(a) = 0, \quad \int_{t_1}^{t_2} h_0(s)ds \le P_k(t_2) - P_k(t_1) \le \int_{t_1}^{t_2} p(s)ds \tag{3.11}$$

for $a \le t_1 \le t_2 \le b$, and therefore the sequence $\{P_k\}_{k=1}^{+\infty}$ is uniformly bounded and equicontinuous on I. Thus by the Arzela-Ascoli lemma, without loss of generality it can be assumed that this sequence uniformly converges, i.e., there exists the function $P \in C(I;R)$ such that

$$\lim_{k\to+\infty} P_k(t) = P(t) \quad \text{uniformly} \quad \text{on} \quad I, \tag{3.12}$$

for which from (3.11) it follows that

$$\int_{t_1}^{t_2} h_0(s)ds \le P(t_2) - P(t_1) \le \int_{t_1}^{t_2} p(s)ds.$$

Consequently, the function P is absolutely continuous, and there exists a function $\widetilde{p} \in L(I;R)$ such that $P(t) = \int_a^t \widetilde{p}(s)ds$, and the inequality

$$h_0(t) \le \widetilde{p}(t) \le p(t) \quad \text{for} \quad t \in I \tag{3.13}$$

holds. Then due to (3.9) and (3.12), from Lemma 2 follows the validity of equality (3.4). Therefore if we integrate equation (3.7) from a to t, and pass to the limit as $k \to +\infty$, due to conditions (3.6), (3.7), (3.9), and (3.4), we find that v_0 is a solution of the problem

$$v_0^{(4)}(t) = \widetilde{p}(t)v_0(t), \qquad v_0^{(j)}(a) = 0, \; v_0^{(j)}(b) = 0 \; (j = 0,1).$$

On the other hand, from the disconjugacy of equation (1.6_1) and inequality (3.13), in view of Theorem 1 we have $v_0 \equiv 0$, which is the contradiction with (3.10). Thus our assumption is invalid and estimation (3.3) holds. $\square$

4. Proof of The Main Results

Proof of Theorem 1. Let $i=1$. From disconjugacy of the equation $u^{(4)}=[p_0]_+u$ on I, by Proposition 1 follows the existence of $p^*\in D_+(I)$ such that $[p_0(t)]_+\preccurlyeq p^*(t)$ for $t\in I$. From the last inequality and (2.1_i) it is clear that

$$[p(t)]_+\le[p_0(t)]_+\preccurlyeq p^*(t)\quad\text{for}\quad t\in I,$$

and therefore due to Proposition 1 equation (1.6_1) is disconjugate on I. But due to Proposition 5 from the disconjugacy of equation (1.6_1) it follows that Green's function of problem (1.6_1), (1.3_1) is totally positive kernel. Now the validity of our theorem immediately follows from Proposition 3.

For $i=2$ the proof is similar with the only difference that instead of Propositions 1 and 3, Propositions 2 and 4 are used. □

Proof of Theorem 2. Set $i=1$, and for $t\in I$ define the functions

$$H(t,x)=\begin{cases}0 & \text{if}\quad |x|>r\\ f(t,x) & \text{if}\quad |x|\le r\end{cases},\quad q^*(t)=\max\{|H(t,x)|:x\in\mathbb{R}\},$$

and

$$F_-(t,x)=\begin{cases}\frac{[f(t,x)\operatorname{sgn}x]_-}{|x|} & \text{if}\quad |x|>r\\ 0 & \text{if}\quad |x|\le r\end{cases},$$

$$F_+(t,x)=\begin{cases}[f(t,x)\operatorname{sgn}x]_+\operatorname{sgn}x & \text{if}\quad |x|>r\\ 0 & \text{if}\quad |x|\le r\end{cases}.$$

Then it is clear that

$$f(t,x)=F_+(t,x)-F_-(t,x)x+H(t,x)\quad\text{for}\quad t\in I,\ x\in\mathbb{R},\tag{4.1}$$

where due to condition (2.4_i) we have the estimations

$$0\le F_-(t,x)\le g(t),\ \ |F_+(t,x)|\le\delta(t,|x|)\ \ \text{for}\ t\in I,\ x\in\mathbb{R}.\tag{4.2}$$

Also due to condition (2.5), there exists a constant $r_0>r$ sauch that

$$\rho_0\Big(||q^*||_L+\int_a^b\delta(s,\rho)ds\Big)<\rho\quad\text{for}\quad\rho\ge r_0,\tag{4.3}$$

where ρ_0 is the constant defined in Lemma 1. Now consider the equation

$$u(t)=F(u)(t)\quad\text{for}\quad t\in I,\tag{4.4}$$

where

$$F(u)(t)=\chi(||u||_C)\int_a^b G(t,s)f(s,u(s))ds,$$

G is Green's function of problem (1.3), (1.3_1), and the function $\chi : R_0^+ \to [0,\,1]$ is defined by the equality

$$\chi(x) = \begin{cases} 1 & \text{if} \quad 0 \le x \le r_0 \\ 2 - \frac{x}{r_0} & \text{if} \quad r_0 < x < 2r_0 \\ 0 & \text{if} \quad x \ge 2r_0 \end{cases}.$$

From the continuity of the functions $G, \partial G/\partial t$ it follows that F maps $C(I,\mathbb{R})$ into the set $\{z \in C(I,\mathbb{R}) : ||z||_C \le r_1\}$, and is the compact operator, where

$$r_1 = \max_{a \le s,t \le b} |G(t,s)| \int_a^b f^*(s)ds, \qquad f^*(t) = \sup\{|f(t,\,x)| : |x| \le 2r_0\}.$$

Then according to Schauder's principle [8] equation (4.4) has a solution u. But from the definition of Green's function G, it is clear that u satisfies the boundary conditions (1.3_1) and is the solution of the equation

$$u^{(4)}(t) = \chi(||u||_C)[p(t)u(t) + f(t,\,u(t))] \quad \text{for} \quad t \in I, \tag{4.5}$$

which in view of (4.1) can be rewritten as equation (3.2) with

$$\widetilde{p}(t) = \chi(||u||_C)[p(t) - F_-(t,\,u(t))], \qquad q(t) = \chi(||u||_C)[F_+(t,\,u(t)) + H(t,\,u(t))].$$

Also from (4.2) and the fact that the function δ is nondecreasing in the second argument, estimations (3.1) and

$$|q(t)| \le \delta(t,\,||u||_C) + q^*(t)$$

follow. Therefore all the assumptions of Lemma 1 are fulfilled and the estimation

$$||u||_C \le \rho_0\Big(||q^*||_L + \int_a^b \delta(s,\,||u||_C)ds\Big)$$

is valid. Now if we assume that $||u||_C > r_0$, we get the contradiction with inequality (4.3). Consequently, $||u||_C \le r_0$, and from (4.4) by the definition of the function χ we get that u is a solution of problem (1.2), (1.3_1).

Let now $i = 2$. If we redefine the functions $F_\pm, \widetilde{p}, q$ by the following way

$$F_-(t,\,x) = \begin{cases} [f(t,\,x)\operatorname{sgn} x]_- \operatorname{sgn} x & \text{if} \quad |x| > r \\ 0 & \text{if} \quad |x| \le r \end{cases},$$

$$F_+(t,\,x) = \begin{cases} \frac{[f(t,x)\operatorname{sgn} x]_+}{|x|} & \text{if} \quad |x| > r \\ 0 & \text{if} \quad |x| \le r \end{cases},$$

$$\widetilde{p}(t) = \chi(||u||_C)[p(t) + F_+(t,\,u(t))],$$
$$q(t) = \chi(||u||_C)[F_-(t,\,u(t)) + H(t,\,u(t))],$$

then

$$f(t,\,x) = F_+(t,\,x)x - F_-(t,\,x) + H(t,\,x) \quad \text{for} \quad t \in I,\; x \in \mathbb{R}, \tag{4.6}$$

and due to condition (2.4_i) we have the estimations

$$0 \le F_+(t,x) \le g(t), \ \ |F_-(t,x)| \le \delta(t,|x|) \qquad \text{for } t \in I,\, x \in \mathbb{R}. \tag{4.7}$$

Now we can prove the theorem in a similar way as for the case $i = 1$ with these new definitions (4.6) and (4.7). □

Proof of Theorem 3 (4). Assume that problem (1.2), (1.3_1) ((1.2), (1.3_2)) has solutions u_1 and u_2 such that $u_1 \not\equiv u_2$. Let now $u = u_1 - u_2$, and

$$p_0(t) = \begin{cases} \frac{f(t,x_1)-f(t,x_2)}{u(t)} & \text{if} \quad u(t) \neq 0 \\ -p(t) & \text{if} \quad u(t) = 0 \end{cases}.$$

Then u admits conditions (1.3_1) ((1.3_2)) and is a solution of the equation

$$u^{(4)}(t) = (p(t) + p_0(t))u(t). \tag{4.8}$$

where due to inequality (2.6_1) ((2.6_2)), the estimations

$$p(t) + p_0(t) < p^*(t) \ \ (p(t) + p_0(t) > p_*(t)) \quad \text{a. e. on} \quad I$$

hold. Therefore from Corollary 1 (2) it follows that problem (4.8), (1.3_1) ((4.8), (1.3_2)) has only the trivial solution, and thus $u_1 \equiv u_2$. □

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