Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 993–1008
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4439

SOLVING CAPUTO FRACTIONAL ORDER DIFFERENTIAL EQUATIONS VIA FIXED POINT RESULTS OF α-GERAGHTY CONTRACTION TYPE MAPPINGS

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Received 05 October, 2022

Abstract. In this paper, by the fixed point results of α–Geraghty contraction type mappings, we prove the existence and uniqueness results of solutions for some Caputo fractional order differential equations in metric spaces. In addition, we conclude the applicability of these results with an illustrative example.

2010 *Mathematics Subject Classification:* 35G05;35G55;46N20

Keywords: fractional differential equation, α-admissible, Caputo fractional derivative, α-Geraghty contraction type mappings, fixed point, metric spaces

1. INTRODUCTION

The theory of fractional order differential equations has emerged as an interesting area to explore in recent years. Note that this theory has a lot of applications in the description of many events in the real world. For example, fractional order differential equations are often applicable in engineering, chemistry, physics, biology, etc. For examples and details see [2, 3, 11–13, 15]. Currently in the mathematical literature, studies on the existence, uniqueness, multiplicity of solutions of nonlinear fractional problems use nonlinear analysis techniques such as fixed point theorems. Fixed point theorems are the basic mathematical tools, to show solutions in different types of equations. [4, 6]. The theory of fixed points is at the heart of nonlinear analysis since it provides the tools for having existence theorems in a lot of different nonlinear problems, tools to have existence theorems in many different nonlinear problems. The fixed point theorems are often based on certain properties (such as complete continuity, monotony, contraction...) that the application considered must satisfy [1, 9, 16, 19, 20]. In this work, we are studying the existence, uniqueness of the solution for some of the nonlinear fractional differential equations with integral boundary conditions, in particular, we have presented the existence and uniqueness for some classes of differential equations of fractional order in the sense of Caputo. For this we used fixed point results of α-Geraghty contraction type mappings.

2. Preliminaries

We start with the definitions of triangular α-admissible and α-admissible mappings, which was introduced by Karapinar, Samet et al [17, 18, 20].

Definition 1 ([20, Definition 2.2])**.** Let $\mathcal{S}\colon \Xi \longrightarrow \Xi$ be a mapping and a function $\alpha\colon \Xi^2 \longrightarrow \mathbb{R}^+$. $\mathcal{S}$ is called a α-admissible if

$$\alpha(\vartheta,\eta) \geq 1 \Longrightarrow \alpha(\mathcal{S}\vartheta,\mathcal{S}\eta) \geq 1 \qquad \text{for all } \vartheta,\eta \in \Xi.$$

Example 1 ([20, Example 2.2]). Let $\Xi = \mathbb{R}_+^*$.
Define $\alpha\colon \Xi^2 \longrightarrow \mathbb{R}^+$ and $\mathcal{S}\colon \Xi \longrightarrow \Xi$, as follows $\mathcal{S}\vartheta = \ln(\vartheta)$ for all $\vartheta \in \Xi$, and

$$\alpha(\vartheta,\eta) = \begin{cases} 0 & \text{if } \vartheta < \eta, \\ 2 & \text{otherwise.} \end{cases}$$

Then, $\mathcal{S}$ is α-admissible.

Definition 2 ([17, Definition 1])**.** A $\mathcal{S}$ map is called a triangular α-admissible if it is α-admissible and α satisfies the following condition:

$$\begin{cases} \alpha(\vartheta,\sigma) \geq 1 \\ \alpha(\sigma,\eta) \geq 1 \end{cases} \quad \text{implies } \alpha(\vartheta,\eta) \geq 1, \text{ for all } \vartheta,\eta,\sigma \in \Xi.$$

Definition 3 ([14, Definition, p. 606])**.** Let $\psi\colon \mathbb{R}^+ \to]0,1[$ be a function. ψ is said to be strong Geraghty function if $\{y_n\} \subset [0,\infty)$ and $\lim_{n\to\infty} \psi(y_n) = 1$ implies $\lim_{n\to\infty} y_n = 0$.

The set of all Geraghty functions is denoted by $\mathcal{G}$.

Definition 4 ([5, Definition 1.8])**.** Let (Ξ,d) be a metric space and a function $\alpha\colon \Xi^2 \longrightarrow [0,\infty)$. Ξ is called a α-regular

$$\text{if for every sequence } (\mu_m)_{m\in\mathbb{N}} \subset \Xi, \text{ such that } \begin{cases} \alpha(\mu_m,\mu_{m+1}) \geq 1, \\ \lim_{m\to\infty} \mu_m = \mu, \end{cases} \quad \text{for all } m \in \mathbb{N}$$

there exist a subsequence $(\mu_{m_l})_{l\in\mathbb{N}}$ of $(\mu_m)_{m\in\mathbb{N}}$, where $\alpha(\mu_{m_l},\mu) \geq 1$, $\forall l \in \mathbb{N}$.

Cho et al. [10] proved the fixed point theorem via α-Geraghty contraction.

Definition 5 ([10, Definition 4])**.** Let (Ξ,d) be a metric space and a function $\alpha\colon \Xi^2 \longrightarrow [0,\infty)$. A map $\mathcal{S}\colon \Xi \longrightarrow \Xi$ is said to be α-Geraghty contraction if there exist $\psi \in \mathcal{G}$, such that

$$\alpha(\vartheta,\eta)d(\mathcal{S}\vartheta,\mathcal{S}\eta) \leq \psi(d(\vartheta,\eta))d(\vartheta,\eta),$$

for all $\vartheta,\eta \in \Xi$.

The Following result plays a key role in our main results.

Theorem 1 ([10, Theorem 4]). *Let (Ξ, d) be a complete metric space and $\alpha\colon \Xi^2 \to \mathbb{R}^+$ be a function. Define a map $\mathcal{S}\colon \Xi \longrightarrow \Xi$. Assume that*

1) $\mathcal{S}$ is a triangular α-admissible;
2) $\mathcal{S}$ is a α-Geraghty contraction;
3) either Ξ is α-regular or $\mathcal{S}$ is continous;
4) there exists $\theta_0 \in \Xi$ with $\alpha(\theta_0, \mathcal{S}\theta_0) \geq 1$.

Then $\mathcal{S}$ has a fixed point. Moreover, if

5) for all fixed point ϑ, η of $\mathcal{S}$, either $\alpha(\vartheta, \eta) \geq 1$ or $\alpha(\eta, \vartheta) \geq 1$,

then $\mathcal{S}$ has a uniques fixed point.

Definition 6. For x a strictly positive real number, the gamma function is defined by

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t}\, dt.$$

Now, we gives the definitions of the Caputo fractional order integral and derivative.

Definition 7. The Caputo fractional order derivative is defined by

$$\mathfrak{D}^{\beta} g(s) = \int_0^s \frac{1}{\Gamma(p-\beta)} \frac{g^{(p)}(\sigma)}{(s-\sigma)^{\beta+1-p}}\, d\sigma, \quad p-1 < \beta < p, \quad p \in \mathbb{N}^*,$$

and the fractional order integral is given by

$$\mathrm{J}^{\beta} g(s) = \int_0^s \frac{(s-\sigma)^{\beta-1} g(\sigma)}{\Gamma(\beta)}\, d\sigma, \quad \beta > 0.$$

3. Main Results

Let $(\mathrm{C}(I), \mathrm{d})$ be a metric space and $\mathrm{d}\colon \mathrm{C}(I)^2 \to \mathbb{R}_+$ is defined by

$$\mathrm{d}(y,z) = \sup_{\xi \in I} |y(\xi) - z(\xi)|.$$

Then, $(\mathrm{C}(I), \mathrm{d})$ is a complete metric space.

Now, we consider the following problem

$$\begin{cases} (\mathfrak{D}^{\beta}\, \mathsf{u})(\xi) = \mathsf{f}\big(\xi, \mathsf{u}(\xi)\big), \quad \xi \in I = [0,T], \quad T > 0, \\ a\mathsf{u}(0) + b\mathsf{u}(T) = c, \end{cases} \tag{3.1}$$

where $\mathfrak{D}^{\beta}$ is a Caputo fracional derivative of order $0 < \beta < 1$, $\mathsf{f} \in \mathrm{C}(I \times \mathbb{R})$ and $a, b, c \in \mathbb{R}$ such that $a + b \neq 0$.

Lemma 1 ([7, Lemma 3.2]). *The problem (3.1) is equivalent with following equation*

$$\mathsf{u}(\xi) = M_c + \frac{1}{\Gamma(\beta)} \int_0^{\xi} (\xi-\sigma)^{\beta-1} \mathsf{f}(\sigma, \mathsf{u}(\sigma))\, d\sigma - \frac{N_b}{\Gamma(\beta)} \int_0^T (T-\sigma)^{\beta-1} \mathsf{f}(\sigma, \mathsf{u}(\sigma))\, d\sigma, \quad (3.2)$$

where $M_c = \frac{c}{a+b}$ *and* $N_b = \frac{b}{a+b}$.

Denote by Φ the class of all continuous and nondecreasing functions $\varphi\colon [0,\infty) \to [0,\infty)$ satisfying

$$\varphi(b\xi) \le b\varphi(\xi) \le b\xi \quad \text{for } b > 1.$$

Theorem 2. *Let* $\beta \in (0,1)$. *Suppose that*

(I) There exists a function $\delta\colon \mathsf{C}(I) \times \mathsf{C}(I) \to \mathbb{R}$, $r \in (0,\beta)$ *and* $m \in L^{1/r}$ *such that*

$$|\mathsf{f}(\xi,x) - \mathsf{f}(\xi,y)| \le m(\xi) \frac{|x-y|}{1 + \|x-y\|_{L^\infty}},$$

where

$$\left| \Big(\int_0^T (\xi-\sigma)^{\frac{\beta-1}{1-r}}\, d\sigma \Big)^{1-r} \frac{M(1+N_b)}{\Gamma(\beta)} \right| \le \varphi(\|x-y\|_{L^\infty}),$$

$M = \|m\|_{L^{1/r}}$, $\varphi \in \Phi$, *and for* $x,y \in \mathsf{C}(I)$ *with* $\delta(x,y) \ge 0$.

(II) There exist $\gamma_0 \in \mathsf{C}(I)$ *such that,* $\delta\big(\gamma_0(\xi), \aleph_{\gamma_0}(\xi)\big)$ *where*

$$\aleph_{\gamma_0}(\xi) = M_c + \frac{1}{\Gamma(\beta)} \left[\int_0^{\xi} (\xi-s)^{\beta-1} \mathsf{f}(s, \gamma_0(s))\, ds - N_b \int_0^T (T-\zeta)^{\beta-1} \mathsf{f}(\zeta, \gamma_0(\zeta))\, d\zeta \right].$$

(III) For each $x,y \in \mathsf{C}(I)$, *we get*

$$\delta\big(x(\xi), y(\xi)\big) \ge 0 \Rightarrow \delta\big(A_x, A_y\big) \ge 0,$$

where

$$A_x = M_c + \frac{1}{\Gamma(\beta)} \left[\int_0^{\xi} (\xi-s)^{\beta-1} \mathsf{f}(s, x(s)) ds - N_b \int_0^T (T-\zeta)^{\beta-1} \mathsf{f}(\zeta, x(\zeta))\, d\zeta \right],$$

$$A_y = M_c + \frac{1}{\Gamma(\beta)} \left[\int_0^{\xi} (\xi-s)^{\beta-1} \mathsf{f}(s, y(s)) ds - N_b \int_0^T (T-\zeta)^{\beta-1} \mathsf{f}(\zeta, y(\zeta))\, d\zeta \right].$$

(V) For each $(z_n)_{n\in\mathbb{N}} \subset \mathsf{C}(I)$ *such that* $\delta(z_n, z_{n+1}) \ge 0$ *and* $\lim_{n\to\infty} z_n = z$, *then* $\delta(z_n, z) \ge 0$.

Then, the problem (3.1) has at least one Solution.

Proof. According to Lemma 1, then $u \in \mathsf{C}^1(I)$ is a soulution of problem (3.1) if only if it is a solution of equation (3.2).

Now, we define $\mathcal{T}\colon \mathsf{C}(I) \to \mathsf{C}(I)$ by

$$\mathcal{T}\mathsf{u}(\xi) = M_c + \frac{1}{\Gamma(\beta)} \int_0^{\xi} (\xi-\sigma)^{\beta-1} \mathsf{f}(\sigma, \mathsf{u}(\sigma))\, d\sigma - \frac{N_b}{\Gamma(\beta)} \int_0^T (T-\sigma)^{\beta-1} \mathsf{f}(\sigma, \mathsf{u}(\sigma))\, d\sigma,$$

where $M_c = \frac{c}{a+b}$ and $N_b = \frac{b}{a+b}$.

For this purpose, the problem reduces to finding a fixed point of $\mathcal{T}$.

Lets $u, v \in \mathrm{C}(I)$ be such that $\delta(x,y) \geq 0$, we have

$$\begin{aligned}\mathcal{T}\mathsf{u}(\xi) - \mathcal{T}\mathsf{v}(\xi) =& \frac{1}{\Gamma(\beta)}\int_0^{\xi}(\xi-\sigma)^{\beta-1}\left[\mathsf{f}(\sigma,\mathsf{u}(\sigma)) - \mathsf{f}(\sigma,\mathsf{v}(\sigma))\right]d\sigma \\ &- \frac{N_b}{\Gamma(\beta)}\int_0^{T}(T-\sigma)^{\beta-1}\left[\mathsf{f}(\sigma,\mathsf{u}(\sigma)) - \mathsf{f}(\sigma,\mathsf{v}(\sigma))\right]d\sigma.\end{aligned}$$

And thus,

$$\begin{aligned}|\mathcal{T}\mathsf{u}(\xi) - \mathcal{T}\mathsf{v}(\xi)| \leq& \frac{1}{\Gamma(\beta)}\int_0^{\xi}(\xi-\sigma)^{\beta-1}|\mathsf{f}(\sigma,\mathsf{u}(\sigma)) - \mathsf{f}(\sigma,\mathsf{v}(\sigma))|\,d\sigma \\ &+ \frac{N_b}{\Gamma(\beta)}\int_0^{T}(T-\sigma)^{\beta-1}|\mathsf{f}(\sigma,\mathsf{u}(\sigma)) - \mathsf{f}(\sigma,\mathsf{v}(\sigma))|\,d\sigma.\end{aligned}$$

We know that

$$\max\left\{\int_0^{\xi}(\xi-\sigma)^{\beta-1}\,d\sigma, \int_0^{T}(T-\sigma)^{\beta-1}\,d\sigma\right\} \leq \int_0^{T}(\xi-\sigma)^{\beta-1}\,d\sigma.$$

Then,

$$|\mathcal{T}\mathsf{u}(\xi) - \mathcal{T}\mathsf{v}(\xi)| \leq \frac{1+N_b}{\Gamma(\beta)}\int_0^{T}(\xi-\sigma)^{\beta-1}|\mathsf{f}(\sigma,\mathsf{u}(\sigma)) - \mathsf{f}(\sigma,\mathsf{v}(\sigma))|\,d\sigma.$$

By using (I), we obtain

$$\begin{aligned}|\mathcal{T}\mathsf{u}(\xi) - \mathcal{T}\mathsf{v}(\xi)| \leq& \frac{1+N_b}{\Gamma(\beta)} \\ &\times \int_0^{T}(\xi-\sigma)^{\beta-1}m(\sigma)\frac{|\mathsf{u}(\sigma)-\mathsf{u}(\sigma)|}{1+\|\mathsf{u}-\mathsf{v}\|_{L^\infty}}\,d\sigma \\ \leq& \frac{\|\mathsf{u}-\mathsf{v}\|_{L^\infty}}{1+\|\mathsf{u}-\mathsf{v}\|_{L^\infty}} \\ &\times \quad \frac{1+N_b}{\Gamma(\beta)}\int_0^{T}(\xi-\sigma)^{\beta-1}\frac{m(\sigma)}{2}\,d\sigma \\ \leq& \frac{\|\mathsf{u}-\mathsf{v}\|_{L^\infty}}{\|\mathsf{u}-\mathsf{v}\|_{L^\infty+1}} \\ &\times \left[\frac{1+N_b}{\Gamma(\beta)}\int_0^{T}(\xi-\sigma)^{\beta-1}m(\sigma)\,d\sigma\right].\end{aligned}$$

On the other hand, it is clear that

$$g(s) = (\xi-s)^{-1+\beta} \in L^{\frac{1}{-q+1}}[0,T], \quad \text{for } q \in [0,\beta).$$

Let

$$b = \frac{-1+\beta}{1-q}, \text{ then } b \in]-1,0[.$$

According to Hölder inequality, we can obtain

$$\begin{aligned}\left|\int_0^{\xi}(\xi-\sigma)^{\beta-1}m(\sigma)\,d\sigma\right| &\le \left(\int_0^{\xi}(\xi-\sigma)^{\frac{\beta-1}{1-q}}\,d\sigma\right)^{1-q}\left(\int_0^{\xi}|m^{1/q}(\sigma)|\,d\sigma\right)^{q}\\ &\le M\left(\int_0^{\xi}(\xi-\sigma)^{\frac{\beta-1}{1-q}}\,d\sigma\right)^{1-q}.\end{aligned}$$

So,

$$\begin{aligned}|\mathcal{T}\mathsf{u}(\xi)-\mathcal{T}\mathsf{v}(\xi)| \le& \frac{\|\mathsf{u}-\mathsf{v}\|_{L^\infty}}{\|\mathsf{u}-\mathsf{v}\|_{L^\infty+1}}\\ &\times\frac{1}{2}\left[\frac{M(1+N_b)}{\Gamma(\beta)}\left(\int_0^{\xi}(\xi-\sigma)^{\frac{\alpha-1}{1-q}}\,d\sigma\right)^{1-q}\right].\end{aligned}$$

From (I), we get

$$|\mathcal{T}\mathsf{u}(\xi)-\mathcal{T}\mathsf{v}(\xi)| \le \varphi(\|\mathsf{u}-\mathsf{v}\|_{L^\infty})\frac{\|\mathsf{u}-\mathsf{v}\|_{L^\infty}}{\|\mathsf{u}-\mathsf{v}\|_{L^\infty+1}}.$$

So,

$$\mathrm{d}(\mathcal{T}\mathsf{u},\mathcal{T}\mathsf{v}) \le \frac{\mathrm{d}(\mathsf{u},\mathsf{v})}{\mathrm{d}(\mathsf{u},\mathsf{v})+1}\varphi(\mathrm{d}(\mathsf{u},\mathsf{v})).$$

Let a function $\alpha\colon \mathrm{C}(I)^2 \to \mathbb{R}^+$ defined by

$$\alpha(\mathsf{u},\mathsf{v}) = \begin{cases}1 & \text{if } \delta(\mathsf{u}(\xi),\mathsf{v}(\xi)) \ge 0 \quad \xi \in I,\\ 0 & \text{otherwise.}\end{cases}$$

Then,

$$\begin{aligned}\mathrm{d}(\mathcal{T}\mathsf{u},\mathcal{T}\mathsf{v})\alpha(\mathsf{u},\mathsf{v}) &\le \frac{\mathrm{d}(\mathsf{u},\mathsf{v})}{\mathrm{d}(\mathsf{u},\mathsf{v})+1}\varphi(\mathrm{d}(\mathsf{u},\mathsf{v}))\alpha(\mathsf{u},\mathsf{v})\\ &\le \frac{\mathrm{d}(\mathsf{u},\mathsf{v})}{\mathrm{d}(\mathsf{u},\mathsf{v})+1}\varphi(\mathrm{d}(\mathsf{u},\mathsf{v})).\end{aligned}$$

Since $\varphi \in \Phi$, we have

$$\begin{aligned}\alpha(\mathsf{u},\mathsf{v})\mathrm{d}(\mathcal{T}\mathsf{u},\mathcal{T}\mathsf{v}) &\le \frac{\mathrm{d}(\mathsf{u},\mathsf{v})}{\mathrm{d}(\mathsf{u},\mathsf{v})+1}\varphi(\mathrm{d}(\mathsf{u},\mathsf{v}))\\ &\le \frac{\mathrm{d}(\mathsf{u},\mathsf{v})}{\varphi(\mathrm{d}(\mathsf{u},\mathsf{v}))+1}\varphi(\mathrm{d}(\mathsf{u},\mathsf{v}))\\ &\le \frac{\varphi(\mathrm{d}(\mathsf{u},\mathsf{v}))}{\varphi(\mathrm{d}(\mathsf{u},\mathsf{v}))+1}\mathrm{d}(\mathsf{u},\mathsf{v}).\end{aligned}$$

Hence,

$$\alpha(\mathsf{u},\mathsf{v})\mathsf{d}(\mathcal{T}\mathsf{u},\mathcal{T}\mathsf{v}) \leq \mathsf{d}(\mathsf{u},\mathsf{v})\psi\big(\mathsf{d}(\mathsf{u},\mathsf{v})\big),$$

where $\psi(\xi) = \frac{\varphi(\xi)}{\varphi(\xi)+1} \in \mathcal{G}, \quad \xi > 0.$

Then, $\mathcal{T}$ is a α-Geraghty contraction.

On the other hand according to (III) and for each $\mathsf{u},\mathsf{v} \in \mathrm{C}(I)$, we have

$$\begin{aligned}\alpha(\mathsf{u},\mathsf{v}) \geq 1 \text{ implies } & \delta(\mathsf{u},\mathsf{v}) \geq 0\\ & \Longrightarrow \delta(\mathcal{T}\mathsf{u},\mathcal{T}\mathsf{v}) \geq 0\\ & \Longrightarrow \alpha(\mathcal{T}\mathsf{u},\mathcal{T}\mathsf{v}) \geq 1.\end{aligned}$$

So, $\mathcal{T}$ is α-admissible. Mooreover, $\mathcal{T}$ is a triangular α-admissible.

By (II), we can obtain

$$\delta\left(\gamma_0(\xi), M_c + \frac{1}{\Gamma(\beta)}\left[\int_0^{\xi}(\xi-s)^{\beta-1}\mathsf{f}(s,\gamma_0(s))\,ds - N_b\int_0^{T}(T-\zeta)^{\beta-1}\mathsf{f}(\zeta,\gamma_0(\zeta))\,d\zeta\right]\right) \geq 0.$$

That implies,

$$\delta(\gamma_0, \mathcal{T}\gamma_0) \geq 0.$$

And thus,

$$\alpha(\gamma_0, \mathcal{T}\gamma_0) \geq 1.$$

So,

$$\alpha(\gamma_0, \mathcal{T}\gamma_0) \geq 1.$$

From (IV), we have $(z_n)_{n\in\mathbb{N}} \subset \mathrm{C}(I)$ with $\alpha(z_n, z_{n+1}) \geq 1$, which gives

$$\delta(z_n, z_{n+1}) \geq 0.$$

Then, $\delta(z_n, z) \geq 0$. And thus,

$$\delta(z_n, z) \geq 0.$$

That implies,

$$\alpha(z_n, b) \geq 1.$$

So, using Theorem 1, then $\mathcal{T}$ has a fixed point in $\mathrm{C}(I)$ is a solution of problem (3.1). □

We denote by $\mathrm{Fix}(\mathcal{T})$ the set of fixed points of $\mathcal{T}$.

(CD) We have $\delta(\mathsf{u},\mathsf{v}) \geq 0$, for each $\mathsf{u},\mathsf{v} \in \mathrm{Fix}(\mathcal{T})$.

Theorem 3. *Adding (CD) to the hypotheses of Theorem 2, we obtain the uniqueness of the fixed point of $\mathcal{T}$.*

Proof. By the contion (CD), if u and v are two solutions of problem (3.1), implies that $\mathsf{u},\mathsf{v} \in \mathrm{Fix}(\mathcal{T})$ and thus

$$\delta(\mathsf{u},\mathsf{v}) \geq 0.$$

So,

$$\alpha(\mathsf{u},\mathsf{v}) \geq 1.$$

According Theorem 1 and Theorem 2, then the mapping $\mathcal{T}$ has a unique fixed point. □

Now, we are discuss the existence, uniqueness of solutions for the following caputo fractional problem

$$\begin{cases} (\mathfrak{D}^{\beta}\mathsf{u})(\xi) = \mathsf{f}\big(\xi,\mathsf{u}(\xi)\big), & \xi \in I = [0,1], \\ a\mathsf{u}'(t_1) + b\mathsf{u}'(t_2) = \theta \int_0^{\eta} \mathsf{u}(s)\,ds, & \mathsf{u}(0) = 0, \end{cases} \tag{3.3}$$

where $\mathfrak{D}^{\beta}$ is a Caputo fracional order derivative of $0 < \beta < 1$, $\mathsf{f} \in C(I \times \mathbb{R})$ and $a, b, \theta \in \mathbb{R}$ with $\theta\frac{\eta^2}{2} - at_1 - bt_2 \neq 0$. The boundary conditions in (3.3) implies that the linear combination of the values of the derivative of the unknown function at nonlocal positions t_1 and t_2 are proportional to the continuous distribution of the unknown function over a strip of an arbitrary length η.

Lemma 2 ([8, Lemma 2.2]). *For all $\xi \in I$. The problem (3.3) is equivalent with following equation*

$$\begin{aligned}\mathsf{u}(\xi) = &\int_0^{\xi} \frac{(\xi-\sigma)^{\beta-1}}{\Gamma(\beta)}\mathsf{f}(\sigma,\mathsf{u}(\sigma))\,d\sigma \\ &+ \frac{\xi}{A_\theta}\Bigg(\int_0^{t_1} \frac{(t_1-\sigma)^{\beta-2}}{\Gamma(\beta-1)}\mathsf{f}(\sigma,\mathsf{u}(\sigma))\,d\sigma + \int_0^{t_2} \frac{(t_2-\sigma)^{\beta-2}}{\Gamma(\beta-1)}\mathsf{f}(\sigma,\mathsf{u}(\sigma))\,d\sigma \\ &\qquad - \theta\int_0^{\eta}\left[\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}\mathsf{f}(\tau,\mathsf{u}(\tau))d\tau\right]d\sigma\Bigg),\end{aligned}$$

where $A_\theta = \theta\frac{\eta^2}{2} - at_1 - bt_2 \neq 0$.

We make the following hypotheses will be used in the sequel.

(H1) There exists a function $\rho\colon C(I)^2 \to \mathbb{R}$, $r \in (0,\beta)$ and $\lambda \in L^{1/r}([0,T])$, $T > 0$ with

$$|\mathsf{f}(\xi,x) - \mathsf{f}(\xi,y)| \leq \lambda(\xi)|x-y|,$$

where

$$\left|\lambda_r(\xi)\left[\frac{B_\xi}{\Gamma(\beta)} + \frac{1}{|A_\theta|}\Big(\frac{B_{t_1}+B_{t_2}}{\Gamma(\beta-1)} + \theta\int_0^{\eta}\frac{B_\sigma}{\Gamma(\beta)}\,d\sigma\Big)\right]\right| \leq \psi(\|x-y\|_{L^\infty}),$$

$\psi \in \mathcal{G}$, $\lambda_r(\xi) = \|\lambda\|_{L^{1/r}([0,\xi])}$, and for $x, y \in C(I)$ with $\rho(x,y) \geq 0$, such that

$$B_\xi = \int_0^{\xi}(\xi-\sigma)^{\beta-1}\,d\sigma, \quad B_{t_1} = \int_0^{t_1}(t_1-\sigma)^{\beta-2}\,d\sigma,$$

$$B_{t_2} = \int_0^{t_2} (t_2 - \sigma)^{\beta-2}\, d\sigma, \quad B_\sigma = \int_0^{\sigma} (\sigma - \tau)^{\beta-1}\, d\tau.$$

(H2) There exist $\gamma_0 \in \mathrm{C}(I)$ such that, $\rho\big(\gamma_0(\xi), \aleph_{\gamma_0}(\xi)\big) \geq 0$ where

$$\begin{aligned}\aleph_{\gamma_0}(\xi) = & \int_0^{\xi} \frac{(\xi-\sigma)^{\beta-1}}{\Gamma(\beta)} \mathsf{f}(\sigma, \gamma_0(\sigma))\, d\sigma \\ & + \frac{\xi}{A_\theta}\Bigg(\int_0^{t_1} \frac{(t_1-\sigma)^{\beta-2}}{\Gamma(\beta-1)} \mathsf{f}(\sigma, \gamma_0(\sigma))\, d\sigma \\ & \quad + \int_0^{t_2} \frac{(t_2-\sigma)^{\beta-2}}{\Gamma(\beta-1)} \mathsf{f}(\sigma, \gamma_0(\sigma))\, d\sigma \\ & \quad - \theta \int_0^{\eta} \left[\frac{1}{\Gamma(\beta)} \int_0^{\sigma} (\sigma-\tau)^{\beta-1} \mathsf{f}(\tau, \gamma_0(\tau)) d\tau \right] d\sigma \Bigg).\end{aligned}$$

(H3) For each $x, y \in \mathrm{C}(I)$, we get

$$\rho\big(x(\xi), y(\xi)\big) \geq 0 \Rightarrow \delta\big(A_x, A_y\big) \geq 0,$$

where

$$\begin{aligned}A_x = & \int_0^{\xi} \frac{(\xi-\sigma)^{\beta-1}}{\Gamma(\beta)} \mathsf{f}(\sigma, x(\sigma))\, d\sigma \\ & + \frac{\xi}{A_\theta}\Bigg(\int_0^{t_1} \frac{(t_1-\sigma)^{\beta-2}}{\Gamma(\beta-1)} \mathsf{f}(\sigma, x(\sigma))\, d\sigma \\ & \quad + \int_0^{t_2} \frac{((t_2-\sigma)^{\beta-2}}{\Gamma(\beta-1)} \mathsf{f}(\sigma, x(\sigma))\, d\sigma \\ & \quad - \theta \int_0^{\eta} \left[\frac{1}{\Gamma(\beta)} \int_0^{\sigma} (\sigma-\tau)^{\beta-1} \mathsf{f}(\tau, x(\tau)) d\tau \right] d\sigma \Bigg),\end{aligned}$$

$$\begin{aligned}A_y = & \int_0^{\xi} \frac{(\xi-\sigma)^{\beta-1}}{\Gamma(\beta)} \mathsf{f}(\sigma, y(\sigma))\, d\sigma \\ & + \frac{\xi}{A_\theta}\Bigg(\int_0^{t_1} \frac{(t_1-\sigma)^{\beta-2}}{\Gamma(\beta-1)} \mathsf{f}(\sigma, y(\sigma))\, d\sigma \\ & \quad + \int_0^{t_2} \frac{(t_2-\sigma)^{\beta-2}}{\Gamma(\beta-1)} \mathsf{f}(\sigma, y(\sigma))\, d\sigma \\ & \quad - \theta \int_0^{\eta} \left[\frac{1}{\Gamma(\beta)} \int_0^{\sigma} (\sigma-\tau)^{\beta-1} \mathsf{f}(\tau, y(\tau)) d\tau \right] d\sigma \Bigg).\end{aligned}$$

(H4) For each $(y_n)_{n\in\mathbb{N}} \subset \mathrm{C}(I)$ such that $\rho(y_n, y_{n+1}) \geq 0$ and $\lim_{n\to\infty} y_n = y$, then $\delta(y_n, y) \geq 0$.

Theorem 4. *Assume that (H1)-(H4) hold. then the problem* (3.3) *has at least one Solution. Morover, if* $\delta(x,y) \geq 0$*, for each* $u,v \in S_{(3.3)}$ *, where* $S_{(3.3)}$ *denotes the set of solutions of problem* (3.3)*. Then the problem* (3.3) *has a unique solution.*

Proof. From Lemma 2, we define an map $\mathcal{S}\colon \mathrm{C}(I) \to \mathrm{C}(I)$ by

$$\begin{aligned}\mathcal{S}\mathrm{u}(\xi) =& \frac{1}{\Gamma(\beta)}\int_0^\xi (\xi-\sigma)^{\beta-1}\mathrm{f}(\sigma,\mathrm{u}(\sigma))\,d\sigma \\ &+\frac{\xi}{A_\theta}\Bigg(\frac{1}{\Gamma(\beta-1)} \\ &\qquad\times\left[\int_0^{t_1}(t_1-\sigma)^{\beta-2}\mathrm{f}(\sigma,\mathrm{u}(\sigma))\,d\sigma+\int_0^{t_2}(t_2-\sigma)^{\beta-2}\mathrm{f}(\sigma,\mathrm{u}(\sigma))\,d\sigma\right] \\ &\qquad-\theta\int_0^\eta\left[\frac{1}{\Gamma(\beta)}\int_0^\sigma(\sigma-\tau)^{\beta-1}\mathrm{f}(\tau,\mathrm{u}(\tau))d\tau\right]d\sigma\Bigg).\end{aligned}$$

We find a fixed point of $\mathcal{S}$. Let $v,w \in \mathrm{C}(I)$, we have

$$\begin{aligned}&\mathcal{S}\mathrm{v}(\xi)-\mathcal{S}\mathrm{w}(\xi) \\ &\quad=\frac{1}{\Gamma(\beta)}\int_0^\xi (\xi-\sigma)^{\beta-1}\mathrm{f}(\sigma,\mathrm{v}(\sigma))\,d\sigma \\ &\qquad+\frac{\xi}{A_\theta}\Bigg(\frac{1}{\Gamma(\beta-1)} \\ &\qquad\quad\times\left[\int_0^{t_1}(t_1-\sigma)^{\beta-2}\mathrm{f}(\sigma,\mathrm{v}(\sigma))\,d\sigma+\int_0^{t_2}(t_2-\sigma)^{\beta-2}\mathrm{f}(\sigma,\mathrm{v}(\sigma))\,d\sigma\right] \\ &\qquad\quad-\theta\int_0^\eta\left[\frac{1}{\Gamma(\beta)}\int_0^\sigma(\sigma-\tau)^{\beta-1}\mathrm{f}(\tau,\mathrm{v}(\tau))d\tau\right]d\sigma\Bigg) \\ &\qquad-\frac{1}{\Gamma(\beta)}\int_0^\xi (\xi-\sigma)^{\beta-1}\mathrm{f}(\sigma,\mathrm{w}(\sigma))\,d\sigma \\ &\qquad-\frac{\xi}{A_\theta}\Bigg(\frac{1}{\Gamma(\beta-1)} \\ &\qquad\quad\times\left[\int_0^{t_1}(t_1-\sigma)^{\beta-2}\mathrm{f}(\sigma,\mathrm{w}(\sigma))\,d\sigma+\int_0^{t_2}(t_2-\sigma)^{\beta-2}\mathrm{f}(\sigma,\mathrm{w}(\sigma))\,d\sigma\right] \\ &\qquad\quad-\theta\int_0^\eta\left[\int_0^\sigma\frac{(\sigma-\tau)^{\beta-1}}{\Gamma(\beta)}\,\mathrm{f}(\tau,\mathrm{w}(\tau))d\tau\right]d\sigma\Bigg) \\ &\quad=\frac{1}{\Gamma(\beta)}\int_0^\xi (\xi-\sigma)^{\beta-1}\Big[-\mathrm{f}(\sigma,\mathrm{w}(\sigma))+\mathrm{f}(\sigma,\mathrm{v}(\sigma))\Big]\,d\sigma\end{aligned}$$

$$
\begin{aligned}
&+\frac{\xi}{A_\theta}\Bigg(\int_0^{t_1}\frac{(t_1-\sigma)^{\beta-2}}{\Gamma(\beta-1)}\big(-\mathsf{f}(\sigma,\mathsf{w}(\sigma))+\mathsf{f}(\sigma,\mathsf{v}(\sigma))\big)\,d\sigma\\
&\quad+\int_0^{t_2}\frac{(t_2-\sigma)^{\beta-2}}{\Gamma(\beta-1)}\big(-\mathsf{f}(\sigma,\mathsf{w}(\sigma))+\mathsf{f}(\sigma,\mathsf{v}(\sigma))\big)\,d\sigma\\
&\quad-\theta\int_0^{\eta}\left[\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}\big(\mathsf{f}(\tau,\mathsf{v}(\tau))-\mathsf{f}(\tau,\mathsf{w}(\tau))\big)\,d\tau\right]d\sigma\Bigg).
\end{aligned}
$$

Then,

$$
\begin{aligned}
|\mathcal{S}\mathsf{v}(\xi)-\mathcal{S}\mathsf{w}(\xi)|\le&\frac{1}{\Gamma(\beta)}\int_0^{\xi}(\xi-\sigma)^{\beta-1}|\mathsf{f}(\sigma,\mathsf{v}(\sigma))-\mathsf{f}(\sigma,\mathsf{w}(\sigma))|\,d\sigma\\
&+\frac{\xi}{|A_\theta|}\Bigg(\int_0^{t_1}\frac{(t_1-\sigma)^{\beta-2}}{\Gamma(\beta-1)}|-\mathsf{f}(\sigma,\mathsf{w}(\sigma))+\mathsf{f}(\sigma,\mathsf{v}(\sigma))|\,d\sigma\\
&\quad+\int_0^{t_2}\frac{(t_2-\sigma)^{\beta-2}}{\Gamma(\beta-1)}|-\mathsf{f}(\sigma,\mathsf{w}(\sigma))+\mathsf{f}(\sigma,\mathsf{v}(\sigma))|\,d\sigma\\
&\quad+\theta\int_0^{\eta}\left[\int_0^{\sigma}\frac{(\sigma-\tau)^{\beta-1}}{\Gamma(\beta)}|\mathsf{f}(\tau,\mathsf{v}(\tau))-\mathsf{f}(\tau,\mathsf{w}(\tau))|\,d\tau\right]d\sigma\Bigg).
\end{aligned}
$$

According to (I), we get

$$
\begin{aligned}
|\mathcal{S}\mathsf{v}(\xi)-\mathcal{S}\mathsf{w}(\xi)|\le&\int_0^{\xi}\frac{(\xi-\sigma)^{\beta-1}}{\Gamma(\beta)}|\lambda(\sigma)||\mathsf{v}(\sigma)-\mathsf{w}(\sigma)|\,d\sigma\\
&+\frac{\xi}{|A_\theta|}\Bigg(\int_0^{t_1}\frac{(t_1-\sigma)^{\beta-2}}{\Gamma(\beta-1)}|\lambda(\sigma)||\mathsf{v}(\sigma)-\mathsf{w}(\sigma)|\,d\sigma\\
&\quad+\int_0^{t_2}\frac{(t_2-\sigma)^{\beta-2}}{\Gamma(\beta-1)}|\lambda(\sigma)||\mathsf{v}(\sigma)-\mathsf{w}(\sigma)|\,d\sigma\\
&\quad+\theta\int_0^{\eta}\left[\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}|\lambda(\tau)||\mathsf{v}(\tau)-\mathsf{w}(\tau)|\,d\tau\right]d\sigma\Bigg).
\end{aligned}
$$

So,

$$
\begin{aligned}
|\mathcal{S}\mathsf{v}(\xi)-\mathcal{S}\mathsf{w}(\xi)|\le&\frac{\|\mathsf{v}-\mathsf{w}\|_\infty}{\Gamma(\beta)}\int_0^{\xi}(\xi-\sigma)^{\beta-1}|\lambda(\sigma)|\,d\sigma\\
&+\frac{\|\mathsf{v}-\mathsf{w}\|_\infty}{|A_\theta|}\Bigg(\int_0^{t_1}\frac{(t_1-\sigma)^{\beta-2}}{\Gamma(\beta-1)}|\lambda(\sigma)|\,d\sigma\\
&\quad+\int_0^{t_2}\frac{(t_2-\sigma)^{\beta-2}}{\Gamma(\beta-1)}|\lambda(\sigma)|\,d\sigma
\end{aligned}
$$

$$+\theta\int_0^\eta\left[\frac{1}{\Gamma(\beta)}\int_0^\sigma(\sigma-\tau)^{\beta-1}|\lambda(\tau)|\,d\tau\right]d\sigma\Bigg).$$

Let

$$b=\frac{\beta-1}{1-q},\text{ it is clear that } b\in]-1,0[,$$

for $q\in[0,\beta)$. On the other hand, we have

$$h(\sigma)=(\xi-\sigma)^{\beta-2}\in L^{\frac{1}{1-q}}[0,\xi].$$

From Hölder inequality, we get

$$\begin{aligned}|\mathcal{S}\mathsf{v}(\xi)-\mathcal{S}\mathsf{w}(\xi)|\leq&\|\mathsf{v}-\mathsf{w}\|_\infty\frac{\lambda_r}{\Gamma(\beta)}\Big(\int_0^\xi(\xi-\sigma)^{\frac{-1+\beta}{-q+1}}\,d\sigma\Big)^{-q+1}\\&+\frac{\|\mathsf{v}-\mathsf{w}\|_\infty}{|A_\theta|}\left(\frac{\lambda_r}{\Gamma(-1+\beta)}\left[\Big(\int_0^{t_1}(t_1-\sigma)^{\frac{-2+\beta}{1q+1}}\,d\sigma\Big)^{-q+1}\right.\right.\\&\left.+\Big(\int_0^{t_2}(t_2-\sigma)^{\frac{-2+\beta}{-q+1}}\,d\sigma\Big)^{-q+1}\right]\\&\left.+\theta\int_0^\eta\big[\frac{\lambda_r}{\Gamma(\beta)}\Big(\int_0^\sigma(\sigma-\tau)^{\frac{-1+\beta}{-q+1}}\,d\tau\Big)^{-q+1}d\sigma\right)\\\leq&\lambda_r\|\mathsf{v}-\mathsf{w}\|_\infty\left[\frac{B_\xi}{\Gamma(\beta)}\frac{1}{|A_\theta|}\Big(\frac{B_{t_1}+B_{t_2}}{\Gamma(\beta-1)}+\theta\int_0^\eta\frac{B_\sigma}{\Gamma(\beta)}\,d\sigma\Big)\right],\end{aligned}$$

where $\lambda_r=\sup_{\xi\in I}\lambda_r(\xi)$.

By (I), we obtain

$$|\mathcal{S}\mathsf{v}(\xi)-\mathcal{S}\mathsf{w}(\xi)|\leq\|\mathsf{v}-\mathsf{w}\|_\infty\varphi\big(\|\mathsf{v}-\mathsf{w}\|_\infty\big).$$

So,

$$\mathsf{d}\big(\mathcal{S}\mathsf{v}-\mathcal{S}\mathsf{w}\big)\leq\mathsf{d}(\mathsf{v}-\mathsf{w})\varphi\big(\mathsf{d}(\mathsf{v}-\mathsf{w})\big).$$

Let $\alpha\colon \mathrm{C}(I)^2\to\mathbb{R}^+$ such that

$$\alpha(\mathsf{v},\mathsf{w})=\begin{cases}1 & \text{if }\ \delta(\mathsf{v}(\xi),\mathsf{w}(\xi))\geq 0\quad \xi\in I,\\0 & \text{otherwise.}\end{cases}$$

Then,

$$\begin{aligned}\alpha(\mathsf{v},\mathsf{w})\mathsf{d}\big(\mathcal{S}\mathsf{v},\mathcal{S}\mathsf{w}\big)&\leq\alpha(\mathsf{v},\mathsf{w})\mathsf{d}(\mathsf{v},\mathsf{w})\frac{1}{3}\varphi\big(\mathsf{d}(\mathsf{v}-\mathsf{w})\big)\\&\leq\mathsf{d}(\mathsf{v},\mathsf{w})\varphi\big(\mathsf{d}(\mathsf{v},\mathsf{w})\big).\end{aligned}$$

Hence,

$$\alpha(\mathsf{v},\mathsf{w})\mathsf{d}\big(\mathcal{S}\mathsf{v},\mathcal{S}\mathsf{w}\big)\leq\mathsf{d}(\mathsf{v}-\mathsf{w})\varphi\big(\mathsf{d}(\mathsf{v},\mathsf{w})\big),$$

where $\psi(\xi) = \varphi(\xi)$, $\xi \in]0,1[$.

Then, $\mathcal{S}$ is a α-Geraghty contraction.

From (III) and for each $\mathsf{v}, \mathsf{w} \in \mathrm{C}(I)$, we get

$$\begin{aligned} \alpha(\mathsf{v},\mathsf{w}) \geq 1 \quad \text{implies that } & \rho(\mathsf{v},\mathsf{w}) \geq 0 \\ & \rho(\mathcal{S}\mathsf{v},\mathcal{S}\mathsf{w}) \geq 0 \\ & \alpha(\mathcal{S}\mathsf{v},\mathcal{S}\mathsf{w}) \geq 1. \end{aligned}$$

Then, $\mathcal{S}$ is α-admissible. Moreover, $\mathcal{S}$ is a triangular α-admissible

By (II), we have

$$\rho(\gamma_0, \mathcal{S}\gamma_0) \geq 0.$$

And thus,

$$\rho(\gamma_0, \mathcal{S}\gamma_0) \geq 1.$$

So,

$$\rho(\gamma_0, \mathcal{S}\gamma_0) \geq 1.$$

According to (IV), we have $(y_n)_{n\in\mathbb{N}} \subset \mathrm{C}(I)$ with $\alpha(y_n, y_{n+1}) \geq 1$, which gives

$$\delta(y_n, y_{n+1}) \geq 0.$$

Then, $\rho(y_n, y) \geq 0$.

And thus,

$$\rho(y_n, y) \geq 0.$$

That implies ,

$$\alpha(y_n, b) \geq 1.$$

So, by Theorem 1, then $\mathcal{S}$ has a fixed point in $\mathrm{C}(I)$ is a solution of problem (3.3).

Finally, if w and v are two solutions of problem (3.3), then

$$\rho(\mathsf{v},\mathsf{w}) \geq 0.$$

So,

$$\alpha(\mathsf{v},\mathsf{w}) \geq 1.$$

From Theorem 1, then the problem (3.3) has a unique solution. □

4. Example

Consider the following fractional boundary value problem

$$\begin{cases} (\mathfrak{D}^{\beta}\,\mathsf{u})(\xi) = \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\frac{\sin|\mathsf{u}(\xi)|}{1+|\mathsf{u}(\xi)|}, & \xi \in I = [0,1], \\ \mathsf{u}(0) = 0, & \\ \mathsf{u}'(1/4) + 3\mathsf{u}'(1/2) = 2\int_0^{1/3} \mathsf{u}(\zeta)\,d\zeta, & \end{cases} \tag{4.1}$$

where $\mathfrak{D}^{\beta}$ is a Caputo fracional order derivative with $\beta = 1/2$ and

$$\mathsf{f}\big(\xi,\mathsf{u}(\xi)\big) = \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\frac{\sin|\mathsf{u}(\xi)|}{\big(1+|\mathsf{u}(\xi)|\big)}.$$

Let $(\mathrm{C}(I),\mathrm{d})$ be a metric space where $\mathrm{d}\colon \mathrm{C}(I)^2 \to \mathbb{R}_+$ is given by

$$\mathrm{d}(y,z) = \|y-z\|_\infty.$$

Lets $\mathsf{u},\mathsf{v} \in \mathrm{C}(I)$, we have

$$\mathsf{f}\big(\xi,\mathsf{u}(\xi)\big) - \mathsf{f}\big(\xi,\mathsf{v}(\xi)\big) = \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\frac{\sin|\mathsf{u}(\xi)|}{1+|\mathsf{u}(\xi)|} - \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\frac{\sin|\mathsf{v}(\xi)|}{1+|\mathsf{v}(\xi)|}.$$

Then,

$$\begin{aligned}
|\mathsf{f}\big(\xi,\mathsf{u}(\xi)\big) - \mathsf{f}\big(\xi,\mathsf{v}(\xi)\big)| &= \left|\frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\frac{\sin|\mathsf{u}(\xi)|}{1+|\mathsf{u}(\xi)|} - \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\frac{\sin|\mathsf{v}(\xi)|}{1+|\mathsf{v}(\xi)|}\right| \\
&\leq \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\left|\frac{\sin|\mathsf{u}(\xi)|}{1+|\mathsf{u}(\xi)|} - \frac{\sin|\mathsf{v}(\xi)|}{1+|\mathsf{v}(\xi)|}\right| \\
&\leq \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}} \\
&\quad \times \left|\frac{\sin|\mathsf{u}| + |\mathsf{v}|\sin|\mathsf{u}| - \sin|\mathsf{v}| - |\mathsf{u}|\sin|\mathsf{v}|}{\big(1+|\mathsf{u}|\big)\big(1+|\mathsf{v}|\big)}\right|.
\end{aligned}$$

Case-1: If $|\mathsf{v}(\xi)| \leq |\mathsf{u}(\xi)|$, we get

$$\begin{aligned}
|\mathsf{f}\big(\xi,\mathsf{u}(\xi)\big) - \mathsf{f}\big(\xi,\mathsf{v}(\xi)\big)| &\leq \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}} \\
&\quad \times \left|\frac{\sin|\mathsf{u}| - \sin|\mathsf{v}| + |\mathsf{v}(\xi)||\sin|\mathsf{u}| - \sin|\mathsf{v}|}{\big(1+|\mathsf{u}(\xi)|\big)\big(1+|\mathsf{v}(\xi)|\big)}\right| \\
&\leq \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}} \\
&\quad \times \left|\sin|\mathsf{u}| - \sin|\mathsf{v}|\frac{1+|\mathsf{v}(\xi)|}{\big(1+|\mathsf{u}(\xi)|\big)\big(1+|\mathsf{v}(\xi)|\big)}\right| \\
&\leq \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}} 2\sin\left(\frac{|\mathsf{u}|-|\mathsf{v}|}{2}\right)\cos\left(\frac{|\mathsf{u}|+|\mathsf{v}|}{2}\right) \\
&\quad \times \frac{1+|\mathsf{v}(\xi)|}{\big(1+|\mathsf{u}(\xi)|\big)\big(1+|\mathsf{v}(\xi)|\big)}.
\end{aligned}$$

Since $\sin u \leq u$ for all $u \geq 0$, then

$$\begin{aligned}|\mathsf{f}(\xi,\mathsf{u}(\xi)) - \mathsf{f}(\xi,\mathsf{v}(\xi))| \leq & \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\left(|\mathsf{u}|-|\mathsf{v}|\right) \\ & \times \frac{1+|\mathsf{v}(\xi)|}{\left(1+|\mathsf{u}(\xi)|\right)\left(1+|\mathsf{v}(\xi)|\right)} \\ \leq & |\mathsf{u}-\mathsf{v}|\frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}} \\ & \times \frac{1+|\mathsf{v}(\xi)|}{\left(1+|\mathsf{u}(\xi)|\right)\left(1+|\mathsf{v}(\xi)|\right)} \\ \leq & |\mathsf{u}-\mathsf{v}|\frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\frac{1}{\left(1+|\mathsf{u}(\xi)|\right)}.\end{aligned}$$

Case-2: If $|\mathsf{v}(\xi)| \geq |\mathsf{u}(\xi)|$, we can obtain

$$\begin{aligned}|\mathsf{f}(\xi,\mathsf{u}(\xi)) - \mathsf{f}(\xi,\mathsf{v}(\xi))| \leq & |\mathsf{u}-\mathsf{v}|\frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}} \\ & \times \frac{1+|\mathsf{u}(\xi)|}{\left(1+|\mathsf{u}(\xi)|\right)\left(1+|\mathsf{v}(\xi)|\right)} \\ \leq & |\mathsf{u}-\mathsf{v}|\frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\frac{1}{\left(1+|\mathsf{v}(\xi)|\right)}.\end{aligned}$$

Finally, we have

$$|\mathsf{f}(\xi,\mathsf{u}(\xi)) - \mathsf{f}(\xi,\mathsf{v}(\xi))| \leq |\mathsf{u}-\mathsf{v}|\frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\frac{1}{\left(1+\max\{|\mathsf{u}(\xi)|,|\mathsf{v}(\xi)|\}\right)}.$$

Then, the hypothesis (H1) is satisfied with

$$|\mathsf{f}(\xi,\mathsf{u}(\xi)) - \mathsf{f}(\xi,\mathsf{v}(\xi))| \leq \lambda(\xi)|\mathsf{u}-\mathsf{v}|,$$

where $\lambda(\xi) = \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\frac{1}{\left(1+\max\{|\mathsf{u}(\xi)|,|\mathsf{v}(\xi)|\}\right)}$,

$$B_\xi = -2\sqrt{\xi}, \qquad B_{t_1} = 4, \qquad B_{t_2} = 2\sqrt{2}, \qquad B_\sigma = -2\sqrt{\sigma},$$

and

$$\left\|\lambda_r(\xi)\left[\frac{B_t}{\Gamma(\beta)} + \frac{1}{|A_\theta|}\left(\frac{B_{t_1}+B_{t_2}}{\Gamma(\beta-1)} + \theta\int_0^\eta \frac{B_\sigma}{\Gamma(\beta)}\,d\sigma\right)\right]\right\|_\infty \leq \psi(\|u-v\|_{L^\infty})$$

with $\psi(\xi) = \frac{1}{1+\xi}$, for all $t > 0$.

Next, we define the function $\alpha\colon \mathrm{C}(I)^2 \to [0,\infty)$ by

$$\alpha(u(\xi),v(\xi)) = \begin{cases} 1 & \text{if } \rho(u(\xi),v(\xi)) \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

with $\rho(u(\xi),v(\xi)) = \sup_{\xi\in I} |u(\xi)-v(\xi)|$.

Then, the hypotheses (H2) and (H3) are satisfied, where $\gamma_0(\xi) = u(0)$. Moreover from the defnition of the ρ , then (H4) holds.

Finally, by Theorem 4, we get the existence of solutions and the uniqueness of problem (4.1).

Acknowledgements

The authors would like to express their sincere apreciation to the referes for their very helpful suggestions and many comments.

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