Miskolc Mathematical Notes
Vol. 26 (2025), No. 1, pp. 243–260

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2025.4435

SOME FIXED POINT RESULTS FOR GENERALIZED MODIFIED F_{α}-CONTRACTIONS IN MODULAR METRIC SPACES AND ITS APPLICATIONS

HAMI GÜNDOĞDU, MAHPEYKER ÖZTÜRK, AND ÖMER FARUK GÖZÜKIZIL

Received 04 October, 2022

Abstract. This paper introduces new generalizations of modified F-contractions through α-admissible mappings. It expands the fixed-point results of modified F-contractions within the framework of modular metric spaces. Several theorems are presented regarding the existence of solutions, along with a condition that ensures the uniqueness of a solution. These theorems are applied to specific mappings to verify the results. Additionally, the paper presents significant outcomes, demonstrating the existence of solutions to a general form of an integral equation. Finally, a particular example of an integral equation is discussed.

2010 *Mathematics Subject Classification:* 47H10; 46A80

Keywords: modular metric space, fixed point theory, modified F-contraction, integral equation

1. INTRODUCTION

Wardowski [20] acquainted the prominent idea of F-contraction.

Definition 1. Let Ω be a nonempty set, and d be a usual metric on Ω. Any mapping $H\colon \Omega \to \Omega$ is called an F-contraction provided that for a given $\kappa > 0$

$$d(H\tau_0, H\tau_1) > 0 \implies \kappa + F(d(H\tau_0, H\tau_1)) \leq F(d(\tau_0, \tau_1)) \tag{1.1}$$

is satisfied for all $\tau_0, \tau_1 \in \Omega$ and the function $F\colon R^+ \to R$ yields the followings:

(1_F) For all $a, b \in R^+$, $a < b$ implies $F(a) < F(b)$, that is, F is an increasing function.

(2_F) $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$ where $\{a_n\}_{n=1}^{\infty}$ is a sequence in R^+.

(3_F) For some $r \in (0,1)$, $\lim_{n\to\infty} a^r F(a) = 0$ holds.

The set of all functions satisfying $(1_F)-(3_F)$ is specified by $\mathfrak{F}$. It is easy to figure out that $A(\xi)=\ln(\xi)$, $B(\xi)=\xi+\ln(\xi)$, and $C(\xi)=-\frac{1}{\sqrt{\xi}}$ are satisfying the conditions $(1_F)-(3_F)$, i.e, $A(\xi),\ B(\xi),\ C(\xi)\in\mathfrak{F}$.

By defining F-contraction, Wardowski [20] generalized the Banach contraction principle [3] and demonstrated the subsequent theorem.

Theorem 1. *Assume that (Ω,d) is a complete metric space. If $H\colon\Omega\to\Omega$ is an F-contraction, then H admits a fixed point.*

There are many studies related to F-contraction; see [9, 12, 15–19, 21].

Definition 2. Let ψ be defined and continuous on $[0,\infty)$. If ψ yields

(1) ψ is non-decreasing,
(2) For all $\xi>0$, $\sum_{i=1}^{\infty}\psi^i(\xi)<\infty$,

then it is named as c-comparison function.

It is clear that $\psi(\xi)<\xi$ and also $\psi(0)=0$. Ψ represents the set of functions providing the conditions $(1)-(2)$. For further properties and examples of comparison functions, see [4, 5, 13].

Since the notion of α-admissible mappings were defined, they have been used in many metric fixed point studies and generalized in various directions.

Definition 3 ([14])**.** Let $\Omega\neq\emptyset$ and $\alpha\colon\Omega\times\Omega\to[0,\infty)$ be a mapping. If for all $a,b\in\Omega$,

$$\alpha(a,b)\geq 1 \implies \alpha(Ha,Hb)\geq 1, \tag{1.2}$$

then $H\colon\Omega\to\Omega$ is said to be α-admissible.

F-contraction has been reconstructed via α-admissible structure as in the following.

Definition 4 ([2])**.** Consider the metric space (Ω,d). If for a given $\kappa>0$ the following expression holds for all $\tau_0,\tau_1\in\Omega$:

$$d(H\tau_0,H\tau_1)>0 \implies \kappa+F(\alpha(\tau_0,\tau_1)d(H\tau_0,H\tau_1))\leq F(\psi(d(\tau_0,\tau_1))), \tag{1.3}$$

where $\psi\in\Psi$ and $F\in\mathfrak{F}$, then the α-admissible mapping $H\colon\Omega\to\Omega$ is called a modified F_α-contraction.

Recall that $F(\xi)=\ln(\xi)\in\mathfrak{F}$. Then, inequality (1.3) turns into

$$\alpha(\tau_0,\tau_1)d(H\tau_0,H\tau_1)\leq e^{-\kappa}\psi(d(\tau_0,\tau_1))\leq\psi(d(\tau_0,\tau_1)) \tag{1.4}$$

for all $\tau_0,\tau_1\in\Omega$ and $H\tau_0\neq H\tau_1$.

In [2] the following result has been verified.

Theorem 2. *Consider a complete metric space (Ω,d) and let $H\colon\Omega\to\Omega$ be a modified F_α-contraction. If the followings are satisfied:*

(1*) *for a given* s_0 $\alpha(s_0, Hs_0) \geq 1$ *holds,*
(2*) H *is* α*-admissible,*
(3*) H *is continuous,*

then H *has a fixed point.*

All the above motivate us to investigate the fixed point results for modified F_α-contraction mappings in the setting modular metric spaces (*MMS*).

The primary purpose of this paper is to give some fixed point theorems and results in a *MMS* for modified F_α-contractions. The existence of solutions for an integral equation is investigated for an application. As an example, a solution for a given integral equation is obtained.

This paper is organized as follows: Section 2 offers a comprehensive introduction to *MMS*, establishing the foundation for subsequent discussions. Section 3 defines modified F-contractions and presents the main results of the paper, along with several important outcomes. Section 4 is dedicated to exploring applications that support the results obtained. Finally, Section 5 provides a summary of the key insights and conclusions of the present work.

2. Fundamentals of modular spaces

We consider an example to understand the theory of modular metric spaces [1]. Take Ω as the set of all points above water on the earth's surface. The average speed needed to travel directly over land from any point τ_0 to another point τ_1 in a time t is denoted by $m_t(\tau_0, \tau_1)$.

Now we shall consider what kind of characteristics the function $m_t(\tau_0, \tau_1)$ should have. If we determine τ_0 and τ_1, then $m_t(\tau_0, \tau_1)$ becomes a non-increasing function of t, and also non-negative. Clearly, $m_t(\tau_0, \tau_1)$ is symmetric in τ_0 and τ_1. Till now, we assume that the points are in same ground. What if τ_0 and τ_1 are in separate landmasses? As it is needed to travel from τ_0 to τ_1 by land, it is not possible to travel in any time t independently of the speed. In all situations, the speed function $m_t(\tau_0, \tau_1)$ should be defined so it can be allowed to get extended real values, i.e., $m_t(\tau_0, \tau_1) = \infty$. In short, the speed function is non-increasing in t, symmetric in $\tau_0, \tau_1 \in \Omega$, and also takes values in $[0, \infty]$ for $t > 0$. Further property of the function $m_t(\tau_0, \tau_1)$ is given by

$$m_{t+s}(\tau_0, \tau_1) \leq m_t(\tau_0, \tau_2) + m_s(\tau_2, \tau_1) \tag{2.1}$$

for all $\tau_0, \tau_1, \tau_2 \in \Omega$ and $t, s > 0$.

Consider a function $m\colon (0, \infty) \times \Omega \times \Omega \to [0, \infty]$ where $\Omega \neq \emptyset$ and $\eta > 0$. It can be rewritten as $m(\eta, \tau_0, \tau_1) = m_\eta(\tau_0, \tau_1)$ for all $\eta > 0$ and $\tau_0, \tau_1 \in \Omega$ so that $m_\eta\colon \Omega \times \Omega \to [0, \infty]$.

Definition 5 ([6, 7]). Assume that $m_\eta : \Omega \times \Omega \to [0,\infty]$ satisfies the followings:

$$\begin{aligned} &m1) \qquad \tau_0 = \tau_1 \iff m_\eta(\tau_0,\tau_1) = 0, \\ &m2) \qquad m_\eta(\tau_0,\tau_1) = m_\eta(\tau_1,\tau_0), \\ &m3) \qquad m_{\eta+\mu}(\tau_0,\tau_1) \le m_\eta(\tau_0,\tau_2) + m_\mu(\tau_2,\tau_1) \end{aligned} \tag{2.2}$$

for all $\tau_0,\tau_1,\tau_2 \in \Omega$, and $\eta,\mu > 0$. Then, m_η is called a modular metric (*MM*) on Ω.

m is called pseudomodular if it satisfies $m_\eta(\tau,\tau) = 0, \quad \forall\eta > 0$ instead of $m1$). If instead of $m3$), m grants the following

$$m_{\eta+\mu}(\tau_0,\tau_1) \le \frac{\eta}{\eta+\mu} m_\eta(\tau_0,\tau_2) + \frac{\mu}{\eta+\mu} m_\mu(\tau_2,\tau_1), \quad \forall\eta,\mu > 0, \tag{2.3}$$

then it is a convex *MM*. Furthermore, any convex *MM* yields

$$m_\eta(\tau_0,\tau_1) \le \frac{\mu}{\eta} m_\mu(\tau_0,\tau_1) \le m_\mu(\tau_0,\tau_1) \quad \forall\eta,\mu > 0 \tag{2.4}$$

for all $\tau_0,\tau_1 \in \Omega$ and $0 < \mu \le \eta$ [6]. In general,

$$m_{\eta_2}(\tau_0,\tau_1) \le m_{\eta_1}(\tau_0,\tau_1) \tag{2.5}$$

holds for $0 < \eta_1 \le \eta_2$ and $\forall\tau_0,\tau_1 \in \Omega$.

Definition 6 ([6, 7]). m is said to be strict on Ω provided that for $\tau_1,\tau_2 \in \Omega$ with $\tau_1 \ne \tau_2$, $m_\eta(\tau_1,\tau_2) > 0$, $\forall\eta > 0$, or equivalently if $m_\eta(\tau_1,\tau_2) = 0$ for some $\eta > 0$, then $\tau_1 = \tau_2$.

Let (Ω,d) be a metric space consisting of at least two elements. We shall define some modular metrics on Ω.

Example 1. Consider the *MM*

$$m_\eta(\tau_0,\tau_1) = \frac{d(\tau_0,\tau_1)}{\eta}, \tag{2.6}$$

where d is a metric on Ω. Here $m_\eta(\tau_0,\tau_1)$ can be taken as the average speed needed to travel from τ_0 to τ_1 in time η. It is easy to see that the *MM* (2.6) is convex.

Example 2. Define the *MM*

$$m_\eta(\tau_0,\tau_1) = \begin{cases} \infty, & \eta < d(\tau_0,\tau_1), \\ 0, & \eta \ge d(\tau_0,\tau_1), \end{cases} \tag{2.7}$$

where d is a metric on Ω. Here $m_\eta(\tau_0,\tau_1)$ can be considered as the extreme condition of the speed metaphor. If $\eta \ge d(\tau_0,\tau_1)$, then we can travel immediately, yet in the case of $\eta < d(\tau_0,\tau_1)$, it is not possible to travel from τ_0 to τ_1. The *MM* (2.7) is also convex.

Definition 7 ([6]). Take an m on Ω and $\tau_0 \in \Omega$. The followings are *MMS* around τ_0:

$$\Omega_m = \Omega_m(\tau_0) = \{\tau \in \Omega\colon\, m_\eta(\tau, \tau_0) \to 0 \text{ as } \eta \to \infty\}, \tag{2.8}$$

$$\Omega_m^* = \Omega_m^*(\tau_0) = \{\tau \in \Omega\colon\, \exists \eta = \eta(\tau) > 0, \text{ s.t. } m_\eta(\tau, \tau_0) < \infty\}. \tag{2.9}$$

Definition 8 ([6, 8]). For Ω_m and Ω_m^* the followings hold.

- The sequence $\{h_n\}$ in Ω_m is m-convergent to a point $h \in \Omega$, named as the modular limit of $\{h_n\}$, if and only if $m_\eta(h_n, h) \to 0$ as $n \to \infty$ for some $\eta > 0$.
- $\{h_n\}$ in Ω_m is m-Cauchy if $m_\eta(h_n, h_m) \to 0$ as $n, m \to \infty$ for some $\eta > 0$.
- Consider S to be a nonempty subset of Ω_m. Provided that every m-Cauchy sequence belonging to S is m-convergent in S, then S is m-complete.

Now, we give some information about metrics on modular sets. If we take a modular m on Ω and a modular set Ω_m defined by m, then

$$d_m(\tau_0, \tau_1) = \inf\{\eta > 0\colon\, m_\eta(\tau_0, \tau_1) \geq \eta\}, \quad \forall \tau_0, \tau_1 \in \Omega_m \tag{2.10}$$

describes a metric on Ω_m [6].

In the case when m is convex, the modular space can be equipped with the following metric:

$$d_m^*(\tau_0, \tau_1) = \inf\{\eta > 0\colon\, m_\eta(\tau_0, \tau_1) \leq 1\}, \quad \forall \tau_0, \tau_1 \in \Omega_m. \tag{2.11}$$

The metrics given above on Ω_m are strongly equivalent, i.e., $d_m \leq d_m^* \leq 2d_m$.

Lemma 1 ([8]). *Suppose that m is convex on Ω_m. For a sequence h_n belonging to Ω_m and $h \in \Omega_m^*$, the following holds:*

$$\lim_{n\to\infty} d_m^*(h_n, h) = 0 \iff \lim_{n\to\infty} m_\eta(h_n, h) = 0 \tag{2.12}$$

for all $\eta > 0$.

In [8], it is demonstrated that modular convergence is strictly weaker than metric convergence.

Lemma 2 ([8]). *For any pseudomodular metric m on Ω, the (2.8)-(2.9) are closed w.r.t m-convergent. Moreover, if m is strict then the limit is unique whenever it exists.*

3. Main results

This part introduces new definitions for generalized modified F_α-contraction mappings. Then, some theorems and related results about fixed points for the mentioned mappings are proved in modular metric spaces.

Definition 9 ([10, 11]). Let m be a modular metric on a nonempty set Ω. Any self-mapping H on Ω_m is named as a generalized modified F_α-contraction of type I if there exists $\kappa > 0$ s.t.

$$\begin{aligned} & m_\eta(H\tau_0, H\tau_1) > 0 \quad \text{implies that} \\ & \kappa + F(\alpha(\tau_0,\tau_1) m_\eta(H\tau_0, H\tau_1)) \leq \\ & F\Big(\psi\Big(\max\Big\{m_\eta(\tau_0,\tau_1), \frac{m_\eta(\tau_0,H\tau_0)(1+m_\eta(\tau_1,H\tau_1))}{1+m_\eta(\tau_0,\tau_1)}, \frac{m_\eta(\tau_1,H\tau_0)(1+m_\eta(\tau_1,H\tau_0))}{1+m_\eta(\tau_0,\tau_1)}\Big\}\Big)\Big) \end{aligned} \tag{3.1}$$

for all $\tau_0, \tau_1 \in \Omega_m$. Here, $F \in \mathfrak{F}$ and $\psi \in \Psi$.

We also introduce the following definition.

Definition 10 ([10, 11]). Let m be a modular metric on a nonempty set Ω. Any self-mapping H on Ω_m is named as a generalized modified F_α-contraction of type II if for all $\tau_0, \tau_1 \in \Omega_m$, there exists $\kappa > 0$ such that the following holds;

$$\begin{aligned} & m_\eta(H\tau_0, H\tau_1) > 0 \quad \text{implies that} \\ & \kappa + F(\alpha(\tau_0,\tau_1) m_\eta(H\tau_0, H\tau_1)) \leq \\ & F\Big(\psi\Big(\max\Big\{m_\eta(\tau_0,\tau_1), \frac{m_\eta(\tau_0,H\tau_0)(1+m_\eta(\tau_1,H\tau_1))}{1+m_\eta(\tau_0,\tau_1)}, \frac{m_\eta(\tau_1,H\tau_1) m_\eta(\tau_0,H\tau_0)}{1+m_\eta(H\tau_0,H\tau_1)}, \\ & \qquad \frac{m_\eta(\tau_1,H\tau_1) m_\eta(\tau_1,H\tau_0)}{1+m_\eta(\tau_1,H\tau_0)+m_\eta(\tau_0,H\tau_1)}\Big\}\Big)\Big), \end{aligned} \tag{3.2}$$

where $F \in \mathfrak{F}$ and $\psi \in \Psi$.

Now, we shall give our main theorems for mappings satisfying the contraction conditions (3.1) and (3.2) in Ω_m (2.8).

Theorem 3. *Let m be a strict modular metric in a complete modular metric space Ω_m, and $H\colon \Omega_m \to \Omega_m$ which ensures condition (3.1). Assume that*

(C_1) for a given s_0, $\alpha(s_0, Hs_0) \geq 1$ holds,
(C_2) H is an α-admissible mapping,
(C_3) H is a continuous mapping,

then H admits a fixed point.

Proof. Take an initial point $s_0 \in \Omega_m$ satisfying $\alpha(s_0, Hs_0) \geq 1$. Then, construct a sequence $\{s_n\} \in \Omega_m$ as $s_n = H^n s_0$, $\forall n \in N$.

Presume that $s_n = s_{n+1}$ holds for any $n \in N$. In this case, s_n is the fixed point of H. Hence, for all $n \in N$ s_n cannot be equal to s_{n+1} which gives $m_\eta(s_n, s_{n+1}) > 0$. Since $m_\eta(s_n, s_{n+1}) = m_\eta(Hs_{n-1}, Hs_n)$, $m_\eta(Hs_{n-1}, Hs_n) > 0$ holds.

As $m_\eta(s_0, Hs_0) = m_\eta(s_0, s_1) \geq 1$ and H is an α-admissible mapping, we reach $\alpha(s_{n-1}, s_n) \geq 1$ for all n.

Putting $\tau_0 = s_{n-1}$ and $\tau_1 = s_n$ in (3.1) yields

$$\kappa + F(\alpha(s_{n-1},s_n)m_\eta(Hs_{n-1},Hs_n)) \le F\Big(\psi\Big(\max\{m_\eta(s_{n-1},s_n), \frac{m_\eta(s_{n-1},Hs_{n-1})(1+m_\eta(s_n,Hs_n))}{1+m_\eta(s_{n-1},s_n)}, \frac{m_\eta(s_n,Hs_{n-1})(1+m_\eta(s_n,Hs_{n-1}))}{1+m_\eta(s_{n-1},s_n)}\}\Big)\Big). \tag{3.3}$$

Owing to $m_\eta(s_n,Hs_{n-1}) = m_\eta(s_n,s_n) = 0$, we have

$$\begin{aligned}&\kappa + F(\alpha(s_{n-1},s_n)m_\eta(s_n,s_{n+1}))\\ &\le F\Big(\psi\Big(\max\{m_\eta(s_{n-1},s_n), \frac{m_\eta(s_{n-1},s_n)(1+m_\eta(s_n,s_{n+1}))}{1+m_\eta(s_{n-1},s_n)}\}\Big)\Big).\end{aligned} \tag{3.4}$$

Assume that $m_\eta(s_n,s_{n+1}) > m_\eta(s_{n-1},s_n)$. Then it is evident that

$$\frac{1+m_\eta(s_n,s_{n+1})}{1+m_\eta(s_{n-1},s_n)} < \frac{m_\eta(s_n,s_{n+1})}{m_\eta(s_{n-1},s_n)}. \tag{3.5}$$

Putting (3.5) in the right side of (3.4) we have

$$\kappa + F(\alpha(s_{n-1},s_n)m_\eta(s_n,s_{n+1})) \le F\Big(\psi\Big(\max\{m_\eta(s_{n-1},s_n), m_\eta(s_n,s_{n+1})\}\Big)\Big). \tag{3.6}$$

By the assumption $m_\eta(s_n,s_{n+1}) > m_\eta(s_{n-1},s_n)$, we attain

$$\kappa + F(\alpha(s_{n-1},s_n)m_\eta(s_n,s_{n+1})) \le F(\psi(m_\eta(s_n,s_{n+1}))). \tag{3.7}$$

Since $\alpha(s_{n-1},s_n) \ge 1$ and by (1_F), F is increasing, we have

$$F(m_\eta(s_n,s_{n+1})) < \kappa + F(m_\eta(s_n,s_{n+1})) \le \kappa + F(\alpha(s_{n-1},s_n)m_\eta(s_n,s_{n+1})). \tag{3.8}$$

Moreover, since $\psi(m_\eta(s_n,s_{n+1})) < m_\eta(s_n,s_{n+1})$ and by (1_F), we gain

$$F(\psi(m_\eta(s_n,s_{n+1}))) < F(m_\eta(s_n,s_{n+1})). \tag{3.9}$$

Combining (3.8) and (3.9) in (3.7) provides

$$F(m_\eta(s_n,s_{n+1})) < F(m_\eta(s_n,s_{n+1})), \tag{3.10}$$

which causes a contradiction. Thus,

$$m_\eta(s_n,s_{n+1}) \le m_\eta(s_{n-1},s_n). \tag{3.11}$$

Putting (3.11) into (3.4) grants

$$\kappa + F(m_\eta(s_n,s_{n+1})) \le F(m_\eta(s_{n-1},s_n)), \quad \forall n \ge 1. \tag{3.12}$$

Here, we take $a_n = m_\eta(s_n,s_{n+1})$. Then, expression (3.12) can be written as

$$\kappa + F(a_n) \le F(a_{n-1}), \quad \forall n \ge 1. \tag{3.13}$$

Furthermore, we reach

$$F(a_n) \le F(a_{n-1}) - \kappa \le F(a_0) - n\kappa, \quad \forall n \ge 1. \tag{3.14}$$

Taking the limit of both sides in (3.14) gives

$$\lim_{n\to\infty} F(a_n) = -\infty. \tag{3.15}$$

From (2_F), we achieve

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} m_\eta(s_n, s_{n+1}) = 0 \tag{3.16}$$

for all $\eta > 0$. For $k \in (0,1)$, we can also write $\lim_{n\to\infty} a_n^k F(a_n) = 0$ by (3_F). Hereby, we get

$$0 \le a_n^k F(a_n) - a_n^k F(a_0) \le a_n^k (F(a_0) - n\kappa) - a_n^k F(a_0) = -n\kappa a_n^k \le 0. \tag{3.17}$$

As $n \to \infty$, we reach $\lim_{n\to\infty} n a_n^k = 0$.

By the definition of the limit, one can find $n^* \in N$ s.t.

$$s_n = m_\eta(s_n, s_{n+1}) \le \frac{1}{n^{\frac{1}{k}}}, \tag{3.18}$$

whenever $n \ge n^*$. Due to $(m3)$, for $m > n \ge n^*$ we write

$$m_\eta(s_n, s_m) \le m_{\eta_n}(s_n, s_{n+1}) + m_{\eta_{n+1}}(s_{n+1}, s_{n+2}) + \cdots + m_{\eta_{m-1}}(s_{m-1}, s_m), \tag{3.19}$$

where $\eta_n + \eta_{n+1} + \cdots + \eta_{m-1} = \eta$.

Because $0 < \eta_i < \eta$ for $i = n, n+1, \ldots, m-1$ and (3.18), we get

$$\begin{aligned} m_\eta(s_n, s_m) &\le \frac{1}{n^{\frac{1}{k}}} + \frac{1}{(n+1)^{\frac{1}{k}}} + \cdots + \frac{1}{(m-1)^{\frac{1}{k}}} \\ &= \sum_{j=n}^{m-1} \frac{1}{j^{\frac{1}{k}}} \\ &< \sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{k}}}. \end{aligned} \tag{3.20}$$

Since $\sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{k}}} < \infty$, we reach $\lim_{m,n\to\infty} m_\eta(s_n, s_m) = 0$, i.e., $\{s_n\}$ is an m-Cauchy sequence. Due to the completeness of Ω_m, $\{s_n\}$ is m-convergent to a point s belonging to Ω_m, that is, $\lim_{n\to\infty} m_\eta(s_n, s) = 0$.

By the continuity of H, we obtain

$$m_\eta(s, Hs) = \lim_{n\to\infty} m_\eta(s_n, Hs_n) = \lim_{n\to\infty} m_\eta(s_n, s_{n+1}) = 0. \tag{3.21}$$

As m is strict, we find $s = Hs$. Hence, the proof is completed. □

The following theorem can be verified similarly to Theorem 3.

Theorem 4. *Presume that m is a strict modular on a complete modular metric space Ω_m, and $H\colon \Omega_m \to \Omega_m$ satisfies condition (3.2). Assume that*

(C_1) *for a given s_0, $\alpha(s_0, Hs_0) \ge 1$ holds,*
(C_2) *H is an α-admissible mapping,*
(C_3) *H is a continuous mapping,*

then H has a fixed point.

Example 3. Take the set $\Omega = [0,\infty)$ with $m_\eta(\tau_0,\tau_1) = \frac{|\tau_0-\tau_1|}{\eta}$ for all $\tau_0,\tau_1 \in [0,\infty)$. Consider a mapping given by

$$H(\tau) = \begin{cases} \frac{\tau+3}{4}, & \tau \in [0,1], \\ 2, & \tau > 1. \end{cases} \tag{3.22}$$

The mapping $H\colon [0,\infty) \to [0,\infty)$ is continuous on Ω_m. Now, we take $\alpha\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ as

$$\alpha(\tau_0,\tau_1) = \begin{cases} 1, & \tau_0,\tau_1 \in [0,1], \\ 0, & \text{otherwise}. \end{cases} \tag{3.23}$$

For $\tau_0,\tau_1 \in \Omega$ s.t. $\alpha(\tau_0,\tau_1) \geq 1$, τ_0,τ_1 must be in $[0,1]$. We get $H\tau_0, H\tau_1 \in [0,1]$. By (3.23), we find $\alpha(H\tau_0,H\tau_1) \geq 1$, that is, H is an α-admissible mapping.

For $s_0 = 0$, then $Hs_0 = \frac{3}{4}$. Thus, $\alpha(s_0,Hs_0) \geq 1$ holds.

Now, we display that (3.1) is satisfied for all $\tau_0,\tau_1 \in [0,\infty)$, where $F(\xi) = \ln(\xi)$ and $\psi(\xi) = \frac{1}{2}\xi \in \Psi$ for $\xi > 0$.

If τ_0 or τ_1 is not in $[0,1]$, then $\alpha(\tau_0,\tau_1) = 0$ and so (3.1) is satisfied.

For $\tau_0,\tau_1 \in [0,1]$ with $H\tau_0 \neq H\tau_1$, we can write

$$\begin{aligned} \alpha(\tau_0,\tau_1)m_\eta(H\tau_0,H\tau_1)) &= \frac{|H\tau_0 - H\tau_1|}{\eta} = \frac{1}{4\eta}|\tau_0-\tau_1| \leq \frac{1}{4}\frac{|\tau_0-\tau_1|}{\eta} \\ &= e^{-0.6}\frac{1}{2}m_\eta(\tau_0,\tau_1) = e^{-0.6}\psi(m_\eta(\tau_0,\tau_1)) \\ &\leq e^{-0.6}\psi(M_1(\tau_0,\tau_1)). \end{aligned} \tag{3.24}$$

Hereby, (3.1) holds for all $\tau_0,\tau_1 \in \Omega$. Thus, all conditions of Theorem 3 are satisfied, which gives the existence of fixed points for H. Here, there exist two fixed points, $H1 = 1$ and $H2 = 2$.

In Theorems 3 and 4, if the condition C_3 is replaced by the following one:
(C_3^*) For the sequence $\{s_n\}$ s.t. $\alpha(s_n,s_{n+1}) \geq 1$ with $\lim_{n\to\infty} m_\eta(s_n,s) = 0$ where $s \in \Omega_m$, there exists a subsequence $\{s_{n_p}\}$ of $\{s_n\}$ s.t. $\alpha(s_{n_p},s) \geq 1$ for all p, then the results found therein are still true.

Theorem 5. *Presume that Ω is a complete MMS and m is a strict MM. Suppose that $H\colon \Omega_m \to \Omega_m$ satisfies the condition (3.1). Assume that (C_1), (C_2) and (C_3^*) hold. Then H admits a fixed point.*

Proof. From Theorem 3, the same sequence $\{s_n\}$ can be constructed in Ω_m so that $\{s_n\}$ is m-Cauchy and m-convergent to $s \in \Omega_m$.

Assume that $s_{n_p} = Hs$ for all $p \in \mathbb{N}$. Then, the uniqueness of the limit gives $s = Hs$ as $n \to \infty$. Hereby, the proof is done.

Hence, assume that there exists $p_0 \in \mathbb{N}$ s.t. $s_{n_p} \neq Hs$, $\forall p \geq p_0$. Then, for all $p \geq p_0$ we find $Hs_{n_p-1} \neq Hs$, i.e., $m_\eta(Hs_{n_p-1}, Hs) > 0$. By (1.3), we can write

$$\kappa + F(\alpha(s_{n_p-1}, s)m_\eta(Hs_{n_p-1}, Hs)) \leq F\Big(\psi\big(\max\{m_\eta(s_{n_p-1}, s), \frac{m_\eta(s_{n_p-1}, Hs_{n_p-1})(1+m_\eta(s, Hs))}{1+m_\eta(s_{n_p-1}, s)}, \frac{m_\eta(s, Hs_{n_p-1})(1+m_\eta(s, Hs_{n_p-1}))}{1+m_\eta(s_{n_p-1}, s)}\}\big)\Big). \tag{3.25}$$

Since $\alpha(s_{n_p-1}, s) \geq 1$ and (1_F), we gain

$$m_\eta(Hs_{n_p-1}, Hs) \leq \psi\big(\max\{m_\eta(s_{n_p-1}, s), \frac{m_\eta(s_{n_p-1}, Hs_{n_p-1})(1+m_\eta(s, Hs))}{1+m_\eta(s_{n_p-1}, s)}, \frac{m_\eta(s, Hs_{n_p-1})(1+m_\eta(s, Hs_{n_p-1}))}{1+m_\eta(s_{n_p-1}, s)}\}\big). \tag{3.26}$$

As $n \to \infty$, we obtain $\lim\limits_{n\to\infty} m_\eta(s_{n_p}, Hs) = 0$. The uniqueness of the limit provides $s = Hs$. □

The subsequent theorem is verified in a similar manner.

Theorem 6. *Presume that Ω_m is a complete MMS and m is a strict MM. Suppose that $H\colon \Omega_m \to \Omega_m$ satisfies condition (3.2), and also the statements (C_1), (C_2) and (C_3^*) hold. Then H has a fixed point.*

Theorems 3-6 indicate the existence of a fixed point of a mapping, while the following theorem emphasizes the uniqueness of the fixed point of the mapping.

Theorem 7. *Suppose that Ω_m is a complete MMS and m is a strict MM. Assume that $H\colon \Omega_m \to \Omega_m$ provides the condition (3.1), and also the statements (C_1), (C_2) and (C_3) hold. If for all $s, r \in Fix(H)$, $\alpha(s, r) \geq 1$ holds where $Fix(H)$ represents the set of fixed points of H. Then the fixed point of H is unique.*

Proof. Assume that $r, s \in \Omega_m$ are two different fixed points of H, i.e., $Hs = s$ and $Hr = r$ with $r \neq s$. Then, $m_\eta(Hs, Hr) > 0$ is granted.
Hence, putting $\tau_0 = s$ and $\tau_1 = r$ in (1.3) gives

$$\kappa + F(\alpha(s, r)m_\eta(Hs, Hr)) \leq F\Big(\psi\big(\max\{m_\eta(s, r), \frac{m_\eta(s, Hs)(1+m_\eta(r, Hr))}{1+m_\eta(s, r)}, \frac{m_\eta(r, Hs)(1+m_\eta(r, Hs))}{1+m_\eta(s, r)}\}\big)\Big). \tag{3.27}$$

Then, we gain

$$\kappa + F(\alpha(s, r)m_\eta(Hs, Hr)) \leq F(\psi(m_\eta(s, r))) \leq F(m_\eta(s, r)). \tag{3.28}$$

Since $s, r \in Fix(H)$, $\alpha(s, r) \geq 1$ by (1_F), we have

$$\kappa + F(\alpha(s, r)m_\eta(s, r)) \leq \kappa + F(m_\eta(s, r)) \leq F(m_\eta(s, r)), \tag{3.29}$$

which gives $\kappa + F(m_\eta(s, r)) \leq F(m_\eta(s, r))$, a contradiction. Hence, the proof is completed. □

Similarly, we have the following theorem.

Theorem 8. *Assume that Ω_m is a complete MMS and m is a strict MM. Suppose that $H\colon \Omega_m \to \Omega_m$ satisfies condition (3.2), and (C_1), (C_2) and (C_3) hold. If for all $s,r \in Fix(H)$, $\alpha(s,r) \geq 1$ holds where $Fix(H)$ represents the set of fixed points of H. Then the fixed point of H is unique.*

An example illustrating the uniqueness of the fixed point is given below.

Example 4. Consider the set $\Omega = [0,\infty)$ with $m_\eta(\tau_0,\tau_1) = \frac{|\tau_0-\tau_1|}{\eta}$ for all $\tau_0,\tau_1 \in [0,\infty)$. Define the mapping

$$H(\tau) = \begin{cases} \frac{\tau^2+\tau}{4}, & \tau \in [0,1], \\ \frac{3}{4}, & \text{otherwise.} \end{cases} \tag{3.30}$$

The mapping $H : [0,\infty) \to [0,\infty)$ is continuous on Ω_m. Now, $\alpha\colon [0,\infty)\times[0,\infty) \to [0,\infty)$ is defined as

$$\alpha(\tau_0,\tau_1) = \begin{cases} 1, & \tau_0,\tau_1 \in [0,1], \\ 0, & \text{otherwise.} \end{cases} \tag{3.31}$$

For $\tau_0,\tau_1 \in \Omega$ with $\alpha(\tau_0,\tau_1) \geq 1$, we must have $\tau_0,\tau_1 \in [0,1]$. Then, we get $H\tau_0, H\tau_1 \in [0,1]$. From (3.31), we find $\alpha(H\tau_0,H\tau_1) \geq 1$, that is, H is an α-admissible mapping.

Here an element $s_0 \in [0,\infty)$ exists s.t. $\alpha(s_0,Hs_0) \geq 1$. If we take $s_0 = \frac{1}{2}$, then $Hs_0 = \frac{3}{16}$. Thus, $\alpha(s_0,Hs_0) = \alpha(\frac{1}{2},\frac{3}{16}) \geq 1$ holds.

Now, we show that (3.1) is satisfied for all $\tau_0,\tau_1 \in [0,\infty)$, where $F(\xi) = \ln(\xi)$ and $\psi(\xi) = \frac{3}{2}\xi \in \Psi$ for $\xi > 0$.

If τ_0 or τ_1 is not in $[0,1]$, then $\alpha(\tau_0,\tau_1) = 0$, and therefore (3.1) is satisfied.

For $\tau_0,\tau_1 \in [0,1]$ with $H\tau_0 \neq H\tau_1$, we write

$$\begin{aligned} \alpha(\tau_0,\tau_1) m_\eta(H\tau_0,H\tau_1)) &= \frac{|H\tau_0 - H\tau_1|}{\eta} \\ &= \frac{1}{4\eta}|\tau_0^2+\tau_0-(\tau_1^2+\tau_1)| \leq \frac{1}{4\eta}(|\tau_0^2-\tau_1^2|+|\tau_0-\tau_1|) \\ &\leq \frac{1}{4\eta}(|\tau_0-\tau_1||\tau_0+\tau_1|+|\tau_0-\tau_1|) \\ &\leq \frac{1}{4\eta}(|\tau_0-\tau_1|(|\tau_0+\tau_1|+1)) \\ &\leq \frac{3}{4}\frac{|\tau_0-\tau_1|}{\eta} \leq e^{-0.6}\frac{3}{2}m_\eta(\tau_0,\tau_1) = e^{-0.6}\psi(m_\eta(\tau_0,\tau_1)) \\ &\leq e^{-0.6}\psi(M_1(\tau_0,\tau_1)). \end{aligned} \tag{3.32}$$

Thus, (3.1) holds for all $\tau_0,\tau_1 \in \Omega$.

For all $\tau_0,\tau_1 \in Fix(H)$, we need to show that $\alpha(\tau_0,\tau_1) \geq 1$. Let $\tau_0,\tau_1 \in Fix(H)$. Then, τ_0,τ_1 must be in $[0,1]$, i.e., $\alpha(\tau_0,\tau_1) \geq 1$. Therefore, all conditions of Theorem 7 are satisfied. Consequently, H has a unique fixed point, i.e., $H0 = 0$.

Now, we introduce some results.

Corollary 1. *Let Ω_m be a complete MMS with a strict MM m. Assume that for a given $\kappa > 0$, $H\colon \Omega_m \to \Omega_m$ satisfies the following:*

$$d(H\tau_0,H\tau_1) > 0 \implies \kappa + F(m_\eta(H\tau_0,H\tau_1)) \leq F(\psi(M_1(\tau_0,\tau_1))) \tag{3.33}$$

for all $\tau_0,\tau_1 \in \Omega_m$, where $\psi \in \Psi$, $F \in \mathfrak{F}$, and

$$M_1(\tau_0,\tau_1) = \max\{m_\eta(\tau_0,\tau_1), \frac{m_\eta(\tau_0,H\tau_0)(1+m_\eta(\tau_1,H\tau_1))}{1+m_\eta(\tau_0,\tau_1)}, \frac{m_\eta(\tau_1,H\tau_0)(1+m_\eta(\tau_1,H\tau_0))}{1+m_\eta(\tau_0,\tau_1)}\}. \tag{3.34}$$

Then, H owns a unique fixed point in Ω_m.

Proof. Taking $\alpha(\tau_0,\tau_1) = 1$ in Theorem 7 concludes the proof. □

Corollary 2. *Let Ω_m be a complete MMS with a strict MM m. Assume that for a given $\kappa > 0$, $H\colon \Omega_m \to \Omega_m$ satisfies the following*

$$d(H\tau_0,H\tau_1) > 0 \implies \kappa + F(d(H\tau_0,H\tau_1)) \leq F(\psi(M_2(\tau_0,\tau_1))) \tag{3.35}$$

for all $\tau_0,\tau_1 \in \Omega_m$, where $\psi \in \Psi$, $F \in \mathfrak{F}$, and

$$\begin{aligned} M_2(\tau_0,\tau_1) = \max\{&m_\eta(\tau_0,\tau_1), \frac{m_\eta(\tau_0,H\tau_0)(1+m_\eta(\tau_1,H\tau_1))}{1+m_\eta(\tau_0,\tau_1)}, \\ &\frac{m_\eta(\tau_1,H\tau_1)m_\eta(\tau_0,H\tau_0)}{1+m_\eta(H\tau_0,F\tau_1)}, \frac{m_\eta(\tau_1,H\tau_1)m_\eta(\tau_1,H\tau_0)}{1+m_\eta(\tau_1,H\tau_0)+m_\eta(\tau_0,H\tau_1)}\}. \end{aligned} \tag{3.36}$$

Then, H owns a unique fixed point.

Proof. Putting $\alpha(\tau_0,\tau_1) = 1$ in Theorem 8 completes the proof. □

Now, we consider a partially ordered set (poset) in MMS. Then, we give some fixed point results for posets in MMS.

Definition 11. Let m be a MM on Ω and $(\Omega,\preceq)$ be a poset. A given map H on Ω_m is said to be nondecreasing w.r.t $\preceq$ whenever $\tau_0 \preceq \tau_1 \implies H\tau_0 \preceq H\tau_1$ for all $\tau_0,\tau_1 \in \Omega_m$. Moreover, a sequence $\{h_n\}$ is nondecreasing w.r.t $\preceq$ provided that $h_n \preceq h_{n+1}$ for all n.

Definition 12. Let $(\Omega,\preceq)$ be a poset and m be a MM on Ω. Suppose that $\lim_{n\to\infty} m_\eta(h_n,h) = 0$ for a given sequence $\{h_n\}$ in Ω_m. If there exists a subsequence $\{h_{n_k}\}$ of $\{h_n\}$ s.t. $\{h_{n_k}\} \preceq h$ for all k, then $(\Omega,\preceq)$ is called regular.

Corollary 3. *Let m be a strict MM in a complete MMS, Ω_m, and $(\Omega_m, \preceq)$ be a poset. Suppose that $H\colon \Omega_m \to \Omega_m$ is a nondecreasing mapping w.r.t $\preceq$, and (3.33) is satisfied with $\tau_0 \preceq \tau_1$. If the following statements hold:*

(C_1^*) *for a given $s_0 \in \Omega$, $s_0 \preceq Hs_0$ holds,*
(C_2^*) *either H is a continuous mapping,*
(C_3^*) *or $(\Omega_m, \preceq)$ is regular,*

then H has a fixed point.

Proof. Initially we define $\alpha : \Omega \times \Omega \to [0, \infty)$ as

$$\alpha(\tau_0, \tau_1) = \begin{cases} 1, & \tau_0 \preceq \tau_1 \text{ or } \tau_1 \preceq \tau_0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.37}$$

Since $\tau_1 \preceq \tau_0$, i.e., $m_\eta(H\tau_0, H\tau_1) \neq 0$, then H is F_α-contraction mapping, that is,

$$\kappa + F(\alpha(\tau_0, \tau_1) d(H\tau_0, H\tau_1)) \leq F(\psi(M_1(\tau_0, \tau_1)), \tag{3.38}$$

where $M_1(\tau_0, \tau_1)$ is given in (3.34).

From (C_1^*), we gain $\alpha(s_0, Hs_0) \geq 1$. Furthermore, if $\alpha(\tau_0, \tau_1) \geq 1$ for all $\tau_0, \tau_1 \in \Omega_m$, then either $\tau_0 \preceq \tau_1$ or $\tau_1 \preceq \tau_0$ holds which gives that $H\tau_0 \preceq H\tau_1$ or $H\tau_1 \preceq H\tau_0$. From the definition of $\alpha(.,.)$, we obtain $\alpha(H\tau_0, H\tau_1) \geq 1$. Hence, H is an α-admissible mapping.

Now, if H is continuous, then Theorem 3 assures that H has a fixed point.

Suppose that $(\Omega_m, \preceq)$ is regular. Consider a nondecreasing sequence $\{s_n\}$ in Ω_m, with $\lim_{n\to\infty} m_\eta(s_n, s) = 0$. Since $s_n \preceq s_{n+1}$ for all n, $\alpha(s_n, s_{n+1}) \geq 1$. In addition, we can find a subsequence $\{s_{n_p}\}$ of $\{s_n\}$ s.t. $s_{n_p} \preceq s$ for all p. Then, we attain $\alpha(s_{n_p}, s) \geq 1$ for all p. Hereby, Theorem 5 provides the existence of a fixed point.

Thus, we conclude the proof.

□

We also give the following result.

Corollary 4. *Let m be a strict MM in a complete MMS, Ω_m, and $(\Omega_m, \preceq)$ be a poset. Suppose that $H\colon \Omega_m \to \Omega_m$ is a nondecreasing mapping w.r.t $\preceq$, and (3.35) is satisfied with $\tau_0 \preceq \tau_1$. If the followings hold:*

(C_1^*) *for a given $s_0 \in \Omega$, $s_0 \preceq Hs_0$ holds,*
(C_2^*) *either H is a continuous mapping,*
(C_3^*) *or $(\Omega_m, \preceq)$ is regular,*

then H has a fixed point.

4. Application

This section investigates the existence of a solution for an integral equation given as

$$g(t) = K(t) + \int_a^b M(t,\zeta)Y(\zeta, g(\zeta))\,\mathrm{d}\zeta, \quad t \in [a,b], \tag{4.1}$$

where the function $Y\colon [0,T] \times R^+ \to R$ is continuous and nondecreasing, $M\colon [a,b] \times [a,b] \to R^+$ and $K\colon [a,b] \to R$ are continuous functions.

We study this equation on the set of a real-valued continuous function on $[a,b]$, i.e., $\Omega = C([a,b])$. We consider the following modular space around g_0:

$$\Omega_m = \Omega_m(g_0) = \{h \in \Omega = C([a,b])\colon m_\eta(g,g_0) \to 0 \quad as \quad \eta \to \infty\}. \tag{4.2}$$

Now, let us take the strict and convex modular metric $m_\eta(g,f) = \frac{d(g,f)}{\eta}$. Here $d(g,f)$ represents the usual metric $d(g,f) = \max_{t\in[a,b]} |g(t) - f(t)|$. Then the space Ω_m (4.2) becomes m-complete. Furthermore, we can equip this space with the partial order defined as $g, f \in C([a,b])$ s.t. $g \leq f \implies g(t) \preceq f(t)$ for all $t \in [a,b]$ so that Ω_m becomes partially ordered m-complete w.r.t $' \leq'$.

Now, let us define $H\colon C([0,T]) \to C([0,T])$ as

$$(Hg)(t) = K(t) + \int_a^b M(t,\zeta)Y(\zeta, g(\zeta))\,\mathrm{d}\zeta, \quad t \in [0,T]. \tag{4.3}$$

It is clear that if $g(t)$ is a fixed point of H, then $g \in C([a,b])$ is a solution of the integral equation (4.1). For this aim, we must show that all hypotheses in Corollary 3 hold.

Theorem 9. *Suppose that the followings hold:*

$(T1)$ $Y(\zeta, g(\zeta))$ *is nondecreasing w.r.t the second variable,*

$(T2)$ *for a given* $\kappa > 0$*,* $A \in (0,1)$ *and* $\eta \in (0,\infty)$*, there exist* $z\colon \Omega_m \times \Omega_m \to [0,\infty)$ *and* $\alpha\colon \Omega_m \times \Omega_m \to [0,\infty)$ *s.t. if* $\alpha(g,f) \geq 1$ *for all* $g, f \in \Omega_m$*, then for all* $\zeta \in [a,b]$

$$|Y(\zeta, g(\zeta)) - Y(\zeta, f(\zeta))| \leq z(g(\zeta), f(\zeta))|\frac{g(\zeta) - f(\zeta)}{\eta}|, \tag{4.4}$$

and

$$|\int_a^b M(t,\zeta)z(g,f)\,\mathrm{d}\zeta| \leq Ae^{-\kappa}, \tag{4.5}$$

$(T3)$ *there exists* $g_0 \in \Omega_m$ *s.t.* $g_0(t) \preceq K(t) + \int_a^b M(t,s\zeta)Y(\zeta, g_0(\zeta))\,\mathrm{d}\zeta$*,*

then H *has a solution in* $C([a,b])$*.*

Proof. In the first instance define $\alpha\colon \Omega_m \times \Omega_m \to [0,\infty)$ as

$$\alpha(g,f) = \begin{cases} 1, & g \preceq f, \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

Consider $g,f \in C([a,b])$ s.t. $g \preceq f \iff g(t) \leq f(t)$ for all $t \in [a,b]$. Since $Y(s,g(s))$ is nondecreasing, it easy to see that $Hg \leq Hf$, that is, F is nondecreasing w.r.t '$\leq$'. In addition, $m_{\eta}(Hg,Gf) > 0$ holds.

If $g \preceq f$, then $\alpha(g,f) \geq 1$. Hence, $(T1)$ provides

$$m_{\eta}(Hg,Hf) = \frac{d(Hg,Hf)}{\eta} = \frac{1}{\eta}\max_{t\in[a,b]}|Hg(t)-Hf(t)| \leq \frac{1}{\eta}\max_{t\in[a,b]}\int_a^b |M(t,\zeta)||Y(\zeta,g(\zeta))-Y(\zeta,f(\zeta))|\,\mathrm{d}\zeta. \tag{4.7}$$

By (4.4), we write

$$\begin{aligned} m_{\eta}(Hg,Hf) = &\leq \frac{1}{\eta}\max_{t\in[a,b]}\int_a^b |M(t,\zeta)|z(g,f)\frac{|h(\zeta)-g(\zeta)|}{\eta}\,\mathrm{d}\zeta \\ &\leq \max_{t\in[a,b]}\frac{|h(\zeta)-g(\zeta)|}{\eta}\int_a^b |M(t,\zeta)|z(g,f)\,\mathrm{d}\zeta \\ &\leq m_{\eta}(g,f)\int_a^b |M(t,\zeta)|z(g,f)\,\mathrm{d}\zeta. \end{aligned} \tag{4.8}$$

From (4.5), we obtain

$$m_{\eta}(Hg,Hf) \leq m_{\eta}(g,f)Ae^{-\kappa}. \tag{4.9}$$

Hereby, we write for all $g,f \in \Omega_m$ s.t. $g \preceq f$ with $Hg \neq Hf$

$$\alpha(g,f)m_{\eta}(Hg,Gf) \leq e^{-\kappa}\psi(m_{\eta}(g,f)) \leq e^{-\kappa}\psi(M_1(g,f)), \tag{4.10}$$

where $\psi(\xi) = A\xi \in \Psi$. Thus, (3.33) is satisfied for $F(\xi) = \ln(\xi)$.

From (4.6), we conclude $\alpha(Hg,Hf) \geq 1$. Thus, H is α-admissible. $(T3)$ guarantees that there exists $g_0 \in \Omega$ s.t. $g_0 \preceq Hg_0$, i.e., $\alpha(g_0,Hg_0) \geq 1$.

We, therefore, have all hypotheses in Corollary 3 fulfilled; that is, H possesses a fixed point. Consequently, $g \in C([a,b])$ is a solution of (4.1).

□

Example 5. Let $\Omega = C([0,1])$ and $m_{\eta}(g,f) = \max_{t\in[0,1]}\frac{|g(t)-f(t)|}{\eta}$. Consider the integral equation defined by

$$g(t) = \frac{1}{2} + \frac{1}{2\eta}\int_0^1 \frac{t}{\zeta+1}\frac{g(\zeta)}{2}\,\mathrm{d}\zeta, \quad t \in [0,1]. \tag{4.11}$$

Here, $K(t) = \frac{1}{2}$, $M(t,\zeta) = \frac{t}{\zeta+1}$, and $Y(\zeta,g(\zeta)) = \frac{g(\zeta)}{2\eta}$.

Let us define a mapping $H\colon C([0,1]) \to C([0,1])$ as

$$(Hg)(t) = \frac{1}{2} + \frac{1}{2\eta}\int_0^1 \frac{t}{\zeta+1}\frac{g(\zeta)}{2}\,\mathrm{d}\zeta, \quad t \in [0,1]. \tag{4.12}$$

We take the function α given in (4.6).

For $g \preceq f$, it is clear that $Y(\zeta, g(\zeta)) \leq Y(\zeta, f(\zeta))$, that is, Y is nondecreasing w.r.t the second variable. So, $(T1)$ is done.

If $g \preceq f$, then $\alpha(g,f) \geq 1$. Hence, for all $\zeta \in [0,1]$

$$|Y(\zeta,g(\zeta)) - Y(\zeta,f(\zeta))| \leq \frac{1}{2}\frac{g(\zeta)-f(\zeta)}{\eta} = z(g,f)|\frac{g(\zeta)-f(\zeta)}{\eta}|, \tag{4.13}$$

where $z(g,f) = \frac{1}{2}$. Moreover, we obtain

$$|\int_0^1 \frac{1}{2}\frac{t}{\zeta+1}\,\mathrm{d}\zeta| \leq \frac{1}{2}\max_{t\in[0,1]}\{t\}\int_0^1 \frac{1}{\zeta+1}\,\mathrm{d}\zeta \leq \frac{1}{2}\ln(2) \leq e^{-0.6}\ln(2). \tag{4.14}$$

Hereby, $T2$ is satisfied for $\kappa = -0.6$ and $A = \ln 2 \in (0,1)$.

We conclude that there exists an element $g_0 = 0$ in $C([0,1])$ s.t.

$$g_0 = 0 \leq \frac{1}{2} + \frac{1}{2\eta}\int_0^1 \frac{t}{\zeta+1}\frac{0}{2}\,\mathrm{d}\zeta = \frac{1}{2}.$$

So, $(T3)$ holds.

Since all conditions of Theorem 9 are satisfied, the integral equation (4.11) has a solution in $C([0,1])$. Indeed, $g(t) = \frac{1}{2} + \frac{t}{2\eta} \in C([0,1])$ is a solution for (4.11).

5. Conclusion

We introduce two generalized modified F_α-contraction of type I and II in *MMS*. For these contractions, we state the fixed point theorems, which give the existence of the fixed point. Moreover, we put another condition (C_3^*) instead of (C_3) in Theorem 3 and 4. In this case, we keep the existence of the fixed point. Furthermore, we consider an additional condition that provides the uniqueness of the fixed point. To support the outcomes attained here, we give some examples. We also put forward some significant results. Consequently, we consider an integral equation in *MMS* and prove the existence of this equation. The outcomes obtained herein extend the results for F_α-contraction mappings in metric spaces to *MMS*. As a final point, in the example $m_\eta(x,y) = \frac{d(x,y)}{\eta}$ if we take $\eta = 1$, all the consequences given are still valid in an ordinary metric structure, which emphasizes that the results obtained here are also a generalizations of the existing literature.

REFERENCES

[1] H. Abobaker and R. R. Ryan, "Modular metric spaces." *Irish Math. Soc. Bulletin.*, vol. 80, pp. 35–44, 2017, doi: 10.33232/BIMS.0080.35.44.

[2] H. Aydi, E. Karapinar, and H. Yazidi, "Modified F-contractions via α-admissible mappings and application to integral equations," *Filomat.*, vol. 31, no. 5, pp. 1141–1148, 2017, doi: 10.2298/FIL1705141A.

[3] B. Banach, "Sur les operations dans les ensembles abstraits et leur application aux equations integrales." *Fundam. Math.*, vol. 3, no. 1, pp. 133–181, 1922.

[4] V. Berinde, *Contracţii Generalizate şi Aplicaţii.* Romania: Editura Cub Press, 1997.

[5] R. M. Bianchini and M. Grandolfi, "Trasformazioni di tipo contraccettivo generalizzato in uno spazio metrico." *Atti. Acad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat.*, vol. 45, pp. 212–216, 1968.

[6] V. V. Chistyakov, "Modular metric spaces I: Basic concepts," *Nonlinear Anal.*, vol. 72, no. 1, pp. 1–14, 2010, doi: 10.1016/j.na.2009.04.057.

[7] V. V. Chistyakov, "Modular metric spaces II: Application to superposition operators," *Nonlinear Anal.*, vol. 72, no. 1, pp. 15–30, 2010, doi: 10.1016/j.na.2009.04.018.

[8] V. V. Chistyakov, "A fixed point theorem for contractions in modular metric spaces." *J. Appl. Anal.*, vol. 10, no. 1, pp. 1–31, 2011, doi: 10.48550/arXiv.1112.5561.

[9] N. V. Dung and V. L. Hang, "A fixed point theorem for generalized F-contractions on complete metric spaces," *Vietnam. J. Math.*, vol. 43, no. 4, pp. 743–753, 2015, doi: 10.1007/s10013-015-0123-5.

[10] Gündoğdu, H. and Öztürk, M. and Gözükızıl, Ö. F., "Fixed point theorems of new generalized c-conditions for (psi; gamma)-mappings in modular metric spaces and its applications," *Nonlinear Analysis: Modelling and Control*, vol. 28, no. 5, p. 949–970, 2023, doi: 10.15388/namc.2023.28.32711.

[11] Gündoğdu, H. and Öztürk, M. and Gözükızıl, Ö. F., "Novel fixed point theorems in partially ordered modular metric spaces with some applications," *Proceedings of the Bulgarian Academy of Sciences*, vol. 77, no. 8, p. 1115–1127, 2024, doi: 10.7546/CRABS.2024.08.01.

[12] H. Piri and P. Kumam, " Some fixed point theorems concerning F-contraction in complete metric spaces," *Fixed Point Theory Appl.*, vol. 210, no. 1, pp. 1–6, 2014, doi: 10.1186/1687-1812-2014-210.

[13] I. A. Rus, *Generalized contractions and applications.* Romania: Cluj University Press, 2001.

[14] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for α-ψ-contractive type mappings." *Fixed Point Theory Appl.*, vol. 75, no. 4, pp. 2154–2165, 2012, doi: 10.1016/j.na.2011.10.014.

[15] T. Senapati, L. K. Dey, and D. D. Dekic, "Extensions of Ciric and Wardowski type fixed point theorems in D-generalized metric spaces," *Fixed Point Theory Appl.*, vol. 2016, no. 33, pp. 1–14, 2016, doi: 10.1186/s13663-016-0522-7.

[16] M. Sgroi and C. Vetro, "Multi-valued F-contractions and the solution of certain functional and integral equations," *Filomat.*, vol. 27, no. 7, pp. 1259–1269, 2013, doi: 10.2298/FIL1307259S.

[17] W. Shatanawi, E. Karapinar, H. Aydi, and A. Fulga, "Wardowski type contractions with applications on Caputo type nonlinear fractional differential equations," *UPB Sci. Bull., Ser. A*, vol. 82, no. 2, pp. 157–170, 2020.

[18] S. Shukla, D. Gopal, and J. M. Moreno, "Fixed points of set-valued F-contractions and its application to non-linear integral equations," *Filomat.*, vol. 31, no. 11, pp. 3377–3390, 2017, doi: 10.2298/FIL1711377S.

[19] S. Shukla and S. Radenovic, "Some common fixed point theorems for F-contraction type mappings in 0-complete partial metric spaces," *Fixed Point Theory Appl.*, vol. 2013, no. 1, pp. 1–7, 2014, doi: 10.1155/2013/878730.

[20] D. Wardowski, “Fixed points of a new type of contractive mappings in complete metric spaces.” *Fixed Point Theory Appl.*, vol. 94, no. 1, pp. 1–6, 2012, doi: 10.1186/1687-1812-2012-94.

[21] D. Wardowski and N. V. Dung, “Fixed points of F-weak contractions on complete metric spaces,” *Demonstr. Math.*, vol. 47, no. 1, pp. 146–155, 2014, doi: 10.2478/dema-2014-0012.

Authors’ addresses

Hami Gündoğdu
(**Corresponding author**) Sakarya University, Department of Mathematics, Sakarya, Turkey
E-mail address: hamigundogdu@sakarya.edu.tr

Mahpeyker Öztürk
Sakarya University, Department of Mathematics, Sakarya, Turkey
E-mail address: mahpeykero@sakarya.edu.tr

Ömer Faruk Gözükızıl
Sakarya University, Department of Mathematics, Sakarya, Turkey
E-mail address: farukg@sakarya.edu.tr