Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 673–684
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4434

THE SOLUTIONS TO SOME DUAL MATRIX EQUATIONS

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Received 03 October, 2022

Abstract. The solvability conditions for the dual matrix equation $AXB = D$, a pair of dual matrix equations $AX = B, XC = D$ and the dual matrix equation $AX + YB = D$ are established by using the generalized inverses and the singular value decompositions of some real matrices, and the expressions of the general solutions to these dual matrix equations are presented. Finally, a numerical experiment is given to validate the accuracy of our result.

2010 *Mathematics Subject Classification:* 15A24; 15B99

Keywords: dual matrix equation, generalized inverse, singular value decomposition

1. INTRODUCTION

Throughout this paper, $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices, and the symbols $A^\top, A^-, A^\dagger$ and $\|A\|$ denote the transpose, the g-inverse, the Moore-Penrose inverse and the Frobenius norm of a real matrix A, respectively. I_n represents the identity matrix of order n. For a matrix $A \in \mathbb{R}^{m\times n}$, E_A and F_A stand for the orthogonal projectors: $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$.

In real and complex matrix spaces, many scholars have considered the following linear matrix equations:

$$AXB = D, \tag{1.1}$$

$$AX = B, \qquad XC = D, \tag{1.2}$$

and

$$AX + YB = D. \tag{1.3}$$

For Eq. (1.1), Penrose [15] obtained the solvability condition and the general solution of Eq. (1.1) by using the generalized inverse. Tian and Wang [18] analyzed the relations between least-squares and least-rank solutions of Eq. (1.1). Yuan and Dai [22] considered the reflexive solutions of Eq. (1.1) and the associated optimal approximation problem by applying the generalized singular value decomposition. For Eq. (1.2), Mitra [11] provided the solvability conditions and the general solution of

Eq. (1.2) by using the g-inverses of matrices. Dajić and Koliha [7] considered the common Hermitian and positive solutions to Eq. (1.2) for Hilbert space operators. For Eq. (1.3), Baksalary and Kala [1] and Chu [2] have been derived its general solution by using the g-inverse and the singular value decomposition (SVD), respectively. For inconsistent equations, l_p and Chebyshev solutions of Eq. (1.3) were considered in [24] and [25].

The dual number originally was introduced by Clifford [3], and then was further developed to represent the dual angle in spatial geometry [17]. For any dual number a can be writhen as $a = a_1 + \varepsilon a_2$, where $a_1, a_2 \in \mathbb{R}$ are the real part and the dual part of the dual number a, respectively, and ε denotes the dual unit with the operational rules:$\varepsilon \neq 0,\ \varepsilon 0 = 0\varepsilon = 0,\ \varepsilon 1 = 1\varepsilon = \varepsilon,\ \varepsilon^2 = 0$. A matrix whose elements are dual numbers is called a dual matrix, that is, for an $m \times n$ dual matrix A can be written as $A = A_1 + \varepsilon A_2$, where the matrices A_1 and $A_2 \in \mathbb{R}^{m\times n}$. Today, dual matrices are used in a various areas of engineering like the kinematic analysis and synthesis of spatial mechanisms [4, 16] and the robotics [8]. The growing applicability of dual matrices in science and engineering has sparked a renewal of interest in the linear algebra and numerical algorithms [12, 20].

The solution to some linear dual equations is also a task often required in the kinematic analysis of many spatial mechanisms [13, 14]. For example, Li et al.[9] used dual-quaternions to solve homogeneous transformation equation $AX = ZB$ for the robot-to-world and hand-eye calibration problem. Wang et al.[21] converted the multi-coordinate calibration problem of a dual-robot system into solving the $AXB = YCZ$ problem. Condurache and Burlacu [5] provided a new approach for solving $AX = XB$ sensor calibration problem by applying orthogonal dual tensor methods. Udwadia [19] explored the solution of the linear dual equation $Ax = \beta$ which is commonly encountered in areas of kinematics and robotics, where A is a p-by-q dual matrix and β is a p-by-1 dual vector.

We observe that the dual matrix equations of (1.1), (1.2) and (1.3) are seldom considered in the literature. In this paper, we will discuss the solvability conditions and general solutions of these dual matrix equations, which can be stated as the following problems:

Problem 1. *Given dual matrices $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m\times n}, B_i \in \mathbb{R}^{p\times q}$ and $D_i \in \mathbb{R}^{m\times q}$. Find a dual matrix $X = X_1 + \varepsilon X_2$, where $X_i \in \mathbb{R}^{n\times p}$ $(i = 1, 2)$ such that Eq. (1.1) is satisfied.*

Problem 2. *Given dual matrices $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2, C = C_1 + \varepsilon C_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m\times n}, B_i \in \mathbb{R}^{m\times p}, C_i \in \mathbb{R}^{p\times q}$ and $D_i \in \mathbb{R}^{n\times q}$. Find a dual matrix $X = X_1 + \varepsilon X_2$, where $X_i \in \mathbb{R}^{n\times p}$ $(i = 1, 2)$ such that Eq. (1.2) is satisfied.*

Problem 3. *Given dual matrices $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m\times n}, B_i \in \mathbb{R}^{p\times q}$ and $D_i \in \mathbb{R}^{m\times q}$. Find dual matrices $X = X_1 + \varepsilon X_2$ and $Y = Y_1 + \varepsilon Y_2$, where $X_i \in \mathbb{R}^{n\times q}$ and $Y_i \in \mathbb{R}^{m\times p}$ $(i = 1, 2)$ such that Eq. (1.3) is satisfied.*

In the following sections, each dual matrix equation will firstly be decomposed into two real matrix equations. Then, by making use of the generalized inverses and the SVDs of matrices, the solvability conditions and explicit solutions of Problems 1, 2 and 3 are derived. A numerical example is given to verify its correctness.

2. The solution of Problem 1

To begin with, we introduce some lemmas.

Lemma 1 ([23, page 907]). *Suppose that* $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{p\times q}$ *and* $D \in \mathbb{R}^{m\times q}$. *Then Eq.* (1.1) *has a solution* $X \in \mathbb{R}^{n\times p}$ *if and only if* $AA^{\dagger}DB^{\dagger}B = D$. *In this case, the general solution is* $X = A^{\dagger}DB^{\dagger} + F_AV_1 + V_2E_B$, *where* V_1 *and* V_2 *are arbitrary matrices.*

Lemma 2 ([6, page 30]). *Let* $A \in \mathbb{R}^{m\times p}, B \in \mathbb{R}^{m\times q}$ *and* $C = (I_m - AA^-)B$. *Then the g-inverse of the matrix* $[A,B]$ *is*

$$[A,B]^- = \begin{bmatrix} A^- - A^-BC^-(I_m - AA^-) \\ C^-(I_m - AA^-) \end{bmatrix}.$$

Lemma 3 ([1, page 145]). *Given* $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, $C \in R^{m\times r}$, $D \in \mathbb{R}^{s\times q}$ *and* $L \in \mathbb{R}^{m\times q}$. *Then the matrix equation* $AXB + CYD = L$ *is consistent if and only if*

$$E_GE_AL = 0, \quad E_ALF_D = 0, \quad E_CLF_B = 0, \quad LF_BF_H = 0,$$

where $G = E_AC$ *and* $H = DF_B$. *In this case, the general solution is*

$$X = A^-(L - CYD)B^- + Z - A^-AZBB^-,$$
$$Y = G^-E_ALD^- + (F_GC^- + F_CG^-E_A)LF_BH^- + W - C^-CF_GWHH^- - G^-GWDD^-,$$

where W *and* Z *are arbitrary matrices.*

Theorem 1. *Let dual matrices* $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2$ *and* $D = D_1 + \varepsilon D_2$, *where* $A_i \in \mathbb{R}^{m\times n}, B_i \in \mathbb{R}^{p\times q}$ *and* $D_i \in \mathbb{R}^{m\times q}$ $(i = 1,2)$, *and let* $H = E_{A_1}A_2F_{A_1}, K = E_{B_1}B_2F_{B_1}$. *Then Eq.* (1.1) *is solvable if and only if*

$$A_1A_1^{\dagger}D_1B_1^{\dagger}B_1 = D_1, \qquad (D_2 - A_1A_1^{\dagger}D_1B_1^{\dagger}B_2)F_{B_1}F_K = 0, \tag{2.1}$$
$$E_{A_1}D_2F_{B_1} = 0, \qquad E_HE_{A_1}(D_2 - A_2A_1^{\dagger}D_1B_1^{\dagger}B_1) = 0. \tag{2.2}$$

In this case, the solution set S_1 *of Problem 1 can be expressed as*

$$S_1 = \left\{X = X_1 + \varepsilon X_2 \middle| X_1, X_2 \in \mathbb{R}^{n\times p}\right\},$$

where

$$\begin{aligned} X_1 =& A_1^{\dagger}D_1B_1^{\dagger} + H^{\dagger}(D_2 - A_2A_1^{\dagger}D_1)B_1^{\dagger} + A_1^{\dagger}(D_2 - D_1B_1^{\dagger}B_2)K^{\dagger} \\ &+ F_{A_1}W_{22} - H^{\dagger}HW_{22}B_1B_1^{\dagger} + W_1E_{B_1} - A_1^{\dagger}A_1W_1KK^{\dagger}, \\ X_2 =& A_1^{\dagger}(I_m - A_2H^{\dagger})(D_2 - A_2A_1^{\dagger}D_1)B_1^{\dagger} - A_1^{\dagger}D_1B_1^{\dagger}B_2B_1^{\dagger} \end{aligned} \tag{2.3}$$

$$-A_1^\dagger(D_2 - D_1B_1^\dagger B_2)K^\dagger B_2B_1^\dagger - A_1^\dagger A_1W_{21}B_1B_1^\dagger + W_{21}$$
$$-A_1^\dagger A_1W_1E_KE_{B_1}B_2B_1^\dagger - A_1^\dagger A_2F_{A_1}F_HW_{22}B_1B_1^\dagger, \tag{2.4}$$

and W_1, W_{21}, W_{22} *are arbitrary matrices.*

Proof. Clearly, Eq. (1.1) can be equivalently written as

$$A_1X_1B_1 = D_1, \tag{2.5}$$
$$A_2X_1B_1 + A_1X_1B_2 + A_1X_2B_1 = D_2. \tag{2.6}$$

By Lemma 1, Eq. (2.5) is solvable if and only if the first condition of (2.1) is satisfied, and the general solution is

$$X_1 = A_1^\dagger D_1B_1^\dagger + F_{A_1}V_1 + V_2E_{B_1}, \tag{2.7}$$

where V_1 and $V_2 \in \mathbb{R}^{n\times p}$ are arbitrary matrices. Inserting (2.7) into Eq. (2.6), we have

$$[A_1, A_2F_{A_1}]\begin{bmatrix} X_2 \\ V_1 \end{bmatrix} B_1 + A_1V_2E_{B_1}B_2 = D_2 - A_2A_1^\dagger D_1B_1^\dagger B_1 - A_1A_1^\dagger D_1B_1^\dagger B_2. \tag{2.8}$$

By Lemma 2, we know that $\begin{bmatrix} A_1^\dagger - A_1^\dagger A_2F_{A_1}H^\dagger E_{A_1} \\ H^\dagger E_{A_1} \end{bmatrix} = \begin{bmatrix} A_1^\dagger - A_1^\dagger A_2H^\dagger \\ H^\dagger \end{bmatrix}$ is one of the g-inverse of $[A_1, A_2F_{A_1}]$. Thus

$$E_{[A_1, A_2F_{A_1}]} = I_m - [A_1, A_2F_{A_1}][A_1, A_2F_{A_1}]^- = E_HE_{A_1}.$$

By Lemma 3, Eq. (2.8) is solvable if and only if the second condition of (2.1) and conditions (2.2) are satisfied. In which case, the general solution is

$$V_2 = A_1^\dagger(D_2 - D_1B_1^\dagger B_2)K^\dagger + W_1 - A_1^\dagger A_1W_1KK^\dagger, \tag{2.9}$$
$$\begin{aligned}\begin{bmatrix} X_2 \\ V_1 \end{bmatrix} =& [A_1, A_2F_{A_1}]^-(D_2 - A_2A_1^\dagger D_1 - A_1^\dagger A_1D_1B_1^\dagger B_2 - A_1V_2E_{B_1}B_2)B_1^\dagger \\ &+ \begin{bmatrix} W_{21} \\ W_{22} \end{bmatrix} - [A_1, A_2F_{A_1}]^-[A_1, A_2F_{A_1}]\begin{bmatrix} W_{21} \\ W_{22} \end{bmatrix} B_1B_1^\dagger,\end{aligned} \tag{2.10}$$

where W_1, W_{21} and $W_{22} \in \mathbb{R}^{n\times p}$ are arbitrary matrices. From (2.9) and (2.10), we can get the relation of (2.4) and

$$V_1 = H^\dagger(D_2 - A_2A_1^\dagger D_1)B_1^\dagger + W_{22} - H^\dagger HW_{22}B_1B_1^\dagger. \tag{2.11}$$

Substituting (2.9) and (2.11) into (2.7), we can get (2.3). □

Clearly, the dual matrix equation $AXA = A$ is a special case of the dual matrix equation (1.1). According to Theorem 1, we can get the following result about $\{1\}$-dual generalized inverse of the dual matrix A, which was discussed by Udwadia in [19].

Corollary 1. *Given the dual matrix $A = A_1 + \varepsilon A_2$, where $A_1, A_2 \in \mathbb{R}^{m\times n}$. Then the dual matrix equation $AXA = A$ is solvable if and only if $H = E_{A_1} A_2 F_{A_1} = 0$. In this case, the general solution is $X = X_1 + \varepsilon X_2$, where*

$$X_1 = A_1^{\dagger} + W_1 E_{A_1} + F_{A_1} W_{22},$$
$$X_2 = -A_1^{\dagger} A_2 A_1^{\dagger} - A_1^{\dagger} A_1 W_1 E_{A_1} A_2 A_1^{\dagger} + W_{21} - A_1^{\dagger} A_1 W_{21} A_1 A_1^{\dagger} - A_1^{\dagger} A_2 F_{A_1} W_{22} A_1 A_1^{\dagger},$$

and W_1, W_{21}, W_{22} are arbitrary matrices.

3. The solution of Problem 2

Before solving Problem 2, we need the following Lemma.

Lemma 4 ([10, page 215])**.** *Suppose that $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{m\times p}, C \in \mathbb{R}^{p\times q}$ and $D \in \mathbb{R}^{n\times q}$. Then Eq. (1.2) has a solution $X \in \mathbb{R}^{n\times p}$ if and only if $E_A B = 0, DF_C = 0$ and $AD = BC$. In which case, the general solution is $X = A^{\dagger} B + F_A D C^{\dagger} + F_A W E_C$, where $W \in \mathbb{R}^{n\times p}$ is an arbitrary matrix.*

Theorem 2. *Assume that dual matrices $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2, C = C_1 + \varepsilon C_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m\times n}, B_i \in \mathbb{R}^{m\times p}, C_i \in \mathbb{R}^{p\times q}$ and $D_i \in \mathbb{R}^{n\times q}$ $(i = 1, 2)$. Let the SVDs of the matrices A_1 and C_1 be*

$$A_1 = P \begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix} Q^{\top}, \qquad C_1 = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^{\top}, \tag{3.1}$$

where $\Omega = \mathrm{diag}(\omega_1, \dots, \omega_r)$, $\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_s)$, $r = \mathrm{rank}(A_1)$, $s = \mathrm{rank}(C_1)$, and $P = [P_1, P_2] \in \mathbb{R}^{m\times m}$, $Q = [Q_1, Q_2] \in \mathbb{R}^{n\times n}$, $U = [U_1, U_2] \in \mathbb{R}^{p\times p}$, $V = [V_1, V_2] \in \mathbb{R}^{q\times q}$ are orthogonal matrices with $P_2 \in \mathbb{R}^{m\times(m-r)}$, $Q_2 \in \mathbb{R}^{n\times(n-r)}$, $U_2 \in \mathbb{R}^{p\times(p-s)}$ and $V_2 \in \mathbb{R}^{q\times(q-s)}$. If write

$$G_1 = B_2 - A_2 A_1^{\dagger} B_1 - A_2 F_{A_1} D_1 C_1^{\dagger}, \qquad G_2 = D_2 - A_1^{\dagger} B_1 C_2 - F_{A_1} D_1 C_1^{\dagger} C_2,$$
$$M_1 = U_2^{\top} C_2 V_2, \quad N_1 = P_2^{\top} A_2 Q_2, \quad H_1 = Q_2 F_{N_1}, \quad H_2 = E_{M_1} U_2^{\top},$$
$$S = Q_2 N_1^{\dagger} P_2^{\top} (B_2 - A_2 A_1^{\dagger} B_1) E_{C_1} + Q_2 F_{N_1} Q_2^{\top} G_2 V_2 M_1^{\dagger} U_2^{\top}.$$

Then Eq. (1.2) is solvable if and only if

$$E_{A_1} B_1 = 0, \qquad D_1 F_{C_1} = 0, \qquad A_1 D_1 = B_1 C_1, \tag{3.2}$$
$$A_1 G_2 = G_1 C_1, \qquad A_1^{\dagger} A_1 G_2 F_{C_1} = 0, \qquad E_{A_1} G_1 C_1 C_1^{\dagger} = 0, \tag{3.3}$$
$$E_{N_1} P_2^{\top} G_1 U_2 = 0, \qquad Q_2^{\top} G_2 V_2 F_{M_1} = 0, \qquad N_1 Q_2^{\top} G_2 V_2 = P_2^{\top} G_1 U_2 M_1. \tag{3.4}$$

The solution set S_2 of Problem 2 can be expressed as

$$S_2 = \left\{ X = X_1 + \varepsilon X_2 \middle| X_1, X_2 \in \mathbb{R}^{n\times p} \right\},$$

where

$$X_1 = A_1^{\dagger} B_1 + F_{A_1} D_1 C_1^{\dagger} + S + H_1 J_1 H_2, \tag{3.5}$$

$$X_2 = A_1^\dagger(G_1 - A_2S) + F_{A_1}G_2C_1^\dagger - SC_2C_1^\dagger - A_1^\dagger A_2H_1J_1H_2 \\ - H_1J_1H_2C_2C_1^\dagger + F_{A_1}R_1E_{C_1}, \tag{3.6}$$

and J_1, R_1 are arbitrary matrices.

Proof. Eq. (1.2) can be equivalently written as

$$A_1X_1 = B_1, \qquad X_1C_1 = D_1, \tag{3.7}$$

$$A_2X_1 + A_1X_2 = B_2, \qquad X_1C_2 + X_2C_1 = D_2. \tag{3.8}$$

By using Lemma 4, we know Eq. (3.7) is solvable if and only if the conditions (3.2) are satisfied, and the general solution is

$$X_1 = A_1^\dagger B_1 + F_{A_1}D_1C_1^\dagger + F_{A_1}W_1E_{C_1}, \tag{3.9}$$

where $W_1 \in \mathbb{R}^{n\times p}$ is an arbitrary matrix. Substituting (3.9) into Eq. (3.8), we can get

$$A_1X_2 = G_1 - A_2F_{A_1}W_1E_{C_1},\ X_2C_1 = G_2 - F_{A_1}W_1E_{C_1}C_2. \tag{3.10}$$

Using Lemma 4 again, Eq. (3.10) with respect to X_2 is solvable if and only if the first condition of (3.3) is satisfied and

$$F_{A_1}W_1E_{C_1}C_2F_{C_1} = G_2F_{C_1}, \tag{3.11}$$

$$E_{A_1}A_2F_{A_1}W_1E_{C_1} = E_{A_1}G_1. \tag{3.12}$$

In this case, the general solution is

$$X_2 = A_1^\dagger(G_1 - A_2F_{A_1}W_1E_{C_1}) + F_{A_1}(G_2 - F_{A_1}W_1E_{C_1}C_2)C_1^\dagger + F_{A_1}R_1E_{C_1}, \tag{3.13}$$

where $R_1 \in \mathbb{R}^{n\times p}$ is an arbitrary matrix. Inserting (3.1) to Eq. (3.11) and Eq. (3.12), we can obtain

$$Q\begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Q^\top W_1 U \begin{bmatrix} 0 & 0 \\ 0 & I_{p-s} \end{bmatrix} U^\top C_2 V \begin{bmatrix} 0 & 0 \\ 0 & I_{q-s} \end{bmatrix} V^\top \\ = G_2 V \begin{bmatrix} 0 & 0 \\ 0 & I_{q-s} \end{bmatrix} V^\top, \tag{3.14}$$

$$P\begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix} P^\top A_2 Q \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Q^\top W_1 U \begin{bmatrix} 0 & 0 \\ 0 & I_{p-s} \end{bmatrix} U^\top \\ = P\begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix} P^\top G_1. \tag{3.15}$$

If let

$$Q^\top W_1 U = \begin{bmatrix} W_{11} & W_{12} \\ W_{13} & W_{14} \end{bmatrix}, \quad U^\top C_2 V = \begin{bmatrix} C_{21} & C_{22} \\ C_{23} & C_{24} \end{bmatrix}, \quad Q^\top G_2 V = \begin{bmatrix} G_{21} & G_{22} \\ G_{23} & G_{24} \end{bmatrix},$$

$$P^\top A_2 Q = \begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix}, \quad P^\top G_1 U = \begin{bmatrix} G_{11} & G_{12} \\ G_{13} & G_{14} \end{bmatrix}.$$

Then Eq. (3.14) and Eq. (3.15) can be equivalently written as

$$G_{22} = 0, \qquad G_{13} = 0, \tag{3.16}$$

$$W_{14}M_1 = Q^\top G_2 V_2, \qquad N_1 W_{14} = P_2^\top G_1 U_2. \tag{3.17}$$

We note that

$$G_{22} = 0 \quad \Leftrightarrow \quad Q_1^\top G_2 V_2 = 0 \quad \Leftrightarrow \quad Q_1 Q_1^\top G_2 V_2 V_2^\top = 0 \quad \Leftrightarrow \quad A_1^\dagger A_1 G_2 F_{C_1} = 0,$$

which is the second condition of (3.3). Similarly, the condition $G_{13} = 0$ is equivalent to the third condition of (3.3). Using Lemma 4, Eq. (3.17) with respect to W_{14} is solvable if and only if conditions (3.4) hold, and the general solution is

$$W_{14} = N_1^\dagger p_2^\top G_1 U_2 + F_{N_1} Q_2^\top G_2 V_2 M_1^\dagger + F_{N_1} J_1 E_{M_1},$$

where $J_1 \in \mathbb{R}^{(n-r)\times(p-s)}$ is an arbitrary matrix. Thus, we have

$$F_{A_1} W_1 E_{C_1} = Q_2 W_{14} U_2^\top = S + H_1 J_1 H_2, \tag{3.18}$$

Substituting (3.18) into (3.9) and (3.13), we can obtain (3.5) and (3.6). □

4. The solution of Problem 3

Theorem 3. *Assume that dual matrices $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m\times n}, B_i \in \mathbb{R}^{p\times q}$ and $D_i \in \mathbb{R}^{m\times q} (i = 1,2)$. Let the SVDs of the matrices A_1 and B_1 be*

$$A_1 = P \begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix} Q^\top, \qquad B_1 = M \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} N^\top, \tag{4.1}$$

where $\Omega = \mathrm{diag}(\omega_1,\ldots,\omega_r)$, $\Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_t)$, $r = \mathrm{rank}(A_1)$, $t = \mathrm{rank}(B_1)$, and $P = [P_1,P_2] \in \mathbb{R}^{m\times m}$, $Q = [Q_1,Q_2] \in \mathbb{R}^{n\times n}$, $M = [M_1,M_2] \in \mathbb{R}^{p\times p}$, $N = [N_1,N_2] \in \mathbb{R}^{q\times q}$ are orthogonal matrices with $P_2 \in \mathbb{R}^{m\times(m-r)}$, $Q_2 \in \mathbb{R}^{n\times(n-r)}$, $M_2 \in \mathbb{R}^{p\times(p-t)}$ and $N_2 \in \mathbb{R}^{q\times(q-t)}$. If write

$$J = D_2 - A_2 A_1^\dagger D_1 - E_{A_1} D_1 B_1^\dagger B_2, \quad K_1 = P_2^\top A_2 Q_2, \quad K_2 = M_2^\top B_2 N_2. \tag{4.2}$$

Then Eq. (1.3) is solvable if and only if

$$E_{A_1} D_1 F_{B_1} = 0, \qquad E_{K_1} P_2^\top J N_2 F_{K_2} = 0. \tag{4.3}$$

The solution set S_3 of Problem 3 can be expressed as

$$S_3 = \{X = X_1 + \varepsilon X_2, Y = Y_1 + \varepsilon Y_2 | X_1, X_2 \in \mathbb{R}^{n\times q}, Y_1, Y_2 \in \mathbb{R}^{m\times p}\},$$

where

$$\begin{aligned} X_1 =& A_1^\dagger D_1 + Q_2 K_1^\dagger P_2^\top J F_{B_1} - A_1^\dagger P_1 V_{11} M_1^\top B_1 + Q_2 V_{23} N_1^\top \\ &+ Q_2 (F_{K_1} W_4 - K_1^\dagger W_3 K_2) N_2^\top, \end{aligned} \tag{4.4}$$

$$\begin{aligned} Y_1 =& E_{A_1} D_1 B_1^\dagger + P_2 E_{K_1} P_2^\top J N_2 K_2^\dagger M_2^\top + P_1 (V_{11} M_1^\top + V_{12} M_2^\top) \\ &+ P_2 (W_3 - E_{K_1} W_3 K_2 K_2^\dagger) M_2^\top, \end{aligned} \tag{4.5}$$

$$\begin{aligned} X_2 =& A_1^\dagger J - A_1^\dagger A_2 Q_2 (K_1^\dagger P_2^\top J N_2 - K_1^\dagger W_3 K_2 + F_{K_1} W_4) N_2^\top \\ & - A_1^\dagger P_1 (V_{11} M_1^\top + V_{12} M_2^\top) B_2 - A_1^\dagger R_1 B_1 + F_{A_1} R_2 \\ & + A_1^\dagger A_2 (A_1^\dagger P_1 V_{11} M_1^\top B_1 - Q_2 V_{23} N_1^\top), \end{aligned} \tag{4.6}$$

$$\begin{aligned} Y_2 =& E_{A_1} J B_1^\dagger + E_{A_1} A_2 (A_1^\dagger P_1 V_{11} M_1^\top B_1 - Q_2 V_{23} N_1^\top) B_1^\dagger \\ & - P_2 E_{K_1} P_2^\top J N_2 K_2^\dagger M_2^\top B_2 B_1^\dagger - P_2 (W_3 - E_{K_1} W_3 K_2 K_2^\dagger) M_2^\top B_2 B_1^\dagger \\ & + R_1 - E_{A_1} R_1 B_1 B_1^\dagger, \end{aligned} \tag{4.7}$$

and V_{11}, V_{12}, V_{23}, W_3, W_4, R_1, R_2 *are arbitrary matrices.*

Proof. Obviously, Eq. (1.3) can be equivalently written as

$$A_1 X_1 + Y_1 B_1 = D_1, \tag{4.8}$$

$$A_2 X_1 + A_1 X_2 + Y_2 B_1 + Y_1 B_2 = D_2. \tag{4.9}$$

From Lemma 3, Eq. (4.8) is solvable if and only if the first condition of (4.3) is satisfied, and the general solution is

$$Y_1 = E_{A_1} D_1 B_1^\dagger + V_1 - E_{A_1} V_1 B_1 B_1^\dagger, \tag{4.10}$$

$$X_1 = A_1^\dagger D_1 - A_1^\dagger V_1 B_1 + F_{A_1} V_2, \tag{4.11}$$

where V_1 and V_2 are arbitrary matrices. Plugging (4.10) and (4.11) into Eq. (4.9), we have

$$A_1 X_2 + Y_2 B_1 = J - A_2 F_{A_1} V_2 + A_2 A_1^\dagger V_1 B_1 - V_1 B_2 + E_{A_1} V_1 B_1 B_1^\dagger B_2. \tag{4.12}$$

Using Lemma 3 again, Eq. (4.12) with respects to X_2 and Y_2 is solvable if and only if

$$E_{A_1} A_2 F_{A_1} V_2 F_{B_1} + E_{A_1} V_1 E_{B_1} B_2 F_{B_1} = E_{A_1} J F_{B_1}. \tag{4.13}$$

In this case, the general solution is

$$\begin{aligned} Y_2 =& E_{A_1} J B_1^\dagger + E_{A_1} A_2 A_1^\dagger V_1 B_1 B_1^\dagger - E_{A_1} V_1 E_{B_1} B_2 B_1^\dagger \\ & - E_{A_1} A_2 F_{A_1} V_2 B_1^\dagger + R_1 - E_{A_1} R_1 B_1 B_1^\dagger, \end{aligned} \tag{4.14}$$

$$X_2 = A_1^\dagger (J - A_2 F_{A_1} V_2 + A_2 A_1^\dagger V_1 B_1 - V_1 B_2 - Y_2 B_1) + F_{A_1} R_2, \tag{4.15}$$

where R_1 and R_2 are arbitrary matrices. Inserting (4.1) to Eq. (4.13), we can get

$$\begin{aligned} & P \begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix} P^\top A_2 Q \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Q^\top V_2 N \begin{bmatrix} 0 & 0 \\ 0 & I_{q-t} \end{bmatrix} N^\top \\ & + P \begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix} P^\top V_1 M \begin{bmatrix} 0 & 0 \\ 0 & I_{p-t} \end{bmatrix} M^\top B_2 N \begin{bmatrix} 0 & 0 \\ 0 & I_{q-t} \end{bmatrix} N^\top \\ & = P \begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix} P^\top J N \begin{bmatrix} 0 & 0 \\ 0 & I_{q-t} \end{bmatrix} N^\top. \end{aligned} \tag{4.16}$$

If let

$$P^\top A_2 Q = \begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix}, \quad Q^\top V_2 N = \begin{bmatrix} V_{21} & V_{22} \\ V_{23} & V_{24} \end{bmatrix}, \quad P^\top V_1 M = \begin{bmatrix} V_{11} & V_{12} \\ V_{13} & V_{14} \end{bmatrix},$$

$$M^\top B_2 N = \begin{bmatrix} B_{21} & B_{22} \\ B_{23} & B_{24} \end{bmatrix}, \quad P^\top J N = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix}.$$

Then Eq. (4.16) can be equivalently written as

$$K_1 V_{24} + V_{14} K_2 = P_2^\top J N_2. \tag{4.17}$$

Using Lemma 3, Eq. (4.17) is solvable if and only if the second condition of (4.3) holds, and the general solution is

$$V_{14} = E_{K_1} P_2^\top J N_2 K_2^\dagger + W_3 - E_{K_1} W_3 K_2 K_2^\dagger, \tag{4.18}$$

$$V_{24} = K_1^\dagger P_2^\top J N_2 - K_1^\dagger W_3 K_2 + F_{K_1} W_4, \tag{4.19}$$

where W_3 and W_4 are arbitrary matrices. Thus,

$$V_1 = P_1 V_{11} M_1^\top + P_2 V_{13} M_1^\top + P_1 V_{12} M_2^\top + P_2 V_{14} M_2^\top, \tag{4.20}$$

$$V_2 = Q_1 V_{21} N_1^\top + Q_2 V_{23} N_1^\top + Q_1 V_{22} N_2^\top + Q_2 V_{24} N_2^\top, \tag{4.21}$$

where V_{11}, V_{12}, V_{13}, V_{21}, V_{22} and V_{23} are arbitrary matrices. Substituting (4.18)–(4.21) into (4.10),(4.11),(4.14),(4.15), and noticing

$$\begin{aligned}
&E_{A_1} P_1 V_{11} M_1^\top = 0, \quad P_1 V_{11} M_1^\top E_{B_1} = 0, \quad A_1^\dagger P_2 V_{13} M_1^\top = 0, \quad P_2 V_{13} M_1^\top E_{B_1} = 0,\\
&E_{A_1} P_1 V_{12} M_2^\top = 0, \quad P_1 V_{12} M_2^\top B_1 = 0, \quad A_1^\dagger P_2 V_{14} M_2^\top = 0, \quad P_2 V_{14} M_2^\top B_1 = 0,\\
&F_{A_1} Q_1 V_{21} N_1^\top = 0, \quad Q_1 V_{21} N_1^\top F_{B_1} = 0, \quad A_1 Q_2 V_{23} N_1^\top = 0, \quad Q_2 V_{23} N_1^\top F_{B_1} = 0,\\
&F_{A_1} Q_1 V_{22} N_2^\top = 0, \quad Q_1 V_{22} N_2^\top B_1^\dagger = 0, \quad A_1 Q_2 V_{24} N_2^\top = 0, \quad Q_2 V_{24} N_2^\top B_1^\dagger = 0,
\end{aligned}$$

we can obtain (4.4)–(4.7). □

Based on Theorem 3, we can formulate the following algorithm to solve Problem 3.

Algorithm 1.

(1) *Input matrices* A_i, B_i *and* D_i $(i = 1, 2)$.
(2) *Compute the SVDs of the matrices* A_1 *and* B_1 *by* (4.1).
(3) *Calculate* J, K_1 *and* K_2 *by* (4.2).
(4) *If the conditions* (4.3) *are satisfied, go to* (5)*; otherwise, the Eq.* (1.3) *has no solution, and stop.*
(5) *Randomly choose the matrices* V_{11}, V_{12}, V_{23}, W_3, W_4, R_1 *and* R_2.
(6) *Compute dual matrix* $X = X_1 + \varepsilon X_2$ *by* (4.4)–(4.7).

Example 1. Let $m=7,\ n=6,\ p=5,\ q=8$ and the matrices $A_i,\ B_i$ and $D_i\ (i=1,2)$ be given by

$$A_1=\begin{bmatrix} 0.9295 & 0.8406 & 0.3948 & 0.8918 & 1.1059 & 0.6344 \\ 1.6982 & 1.7141 & 0.7129 & 1.7326 & 1.6930 & 1.3104 \\ 1.0853 & 1.1207 & 0.5291 & 0.9446 & 0.9248 & 0.7579 \\ 1.3420 & 1.2970 & 0.7564 & 0.8606 & 1.1432 & 0.8302 \\ 1.1715 & 1.2305 & 0.4552 & 1.3524 & 1.1679 & 0.7921 \\ 0.7213 & 0.8033 & 0.3125 & 0.7792 & 0.5855 & 0.4736 \\ 1.0188 & 1.1001 & 0.3871 & 1.2182 & 0.9771 & 0.6932 \end{bmatrix},$$

$$A_2=\begin{bmatrix} 0.4243 & 0.7691 & 0.9493 & 0.8620 & 0.4070 & 0.1117 \\ 0.2703 & 0.3968 & 0.3276 & 0.9899 & 0.7487 & 0.1363 \\ 0.1971 & 0.8085 & 0.6713 & 0.5144 & 0.8256 & 0.6787 \\ 0.8217 & 0.7551 & 0.4386 & 0.8843 & 0.7900 & 0.4952 \\ 0.4299 & 0.3774 & 0.8335 & 0.5880 & 0.3185 & 0.1897 \\ 0.8878 & 0.2160 & 0.7689 & 0.1548 & 0.5341 & 0.4950 \\ 0.3912 & 0.7904 & 0.1673 & 0.1999 & 0.0900 & 0.1476 \end{bmatrix},$$

$$B_1=\begin{bmatrix} 0.6079 & 1.1159 & 0.5956 & 0.6270 & 1.2325 & 1.0054 & 0.4038 & 0.9286 \\ 1.1726 & 1.7527 & 1.2573 & 1.0184 & 1.7030 & 1.8189 & 1.2731 & 1.1029 \\ 1.1426 & 1.8468 & 1.1734 & 1.0666 & 1.8938 & 1.8022 & 1.0471 & 1.3055 \\ 1.2331 & 1.8517 & 1.3963 & 1.0324 & 1.8061 & 1.9805 & 1.4456 & 1.2211 \\ 0.7972 & 1.0332 & 0.6332 & 0.7645 & 0.8880 & 0.9636 & 0.6552 & 0.3169 \end{bmatrix},$$

$$B_2=\begin{bmatrix} 0.8449 & 0.6147 & 0.1231 & 0.6352 & 0.5358 & 0.8739 & 0.4170 & 0.1420 \\ 0.2094 & 0.3624 & 0.2055 & 0.2819 & 0.4452 & 0.2703 & 0.2060 & 0.1665 \\ 0.5523 & 0.0495 & 0.1465 & 0.5386 & 0.1239 & 0.2085 & 0.9479 & 0.6210 \\ 0.6299 & 0.4896 & 0.1891 & 0.6952 & 0.4904 & 0.5650 & 0.0821 & 0.5737 \\ 0.0320 & 0.1925 & 0.0427 & 0.4991 & 0.8530 & 0.6403 & 0.1057 & 0.0521 \end{bmatrix},$$

$$D_1=\begin{bmatrix} 6.1406 & 8.0527 & 7.0589 & 5.8597 & 9.0681 & 8.7133 & 6.2116 & 7.1038 \\ 5.9687 & 6.8809 & 7.6271 & 6.0367 & 8.8409 & 8.1612 & 6.6011 & 7.8182 \\ 5.1132 & 6.4170 & 5.9975 & 5.0679 & 7.5120 & 7.2419 & 5.5252 & 6.0488 \\ 3.5206 & 3.7180 & 4.3838 & 3.6134 & 5.0063 & 4.7191 & 3.9893 & 4.7616 \\ 5.1204 & 6.1849 & 6.2281 & 5.2350 & 7.5842 & 7.1172 & 5.4687 & 6.3310 \\ 3.2391 & 4.1829 & 3.8441 & 3.4011 & 5.1168 & 4.6853 & 3.3821 & 4.2400 \\ 3.3819 & 3.8221 & 4.3239 & 3.6439 & 5.1230 & 4.6124 & 3.6780 & 4.5859 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 8.0148 & 8.6320 & 7.9558 & 8.7138 & 10.4261 & 10.2622 & 7.6799 & 8.7502 \\ 8.0950 & 8.6298 & 9.7167 & 8.8261 & 9.6116 & 9.5072 & 7.0522 & 8.1786 \\ 6.9905 & 7.4758 & 7.4314 & 7.6470 & 9.1041 & 8.3395 & 5.7783 & 6.9511 \\ 7.2252 & 8.3537 & 9.3238 & 7.8794 & 9.5693 & 9.1666 & 7.2035 & 7.9500 \\ 6.2462 & 6.5371 & 6.6738 & 6.9087 & 7.8271 & 7.9660 & 5.7225 & 6.5788 \\ 6.5421 & 7.5126 & 6.5417 & 6.4456 & 8.0466 & 8.4030 & 6.0659 & 6.3854 \\ 6.2975 & 7.3790 & 6.7705 & 6.7915 & 8.1127 & 8.4531 & 5.7726 & 5.9501 \end{bmatrix}.$$

It is easy to verify that the solvability conditions (4.3) are satisfied:

$$\|E_{A_1} D_1 F_{B_1}\| = 2.1298e-15, \quad \|E_{K_1} P_2^\top J N_2 F_{K_2}\| = 1.9240e-15.$$

According to Algorithm 1, and taking the matrices V_{11}, V_{12}, V_{23}, W_3, W_4, R_1 and R_2 as zero matrices, we can obtain a pair of solution $X = X_1 + \varepsilon X_2$ and $Y = Y_1 + \varepsilon Y_2$ of Problem 3 as follows:

$$X_1 = \begin{bmatrix} 0.8679 & 1.5792 & 1.0708 & 1.1229 & 1.3338 & 1.4621 & 1.0616 & 1.2846 \\ -1.1069 & -2.0598 & -1.2459 & -0.5771 & -1.3804 & -1.5612 & -1.0286 & -0.8204 \\ 0.4778 & 0.1432 & -0.1177 & 0.4213 & 0.3441 & 0.4644 & 0.4361 & 0.5040 \\ 1.3293 & 1.9433 & 1.5302 & 1.7165 & 2.4117 & 1.9648 & 1.1896 & 1.7743 \\ 6.3263 & 8.7531 & 6.6704 & 5.4077 & 8.9556 & 8.9486 & 5.6859 & 6.2824 \\ -4.8304 & -7.3331 & -4.0270 & -5.1423 & -7.4170 & -7.3118 & -3.6731 & -4.9208 \end{bmatrix},$$

$$X_2 = \begin{bmatrix} 3.9555 & 5.5750 & 4.1899 & 3.9725 & 6.0625 & 5.9144 & 3.9358 & 3.9953 \\ 5.4489 & 8.3834 & 5.1434 & 5.5083 & 8.5723 & 8.3644 & 4.1819 & 4.6228 \\ 3.6170 & 5.4246 & 3.9892 & 3.6878 & 5.9108 & 5.5483 & 3.6692 & 3.7562 \\ 6.1274 & 8.2455 & 4.9003 & 5.4792 & 8.5528 & 9.4694 & 5.3390 & 5.3433 \\ 2.9899 & 2.4317 & 3.4133 & 2.4154 & 3.5625 & 4.2695 & 5.2274 & 4.4004 \\ -23.5531 & -33.3170 & -21.8236 & -21.3747 & -36.1257 & -37.4695 & -24.4449 & -23.8245 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} 0.2731 & 0.1583 & 0.2593 & 0.1392 & 0.1734 \\ -0.0619 & -0.1359 & -0.1246 & -0.1256 & -0.1735 \\ 0.1409 & 0.3755 & 0.3286 & 0.3476 & 0.4810 \\ -0.1599 & -0.1836 & -0.2151 & -0.1660 & -0.2172 \\ -0.2442 & 0.0414 & -0.0900 & 0.0411 & 0.0510 \\ 0.4746 & 0.0933 & 0.3429 & 0.0674 & 0.0352 \\ -0.1406 & -0.1897 & -0.2349 & -0.1644 & -0.1789 \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} 0.5453 & -0.0005 & 0.4083 & -0.2108 & 0.3890 \\ -0.4241 & 0.0642 & -0.2844 & 0.2672 & -0.3632 \\ 1.1377 & -0.2030 & 0.7443 & -0.7519 & 0.9533 \\ -0.5297 & 0.0924 & -0.3426 & 0.3190 & -0.3464 \\ -0.3304 & -0.3287 & -0.4576 & -0.1886 & -0.6673 \\ 0.1814 & -0.0026 & 0.1125 & 0.0463 & -0.2845 \\ -0.0533 & 0.3677 & 0.2065 & 0.3118 & 0.6601 \end{bmatrix}.$$

The absolute errors are estimated by

$$\|A_1 X_1 + Y_1 B_1 - D_1\| = 4.0468e-14,$$

$$\|A_2X_1 + A_1X_2 + Y_2B_1 + Y_1B_2 - D_2\| = 5.6005e - 14,$$

which implies that X and Y is a pair of solution of Problem 3.

5. Acknowledgments

The authors are grateful to an anonymous referee for useful comments and suggestions which helped to improve the presentation of this paper.

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