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ON TWO AND THREE AND FOUR POINT BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL INCLUSIONS WITH $\phi-$LAPLACIAN

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Abstract. We show the existence of solutions satisfying two point, three point or four point boundary conditions, for the differential inclusion

$$(\phi(x'(t)))' \in F(t,x(t)),$$

where $F(\cdot,\cdot)$ is a compact lower semi-continuous multi-valued map and ϕ is a homeomorphism function.

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1. Introduction

In this paper, we consider the following second-order non linear differential inclusion problems:

$$\begin{cases} \big(\phi(x'(t))\big)' \in F(t,x(t)) \text{ a.e. on } [0,T]; \\ x(0) = x'(\eta), \end{cases} \tag{1.1}$$

$$\begin{cases} \big(\phi(x'(t))\big)' \in F(t,x(t)) \text{ a.e. on } [0,T]; \\ x(0) = 0,\ x(T) = x(\eta) \end{cases} \tag{1.2}$$

and

$$\begin{cases} \big(\phi(x'(t))\big)' \in F(t,x(t)) \text{ a.e. on } [0,T]; \\ x(0) = x'(\eta),\ x(\tau) = x(T) \end{cases} \tag{1.3}$$

where $F : [0,T]\times\mathbb{R} \to 2^{\mathbb{R}}$ is a compact multi-valued map, measurable and lower semi-continuous with respect to the second argument, $\phi :]-a,a[\to \mathbb{R}$ is a homeomorphism, $T > 0$ and $(\eta,\tau) \in]0,T[^2$.

This kind of problems has been studied by several authors in the last few years. Sun, Yang and Ge [14], have proved the existence of a positive concave solution of the following two classes of three point p-Laplacian boundary value problems

$$\begin{aligned}&\big(\phi_p(u'(t))\big)'+q(t)f(t,u(t),u'(t))=0, \quad t\in]0,1[,\\&\big(\phi_p(u'(t))\big)'+q(t)f(t,u(t))=0, \quad t\in]0,1[,\\&u(0)=0,\ u(T)=u(\eta),\end{aligned}$$

where f is continuous, $q(.)$ is a nonnegative continuous function defined on $]0,1[$, $\phi_p(s)=|s|^{p-2}s$, with $p>1$, is a p-Laplacian operator, $\phi_p^{-1}=\phi_q$ and $\frac{1}{p}+\frac{1}{q}=1$. We recall here that the ϕ-Laplacian operator generalizes the p-Laplacian operator.

Ayyadi, El Khattabi and Frigon, see [11], have applied the method of upper and lower solutions, combined with the topological degree, to prove the existence of solutions for the problem $\big(\phi(x'(t))\big)'=f(t,x(t),x'(t))$, satisfying periodic, Dirichlet or Neumann boundary conditions. The right-hand side is a Carathéodory function satisfying a growth condition of Wintner-Nagumo type.

Chinní, Di Bella, Jebelean and Precup, see [8], proved the existence of solutions, satisfying four point value problem, for the differential equation

$$-\big(\phi(u'(t))\big)'=f(t,u(t),u'(t)),\ u(0)=\alpha u(\xi),\ u(T)=\beta u(\eta),$$

where $f:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ is continuous, $\alpha,\beta\in[0,1[$, $0<\xi<\eta<T$ and ϕ is an increasing homeomorphism such that $\phi(0)=0$. Note that, if we take $\alpha=0$ and $\beta=1$ in the last problem, we obtain the same boundary conditions of problem (1.2).

The problems (1.2) and (1.3) are solved by Benchohra and Ntouyas in the case where $T=1$, $\phi=id$ (id denotes identity mapping) and F is a multi-valued map. Their work is based on a fixed point theorem for contraction multi-valued maps due to Covitz and Nadler. For more details, see [5].

In [2,3], Aitalioubrahim and Sajid solved the problems (1.1), (1.2) and (1.3) in the case where $T=1$, $\phi=id$ and F is a compact convex L^1-Carathéodory multifunction or is a closed multifunction, measurable in t and Lipschitz continuous in x.

Recently, Aitalioubrahim and Tebbaa [4] have applied the method of upper and lower solutions, combined with the topological degree, to prove the existence of solutions for the problem $\big(\phi(x'(t))\big)'\in F(t,x(t))$, satisfying periodic, Dirichlet boundary conditions. The set valued-map $F(\cdot,\cdot)$ is compact and lower semi-continuous. The function ϕ is a homeomorphism. For a review of results on this kind of boundary value problems for differential inclusions, we refer the reader to the references [1,6,10].

In this paper, we establish the existence results for the problems (1.1) , (1.2) and (1.3). Our approach is based on the topological degree. We study the case where the right hand side is lower semicontinuous. The main results of this work extend, to the multi-valued case, some existence results in [6,11,14] and in the literature related to this kind of problems.

2. Preliminaries and statement of the main results

In this section, we introduce notations, definitions, and preliminary facts from multi-valued analysis which are used throughout this paper. Let E be a Banach space equipped with the norm $\|.\|$. By $\mathcal{C}([0,T],E)$ we denote the Banach space of all continuous functions from $[0,T]$ into E equipped with the norm

$$\|x\|_\infty := \sup\big\{\|x(t)\|; t\in[0,T]\big\}.$$

$\mathcal{C}^1([0,T],E)$ denotes the Banach space of continuously differentiable functions on $[0,T]$ equipped with the norm

$$\|x\|_1 := \|x\|_\infty + \|x'\|_\infty.$$

The set $L^1([0,T],\mathbb{R})$ refers to the Banach space of Lebesgue integrable functions from $[0,T]$ into $\mathbb{R}$ equipped with the norm

$$\|x\|_{L^1} = \int_0^T |x(s)|ds.$$

We say that a subset A of $[0,T]\times\mathbb{R}$ is $\mathcal{L}\otimes\mathcal{B}$-measurable if A belongs to the σ-algebra generated by all sets of the form $I\times D$ where I is Lebesgue measurable in $[0,T]$ and D is Borel measurable in $\mathbb{R}$. A multifunction is said to be measurable if its graph is measurable.

Definition 1. A subset B of $L^1([0,T],\mathbb{R})$ is decomposable if for all $u,v\in B$ and $I\subset[0,T]$ measurable, the function $u\chi_I + v\chi_{[0,T]\setminus I}\in B$, where χ denotes the characteristic function.

Definition 2. Let E be a separable Banach space, X a nonempty closed subset of E and $G : X\to 2^E$ a multi-valued map with nonempty closed values. We say that G is lower semi-continuous if the set $\{x\in X : G(x)\cap C\neq\varnothing\}$ is open for any open set C in E.

Definition 3. Let $F : [0,T]\times\mathbb{R}\to 2^{\mathbb{R}}$ be a multi-valued map with nonempty compact values. Assign to F the multi-valued operator,

$$\mathcal{F} : \mathcal{C}([0,T],\mathbb{R})\to 2^{L^1([0,T],\mathbb{R})},$$

defined by

$$\mathcal{F}(x) = \left\{ y\in L^1([0,T],\mathbb{R}) : y(t)\in F(t,x(t)) \text{ for a.e. } t\in[0,T] \right\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F. We say that F is of the lower semi-continuous type if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

For multi-valued operator which is lower semi-continuous and has nonempty closed and decomposable values, we have the following well-known result.

Lemma 1 ([7]). *Let E be a separable metric space and let $\Gamma : E \to 2^{L^1([0,T],\mathbb{R})}$ be a multi-valued operator which is lower semi-continuous and has nonempty closed and decomposable values. Then Γ has a continuous selection, i.e. there exists a continuous function $g : E \to L^1([0,T],\mathbb{R})$ such that $g(y) \in \Gamma(y)$ for every $y \in E$.*

Now, we give the definition of compact and completely continuous function.

Definition 4. Let X and Y be topological spaces.

(1) A function $f : X \to Y$ is said compact if $f(X)$ is relatively compact.
(2) If X is a metric space, $f : X \to Y$ is called completely continuous if for any bounded subset $B \subset X$, $f(B)$ is relatively compact.

The notion of the topological degree is widely used in the sequel. Here we give its definition and some related results. For more details on topological degree, we refer the reader to [9, 13].

Definition 5. Let E be a real Banach space and id be the identity on E. A degree is an application, which associates to any open bounded set $U \subset E$ and any continuous compact function $f : \overline{U} \to E$, an integer $\deg(id - f, U)$ with the following properties:

(i) (Existence): If $0 \notin (id - f)(\partial U)$, where ∂U is the boundary of U, and $\deg(id - f, U) \neq 0$, then there exists $x \in U$ such that $x = f(x)$.
(ii) (Normalisation): If $0 \notin \partial U$, then $\deg(id, U) = 1$ if and only if $0 \in U$.
(iii) (Additivity): If $0 \notin (id - f)(\overline{U} \setminus U_1 \cup U_2)$, where U_1 and U_2 are disjoint open subsets of U and $\overline{U} \setminus U_1 \cup U_2$ is the relative complement of $U_1 \cup U_2$ in $\overline{U}$, then
$$\deg(id - f, U) = \deg(id - f, U_1) + \deg(id - f, U_2).$$
(iv) (Homotopy): If $H : [0,1] \times \overline{U} \to E$ is a continuous compact function such that $0 \notin (id - H(\lambda, \cdot))(\partial U)$, for every $\lambda \in [0,1]$, then
$$\deg(id - H(\lambda, \cdot), U) = \deg(id - H(0, \cdot), U), \quad \forall \lambda \in [0,1].$$
(v) (Excision): If $V \subset U$ is open and $0 \notin (id - f)(\overline{U} \setminus V)$, then
$$\deg(id - f, U) = \deg(id - f, V).$$

In this work, we will assume the following assumptions.

(H1) $F : [0,T] \times \mathbb{R} \to 2^{\mathbb{R}}$ is a set-valued map with nonempty compact values satisfying
  (i) $(t,x) \mapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}-$ measurable,
  (ii) $x \mapsto F(t,x)$ is lower semi-continuous for almost all $t \in [0,T]$.

(H2) There exists a function $m \in L^1([0,T],\mathbb{R}^+)$ such that for almost all $t \in [0,T]$ and all $x \in \mathbb{R}$
$$\|F(t,x)\| := \sup\Big\{|y| : y \in F(t,x)\Big\} \leq m(t).$$

(H3) $\phi :]-a, a[\to \mathbb{R}$, $0 < a \leq +\infty$, is an increasing homeomorphism such that $\phi(0) = 0$.

We denote by $W_a^{2,1}([0,T],\mathbb{R})$ the class of all functions $x \in C^1([0,T],\mathbb{R})$ such that $\|x'\|_\infty < a$, $\phi(x')$ is absolutely continuous and $(\phi(x'))' \in L^1([0,T],\mathbb{R})$.

Definition 6. A function $x : [0,T] \to \mathbb{R}$ is said to be solution of (1.1) (resp. (1.2) and (1.3)) if $x \in W_a^{2,1}([0,T],\mathbb{R})$ and x satisfies the conditions of (1.1) (resp. (1.2) and (1.3)).

In the sequel, we will use the following important Lemmas.

Lemma 2 ([12]). *If assumptions (H1)-(H2) are satisfied, then F is of the lower semi-continuous type.*

Lemma 3. *Assume that (H3) holds. Let $\rho \in \{\tau, \eta\}$ and*

$$K : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R}),\ Kv(t) = \int_0^t v(s)\mathrm{d}s, \quad \forall t \in [0,T]$$

be a linear operator. For any $c \in \mathbb{R}$ and $v \in C([0,T],\mathbb{R})$, we set

$$\omega(c,v;t) = \phi^{-1}(c + Kv(t)), \quad \forall t \in [0,T] \text{ and } E_\rho(c,v) = \int_\rho^T \omega(c,v;t)\mathrm{d}t.$$

For each $v \in C([0,T],\mathbb{R})$, there exists an unique $Q(v) \in \mathbb{R}$, such that

$$E_\rho(Q(v),v) = 0.$$

Moreover, the map $Q : C([0,T],\mathbb{R}) \to \mathbb{R}$ is continuous and satisfies the following

$$|Q(v)| \le T\|v\|_\infty, \quad \forall v \in C([0,T],\mathbb{R}). \tag{2.1}$$

Proof. We use the same proof of Lemma 2.1 in [8]. From $|Kv(t)| \le T\|v\|_\infty$ and (H_3), we get

$$\omega(-T\|v\|_\infty, v;t) \le 0 \le \omega(T\|v\|_\infty, v;t), \quad \forall t \in [0,T],$$

which yields

$$E_\rho(-T\|v\|_\infty, v) \le 0 \le E_\rho(T\|v\|_\infty, v). \tag{2.2}$$

Since $\omega(\cdot, v; \cdot) : \mathbb{R} \times [0,T] \to \mathbb{R}$ is continuous, using Lebesgue's dominated convergence theorem, it is easy to see that, for any $0 \le b_1 < b_2 \le T$, the mapping

$$c \mapsto \int_{b_1}^{b_2} \omega(c,v;t)\mathrm{d}t$$

is continuous too. This implies that $E_\rho(\cdot, v)$ is continuous on $\mathbb{R}$. Then, the existence of some $Q(v)$ with $E_\rho(Q(v),v) = 0$ and satisfying (2.1). For the uniqueness of $Q(v)$, by the monotonicity of ϕ^{-1}, for any $c_1 < c_2$, one has

$$\omega(c_1,v;t) < \omega(c_2,v;t), \quad \forall t \in [0,T].$$

So $E_\rho(\cdot, v)$ is strictly increasing on $\mathbb{R}$. To see that Q is continuous, let $\{v_n\}_n$ be a sequence, in $C([0,T],\mathbb{R})$, convergent to some $v \in C([0,T],\mathbb{R})$. We want to show that $Q(v_n) \to Q(v)$, as $n \to \infty$. We have

$$\int_\rho^T \phi^{-1}(Q(v_n) + Kv_n(t))\mathrm{d}t = 0.$$

Then, there exists $t_0 \in [\rho, T]$ such that

$$\phi^{-1}(Q(v_n) + Kv_n(t_0)) = 0.$$

We deduce from the last equality that $\{Q(v_n)\}_n$ converges to some $q \in \mathbb{R}$. It remains to show that $q = Q(v)$. Since

$$Q(v_n) + Kv_n \to q + Kv \text{ in } C([0,T],\mathbb{R}), \text{ as } n \to \infty,$$

we infer that $\left\{\phi^{-1}(Q(v_n) + Kv_n)\right\}_n$ is bounded in $C([0,T],\mathbb{R})$ and

$$\omega(Q(v_n), v_n; t) = \phi^{-1}(Q(v_n) + Kv_n(t)) \to \phi^{-1}(q + Kv(t)) = \omega(q, v; t),$$

for all $t \in [0,T]$. Then, using Lebesgue's dominated convergence theorem, we deduce the following

$$0 = E_\rho(Q(v_n), v_n) \to E_\rho(q, v) \text{ as } k \to \infty.$$

Then $E_\rho(q,v) = 0$. However, by the uniqueness part, this means that $q = Q(v)$ and the proof is complete. □

Now, we are ready to give the main results of this paper.

Theorem 1. *If assumptions (H1), (H2) and (H3) are satisfied, then the problem (1.1) has at least one solution x on* $[0,T]$.

Theorem 2. *If assumptions (H1), (H2) and (H3) are satisfied, then the problem (1.2) has at least one solution x on* $[0,T]$.

Theorem 3. *If assumptions (H1), (H2) and (H3) are satisfied, then the problem (1.3) has at least one solution x on* $[0,T]$.

3. Proof of Theorem 1

By Lemma 2, F is of the lower semi-continuous type. Then, by Lemma 1, there exists a continuous function $g : \mathcal{C}([0,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ such that $g(y) \in \mathcal{F}(y)$ for all $y \in \mathcal{C}([0,T],\mathbb{R})$, where $\mathcal{F}$ is the Niemytzki operator associated with F. Consider the problem

$$\begin{cases} (\phi(y'(t)))' = g(y)(t) \text{ a.e. on } [0,T]; \\ y(0) = y'(\eta). \end{cases} \tag{3.1}$$

Remark that, any solution of the problem (3.1) is a solution of the problem (1.1). Now, for all $\lambda \in [0,1]$, consider the following modified problem

$$\begin{cases} (\phi(y'(t)))' = \lambda g(y)(t) \text{ a.e. on } [0,T]; \\ y(0) = y'(\eta). \end{cases} \tag{3.2}$$

Let us consider the operators

$$N_g : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$$

and

$$\mathcal{H} : [0,1] \times C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R}),$$

defined, for all $t \in [0,T]$, by

$$N_g(u)(t) = \int_0^t g(u)(s)ds$$

and

$$\mathcal{H}(\lambda,u)(t) = \phi^{-1}\big(\lambda N_g(u)(\eta)\big) + \int_0^t \phi^{-1}\big(\lambda N_g(u)(s)\big)ds.$$

From the assumptions (H1), (H2) and (H3), the function $\mathcal{H}$ is continuous and completely continuous. Also, observe that, the fixed points of $\mathcal{H}(\lambda,.)$ are solutions of (3.2). Indeed, if $u = \mathcal{H}(\lambda,u)$, then, for all $t \in [0,T]$, we have

$$u(t) = \phi^{-1}\big(\lambda N_g(u)(\eta)\big) + \int_0^t \phi^{-1}\big(\lambda N_g(u)(s)\big)ds.$$

Thus, in particular for $t = 0$, one has $u(0) = \phi^{-1}\big(\lambda N_g(u)(\eta)\big)$. Also by derivation, we get

$$u'(t) = \phi^{-1}\big(\lambda N_g(u)(t)\big) \text{ for almost all } t \in [0,T],$$

then $u'(\eta) = \phi^{-1}\big(\lambda N_g(u)(\eta)\big) = u(0)$. In addition, we have

$$\phi(u'(t)) = \lambda N_g(u)(t) \text{ for almost all } t \in [0,T],$$

which implies

$$(\phi(u'(t))' = \lambda g(u)(t) \text{ for almost all } t \in [0,T].$$

Hence u is a solution of (3.2).

Proposition 1. *Assume that (H1), (H2) and (H3) hold. Then, there exists $R > 0$ such that $deg(id - \mathcal{H}(\lambda,\cdot),\mathcal{U}) = 1$ for every $\lambda \in [0,1]$, where $\mathcal{U} = \{u \in C([0,T],\mathbb{R}) : \|u\|_\infty < R\}$. In particular, the problem (3.2) has at least one solution for every $\lambda \in [0,1]$.*

Proof. Fix $R > a + Ta$. First note that $\|u\|_\infty < R$ for any fixed point u of $\mathcal{H}$. Indeed, let u be a fixed point of $\mathcal{H}$. Then

$$u(t) = \phi^{-1}\big(\lambda N_g(u)(\eta)\big) + \int_0^t \phi^{-1}\big(\lambda N_g(u)(s)\big)ds, \text{ for all } t \in [0,T].$$

which implies that $\|u\|_\infty \leq a + Ta < R$. Now, set

$$\mathcal{U} = \{u \in C([0,T],\mathbb{R}) : \|u\|_\infty < R\}.$$

Since

$$u \neq \mathcal{H}(\lambda, u), \quad \forall (\lambda, u) \in [0,1] \times \partial \mathcal{U},$$

by the homotopy property of the topological degree, we get

$$\deg(id - \mathcal{H}(\lambda,\cdot), \mathcal{U}) = \deg(id - \mathcal{H}(0,\cdot), \mathcal{U}), \quad \forall \lambda \in [0,1],$$

which gives

$$\deg(id - \mathcal{H}(\lambda,\cdot), \mathcal{U}) = \deg(id, \mathcal{U}) = 1, \quad \forall \lambda \in [0,1].$$

Therefore, for every $\lambda \in [0,1]$, $\mathcal{H}(\lambda,\cdot)$ has a fixed point, and hence (3.2) has a solution. □

Now, Proposition 1 assures the existence of a solution $u \in W_a^{2,1}\big([0,T],\mathbb{R}\big)$ of (3.2) for $\lambda = 1$. Hence the problem (1.1) has at least one solution x on $[0,T]$.

4. Proof of Theorem 2

In order to establish the existence of a solution to (1.2), we consider the following family of problems defined, for $\lambda \in [0,1]$, by

$$\begin{cases} (\phi(u'(t)))' = \lambda g(u)(t) \text{ for a.e. } t \in [0,T], \\ u(0) = 0,\ u(T) = u(\eta), \end{cases} \tag{4.1}$$

where $g : C([0,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ is defined in the previous section. We are going to transform the problem (4.1) to a fixed point problem. We consider the operator

$$\mathcal{D} : [0,1] \times C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$$

defined by

$$\mathcal{D}(\lambda,u)(t) = \int_0^t \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds, \tag{4.2}$$

with $Q_{\lambda,u}$ is uniquely determined by Lemma 3 and satisfies the following

$$\int_\eta^T \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds = 0. \tag{4.3}$$

Note that, by taking $\lambda = 0$ in (4.3), we get

$$Q_{0,u} = 0, \quad \forall u \in C([0,T],\mathbb{R}). \tag{4.4}$$

It follows from the assumptions (H1), (H2) and (H3) that $\mathcal{D}$ is continuous and completely continuous. On the other hand, if $u = \mathcal{D}(\lambda,u)$, we have

$$u(t) = \int_0^t \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds, \quad \text{for all} \quad t \in [0,T].$$

Hence $u(0)=0$. For $t=T$ and $t=\eta$, we have

$$u(T)=\int_0^T \phi^{-1}\big(Q_{\lambda,u}+\lambda N_g(u)(s)\big)ds \quad \text{and} \quad u(\eta)=\int_0^\eta \phi^{-1}\big(Q_{\lambda,u}+\lambda N_g(u)(s)\big)ds.$$

So

$$u(T)-u(\eta)=\int_0^T \phi^{-1}\big(Q_{\lambda,u}+\lambda N_g(u)(s)\big)ds-\int_0^\eta \phi^{-1}\big(Q_{\lambda,u}+\lambda N_g(u)(s)\big)ds.$$

Then, by (4.3), one has

$$u(T)-u(\eta)=\int_\eta^T \phi^{-1}\big(Q_{\lambda,u}+\lambda N_g(u)(s)\big)ds=0.$$

Hence $u(T)=u(\eta)$. Also, we get

$$u'(t)=\phi^{-1}\big(Q_{\lambda,u}+\lambda N_g(u)(t)\big), \quad \text{for almost all} \quad t\in[0,T].$$

So

$$(\phi(u'(t)))'=\lambda g(u)(t), \quad \text{for almost all} \quad t\in[0,T].$$

It follows that the fixed points of $\mathcal{D}(\lambda,.)$ are solutions of (4.1).

Proposition 2. *Assume that (H1), (H2) and (H3) hold. Then, there exists* $R>0$ *such that* $deg(id-\mathcal{D}(\lambda,\cdot),\mathcal{U})=1$ *for every* $\lambda\in[0,1]$, *where* $\mathcal{U}=\{u\in C([0,T],\mathbb{R}):\|u\|_\infty<R\}$. *In particular, (4.1) has at least one solution for every* $\lambda\in[0,1]$.

Proof. Fix $R>aT$. It is clair that $\|u\|_\infty<R$ for any fixed point u of $\mathcal{D}$. Set

$$\mathcal{U}=\{u\in C([0,T],\mathbb{R}):\|u\|_\infty<R\}.$$

Then, one has

$$u\neq\mathcal{D}(\lambda,u), \quad \forall(\lambda,u)\in[0,1]\times\partial\mathcal{U}.$$

Thus, by the homotopy property of the topological degree,

$$\deg(id-\mathcal{D}(\lambda,\cdot),\mathcal{U})=\deg(id-\mathcal{D}(0,\cdot),\mathcal{U}), \quad \forall\lambda\in[0,1].$$

Using (4.4), we get

$$\deg(id-\mathcal{D}(\lambda,\cdot),\mathcal{U})=\deg(id,\mathcal{U})=1, \quad \forall\lambda\in[0,1].$$

We conclude that, for every $\lambda\in[0,1]$, $D(\lambda,\cdot)$ has a fixed point, and hence (4.1) has a solution. □

Therefore, we showed the existence of a solution $u\in W_a^{2,1}\big([0,T],\mathbb{R}\big)$ of (4.1) for $\lambda=1$. Hence the problem (1.2) has at least one solution x on $[0,T]$.

5. Proof of Theorem 3

We consider the following family of problems defined, for $\lambda \in [0,1]$, by

$$\begin{cases} (\phi(u'(t)))' = \lambda g(u)(t) \text{ for a.e. } t \in [0,T], \\ u(0) = u'(\eta),\ u(\tau) = u(T). \end{cases} \tag{5.1}$$

Let us transform the problem (5.1) to a fixed point problem. We consider the operator

$$\mathcal{P} : [0,1] \times C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$$

defined by

$$\mathcal{P}(\lambda,u)(t) = \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(\eta)\big) + \int_0^t \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds, \tag{5.2}$$

with $Q_{\lambda,u}$ is uniquely determined by Lemma 3 and satisfies the following

$$\int_\tau^T \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds = 0. \tag{5.3}$$

Taking $\lambda = 0$ in (5.3), we get

$$Q_{0,u} = 0, \quad \forall u \in C([0,T],\mathbb{R}). \tag{5.4}$$

It follows from the assumptions (H1), (H2) and (H3) that $\mathcal{P}$ is continuous and completely continuous. On the other hand, if $u = \mathcal{P}(\lambda,u)$, we have

$$u(t) = \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(\eta)\big) + \int_0^t \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds, \quad \text{for all} \quad t \in [0,T].$$

Then for $t = 0$, we have $u(0) = \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(\eta)\big)$, and for $t = T$ and $t = \tau$, we get

$$u(T) = \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(\eta)\big) + \int_0^T \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds$$

and

$$u(\tau) = \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(\eta)\big) + \int_0^\tau \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds.$$

So

$$u(T) - u(\tau) = \int_0^T \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds - \int_0^\tau \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds.$$

Then, by (5.3),

$$u(T) - u(\tau) = \int_\tau^T \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds = 0.$$

Hence $u(T) = u(\tau)$. On the other hand, we have

$$u'(t) = \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(t)\big), \quad \text{for almost all} \quad t \in [0,T].$$

So

$$(\phi(u'(t)))' = \lambda g(u)(t), \quad \text{for almost all} \quad t \in [0,T].$$

In addition, we have $u'(\eta) = \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(\eta)\big) = u(0)$. It follows that the fixed points of $\mathcal{P}(\lambda,.)$ are solutions of (5.1).

Proposition 3. *Assume that (H1), (H2) and (H3) hold. Then, there exists $R > 0$ such that $deg(id - \mathcal{P}(\lambda,\cdot), \mathcal{U}) = 1$ for every $\lambda \in [0,1]$, where $\mathcal{U} = \{u \in \mathcal{C}([0,T],\mathbb{R}) : \|u\|_\infty < R\}$. In particular, (5.1) has at least one solution for every $\lambda \in [0,1]$.*

Proof. Fix $R > a + aT$. Set

$$\mathcal{U} = \{u \in \mathcal{C}([0,T],\mathbb{R}) : \|u\|_\infty < R\}.$$

We have $\|u\|_\infty < R$ for any fixed point u of $\mathcal{P}$. Then

$$u \neq \mathcal{P}(\lambda,u), \quad \forall(\lambda,u) \in [0,1] \times \partial\mathcal{U}.$$

By the homotopy property of the topological degree,

$$\deg(id - \mathcal{P}(\lambda,\cdot), \mathcal{U}) = \deg(id - \mathcal{P}(0,\cdot), \mathcal{U}), \quad \forall\lambda \in [0,1].$$

Using (5.4), we get

$$\deg(id - \mathcal{P}(\lambda,\cdot), \mathcal{U}) = \deg(id, \mathcal{U}) = 1, \quad \forall\lambda \in [0,1].$$

We conclude that, for every $\lambda \in [0,1]$, $P(\lambda,\cdot)$ has a fixed point, and hence (5.1) has a solution. □

Therefore, we showed the existence of a solution $u \in W_a^{2,1}\big([0,T],\mathbb{R}\big)$ of (5.1) for $\lambda = 1$. Hence the problem (1.3) has at least one solution x on $[0,T]$.

6. Illustrative example

Consider the ϕ-Laplacian differential inclusions with two, three and four boundary conditions

$$\begin{cases} \big(x'(t)^3\big)' \in F(t,x(t)) \text{ a.e. on } [0,4\pi]; \\ x(0) = x'(\frac{\pi}{2}), \end{cases} \tag{6.1}$$

$$\begin{cases} \big(x'(t)^3\big)' \in F(t,x(t)) \text{ a.e. on } [0,4\pi]; \\ x(0) = 0,\ x(4\pi) = x(2\pi) \end{cases} \tag{6.2}$$

and

$$\begin{cases} \big(x'(t)^3\big)' \in F(t,x(t)) \text{ a.e. on } [0,4\pi]; \\ x(0) = x'(\frac{\pi}{2}),\ x(2\pi) = x(4\pi), \end{cases} \tag{6.3}$$

where $F : [0,4\pi] \times \mathbb{R} \to 2^{\mathbb{R}}$ is a multi-valued map defined by

$$F(t,x) = \Big\{v \in \mathbb{R} : f_1(t,x) \leq v \leq f_2(t,x)\Big\}, \quad \forall t \in [0,4\pi],$$

and $f_1, f_2 : [0,4\pi]\times\mathbb{R}\to\mathbb{R}$ are single-valued functions such that, for each $t\in[0,4\pi]$,

$$f_1(t,x) = \frac{-3(1+3\sin(t))\cos^2(t)e^{(\ln(3+\sin(t))-\ln(3))^2}}{(3+\sin(t))^4 e^{x^2}} - 1,$$

$$f_2(t,x) = \frac{-3(1+3\sin(t))\cos^2(t)e^{(\ln(3+\sin(t))-\ln(3))^2}}{(3+\sin(t))^4 e^{x^2}} + 1.$$

It is clear that F has nonempty compact values and is measurable. For each $t\in[0,4\pi]$, $f_1(t,.)$ and $f_2(t,.)$ are continuous on $\mathbb{R}$. Then, for each $t\in[0,4\pi]$, $f_1(t,.)$ is upper semi-continuous on $\mathbb{R}$ and $f_2(t,.)$ is lower semi-continuous on $\mathbb{R}$. Hence, for each $t\in[0,4\pi]$, $F(t,.)$ is lower semi-continuous on $\mathbb{R}$. Moreover, for almost all $t\in[0,4\pi]$ and all $x\in\mathbb{R}$

$$\begin{aligned}\|F(t,x)\| &= \sup\Big\{|y| : y\in[f_1(t,x), f_2(t,x)]\Big\}\\ &\le max\{|f_1(t,x)|,|f_2(t,x)|\}\\ &\le \left|\frac{-3(1+3\sin(t))\cos^2(t)e^{(\ln(3+\sin(t))-\ln(3))^2}}{(3+\sin(t))^4 e^{x^2}}\right| + 1\\ &\le \frac{3}{4}e^{(\ln(3+\sin(t))-\ln(3))^2} + 1.\end{aligned}$$

Set

$$m(t) = \frac{3}{4}e^{(\ln(3+\sin(t))-\ln(3))^2} + 1, \quad \forall t\in[0,4\pi].$$

Then, for almost all $t\in[0,4\pi]$ and all $x\in\mathbb{R}$

$$\|F(t,x)\| \le m(t),$$

with $m\in L^1([0,4\pi],\mathbb{R}^+)$. In this example $\phi(x)=x^3$. It is clear that ϕ is an increasing homeomorphism such that $\phi(0)=0$. We conclude that all assumptions of Theorems 1, 2 and 3 are verified. Thus Problem (6.1) (resp. (6.2) and (6.3)) has at least one solution on $[0,4\pi]$. Set

$$u(t) = \ln\left(\frac{3+\sin(t)}{3}\right), \quad \forall t\in[0,4\pi].$$

u is a solution of Problem (6.1). Indeed, for almost all $t\in[0,4\pi]$,

$$u'(t) = \frac{\cos(t)}{3+\sin(t)} \text{ and } u''(t) = \frac{-1-3\sin(t)}{(3+\sin(t))^2}.$$

Then, for almost all $t\in[0,4\pi]$,

$$\begin{aligned}\big(\phi(u'(t))\big)' &= \big(u'(t)^3\big)'\\ &= 3u''(t)u'(t)^2\end{aligned}$$

$$
\begin{aligned}
&= 3 \times \frac{-1-3\sin(t)}{(3+\sin(t))^2} \times \frac{\cos^2(t)}{(3+\sin(t))^2} \\
&= \frac{-3(1+3\sin(t))\cos^2(t)}{(3+\sin(t))^4} \\
&= \frac{-3(1+3\sin(t))\cos^2(t)e^{(\ln(3+\sin(t))-\ln(3))^2}}{(3+\sin(t))^4 e^{(\ln(3+\sin(t))-\ln(3))^2}} \\
&= \frac{-3(1+3\sin(t))\cos^2(t)e^{(\ln(3+\sin(t))-\ln(3))^2}}{(3+\sin(t))^4 e^{u^2(t)}}.
\end{aligned}
$$

Hence, for almost all $t \in [0, 4\pi]$

$$
\big(\phi(u'(t))\big)' \in [f_1(t,u(t)), f_2(t,u(t))] = F(t,u(t)).
$$

In addition $u(0) = u'(\frac{\pi}{2}) = 0$. On the other hand, we have $u(0) = 0$ and $u(4\pi) = u(2\pi) = 0$. Then u is a solution of Problem (6.2). Also, we have $u(0) = u'(\frac{\pi}{2}) = 0$ and $u(2\pi) = u(4\pi)$. Then u is a solution of Problem (6.3).

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