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AMPLY SOCLE SUPPLEMENTED MODULES

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Abstract. In this work, amply socle supplemented modules are defined and some properties of these modules are investigated. We prove that every $\pi-$projective and socle supplemented module is amply socle supplemented. We also prove that every factor module and every homomorphic image of an amply socle supplemented module are amply socle supplemented. Let M be a projective and socle supplemented $R-$module. Then every finitely $M-$generated $R-$module is amply socle supplemented.

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1. Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an $R-$module. We will denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* (or *superfluous*) submodule of M and denoted by $N \ll M$. Let M be an R -module. M is called a *hollow* module if every proper submodule of M is small in M. M is called a *local* module if M has the largest submodule, i.e. a proper submodule which contains all other proper submodules. A submodule N of an R -module M is called an *essential* submodule of M and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equvalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. Let M be an $R-$module and $U,V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is called a *supplemented* module if every submodule of M has a supplement in M. If every essential submodule of M has a supplement in M, then M is called an *essential supplemented* (or briefly, *e-supplemented*) module. Let M be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a supplement V' with $V' \leq V$, we say U has *ample supplements* in M. If every submodule of M has ample supplements in

M, then M is called an *amply supplemented* module. If every essential submodule of M has ample supplements in M, then M is called an *amply essential supplemented* (or briefly, *amply e-supplemented*) module. The intersection of all maximal submodules of an $R-$module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. The sum of all simple submodules of an $R-$module M is called the *socle* of M and denoted by $SocM$. Let M be an $R-$module. It is defined the relation $'\beta^{*}\,'$ on the set of submodules of an $R-$module M by $X\beta^{*}Y$ if and only if $Y+K=M$ for every $K\leq M$ such that $X+K=M$ and $X+T=M$ for every $T\leq M$ such that $Y+T=M$. Let M be an $R-$module and $K\leq V\leq M$. We say V *lies above* K in M if $V/K\ll M/K$ (see [15, 16]).

More informations about (amply) supplemented modules are in [2, 13–16]. More details about (amply) essential supplemented modules are in [6, 7, 9–11]. The definition of β^{*} relation and some properties of this relation are in [1].

Lemma 1. *Let M be an R -module. The following statements hold.*

(i) *$SocM = \bigcap\limits_{L\trianglelefteq M} L$.*
(ii) *For $K\leq M$, $SocK = K\cap SocM$.*
(iii) *$SocM \trianglelefteq M$ if and only if $SocK\neq 0$ for every nonzero submodule K of M.*
(iv) *Let N be an $R-$module and $f: M\longrightarrow N$ be an $R-$module homomorphism. Then $f(SocM)\subset Socf(M)$.*
(v) *For $K\leq M$, $(SocM+K)/K\subset Soc(M/K)$.*
(vi) *If $M=\bigoplus\limits_{\Lambda} M_{\lambda}$, then $SocM=\bigoplus\limits_{\Lambda} SocM_{\lambda}$.*

Proof. See [16, 21.2]. □

Definition 1 ([3–5])**.** Let M be an $R-$module. If every $U\leq M$ with $SocM\leq U$ has a supplement in M, then M is called a socle supplemented (or briefly, s-supplemented) module.

Definition 2 ([4, 5])**.** Let M be an $R-$module and $X\leq M$. If X is a supplement of a submodule U of M with $SocM\leq U$, then X is called a s-supplement submodule in M.

Lemma 2 ([4, 5])**.** *Let M be a socle supplemented module. Then every finitely $M-$generated $R-$module is socle supplemented.*

2. Amply Socle Supplemented Modules

Definition 3. Let M be an $R-$module. If every submodule of M which contains $SocM$ has ample supplements in M, then M is called an amply socle supplemented (or briefly, amply s-supplemented) module. (See also [8])

Clearly, every amply socle supplemented module is socle supplemented.

Lemma 3. *Let M be an amply s-supplemented module. Then M is amply e-supplemented.*

Proof. Let $U \trianglelefteq M$. By Lemma 1, $SocM \leq U$ and since M is amply s-supplemented, U has ample supplements in M. Hence M is amply e-supplemented. □

Corollary 1. *Let M be an amply s-supplemented module. Then M is e-supplemented.*

Proof. Clear from Lemma 3. □

Proposition 1. *Let M be an amply e-supplemented module and $SocM \trianglelefteq M$. Then M is amply s-supplemented.*

Proof. Let $SocM \leq U \leq M$. Since $SocM \trianglelefteq M$, $U \trianglelefteq M$. Since M is amply e-supplemented, U has ample supplements in M. Hence M is amply s-supplemented. □

Proposition 2. *Let M be an amply s-supplemented module. Then $M/RadM$ have no proper essential submodules.*

Proof. Since M is amply s-supplemented, by Corollary 1, M is e-supplemented. Then by [11, Proposition 2.5], $M/RadM$ have no proper essential submodules. □

Lemma 4. *Let M be an amply s-supplemented module. Then every factor module of M is amply s-supplemented.*

Proof. Let M/K be any factor module of M. Let $Soc(M/K) \leq U/K \leq M/K$ and $M/K = U/K + V/K$. By Lemma 1, $(SocM+K)/K \subset Soc(M/K)$. Hence $(SocM+K)/K \leq U/K$ and $SocM \leq U$. Since $M/K = U/K + V/K$, $M = U + V$. Since M is amply s-supplemented, U has a supplement X in M with $X \leq V$. Since $K \leq U$, by [16, 41.1], $(X+K)/K$ is a supplement of U/K in M/K. Moreover, $(X+K)/K \leq V/K$. Hence M/K is amply s-supplemented. □

Corollary 2. *Every homomorphic image of an amply s-supplemented module is amply s-supplemented.*

Proof. Clear from Lemma 4. □

Let M be an $R-$module. If for every $U,V \leq M$ with $M = U+V$, there exists an $R-$module homomorphism $f : M \longrightarrow M$ with $f(M) \leq U$ and $(1-f)(M) \leq V$, then M is called a π-*projective* module. (See [12, 16].)

Lemma 5. *If M is a π-projective and s-supplemented module, then M is an amply s-supplemented module.*

Proof. Let $SocM \leq U \leq M$, $M = U+V$ and X be a supplement of U in M. Since M is π -projective and $M = U+V$, there exists an R -module homomorphism $f : M \to M$ such that $Imf \subset V$ and $Im(1-f) \subset U$. So, we have $M = f(M) + (1-f)(M) = f(U) + f(X) + U = U + f(X)$. Suppose that $a \in U \cap f(X)$. Since $a \in f(X)$, then there exists $x \in X$ such that $a = f(x)$. Since $a = f(x) = f(x) - x + x = x - (1-f)(x)$ and $(1-f)(x) \in U$, we have $x = a + (1-f)(x) \in U$. Thus $x \in U \cap X$ and so,

$a = f(x) \in f(U \cap X)$. Therefore we have $U \cap f(X) \leq f(U \cap X) \ll f(X)$. This means that $f(X)$ is a supplement of U in M. Moreover, $f(X) \subset V$. Therefore M is amply s-supplemented. □

Corollary 3. *If M is projective and s-supplemented, then M is amply s-supplemented.*

Proof. Clear from Lemma 5. □

Lemma 6. *Let M be an $R-$module. If every submodule of M which contains $SocM$ is β^* equivalent to a s-supplement submodule in M, then M is s-supplemented.*

Proof. Let $SocM \leq U \leq M$. By hypothesis, there exists a s-supplement submodule X in M such that $U\beta^*X$. Since X is a s-supplement submodule in M, there exists a submodule Y of M such that $SocM \leq Y$ and X is a supplement of Y in M. Since $SocM \leq Y$, by hypothesis, there exists a s-supplement submodule V in M such that $Y\beta^*V$. Since X is a supplement of Y in M and $Y\beta^*V$, by [1, Theorem 2.6 (ii)], X is a supplement of V in M. Since V is a supplement submodule in M, we can see that V is a supplement of X in M and since $U\beta^*X$, by [2, Theorem 2.6 (ii)], V is a supplement of U in M. □

Lemma 7. *Let M be a π-projective $R-$module. If every submodule of M which contains $SocM$ is β^* equivalent to a s-supplement submodule in M, then M is amply s-supplemented.*

Proof. By Lemma 6, M is s-supplemented. Then by Lemma 5, M is amply s-supplemented. □

Corollary 4. *Let M be a projective $R-$module. If every submodule of M which contains $SocM$ is β^* equivalent to a s-supplement submodule in M, then M is amply s-supplemented.*

Proof. Clear from Lemma 7. □

Corollary 5. *Let M be a π-projective $R-$module. If every submodule of M which contains $SocM$ lies above a s-supplement submodule in M, then M is amply s-supplemented.*

Proof. Clear from Lemma 7. □

Lemma 8. *Let Λ be a finite index set and $\{M_\lambda\}_\Lambda$ be a family of projective $R-$modules. If M_λ is s-supplemented for every $\lambda \in \Lambda$, then $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is amply s-supplemented.*

Proof. Since M_λ is s-supplemented for every $\lambda \in \Lambda$, we can see that $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is s-supplemented. Since M_λ is projective for every $\lambda \in \Lambda$, by [16, 18.1], $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is projective. Since $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is projective and s-supplemented, by Corollary 3, $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is amply s-supplemented. □

Corollary 6. *Let M be a projective $R-$module. If M is s-supplemented, then $M^{(\Lambda)}$ is amply s-supplemented for every finite index set Λ.*

Proof. Clear from Lemma 8. □

Corollary 7. *Let M be a projective $R-$module. If M is s-supplemented, then every finitely $M-$generated $R-$module is amply s-supplemented.*

Proof. Let N be a finitely $M-$generated $R-$module. Then there exist a finite index set Λ and an $R-$module epimorphism $f : M^{(\Lambda)} \longrightarrow N$. Since M is projective and s-supplemented, by Corollary 6, $M^{(\Lambda)}$ is amply s-supplemented. Then by Corollary 2, N is amply s-supplemented. □

Lemma 9. *Let M be an $R-$module. If every submodule of M is s-supplemented, then M is amply s-supplemented.*

Proof. Let $SocM \leq U \leq M$ and $M = U+V$ with $V \leq M$. Since $SocM \leq U$, by Lemma 1, $SocV = V \cap SocM \leq U \cap V$. By hypothesis, V is s-supplemented. Then $U \cap V$ has a supplement X in V. By this, $V = U \cap V + X$ and $U \cap X = U \cap V \cap X \ll X$. Then $M = U+V = U+U \cap V + X = U+X$ and $U \cap X \ll X$. Hence M is amply s-supplemented. □

Lemma 10. *Let R be any ring. Then every $R-$module is s-supplemented if and only if every $R-$module is amply s-supplemented.*

Proof. ($\Longrightarrow$) Let M be any $R-$module. Since every $R-$module is s-supplemented, every submodule of M is s-supplemented. Then by Lemma 9, M is amply s-supplemented.

($\Longleftarrow$) Clear. □

Proposition 3. *Let R be a ring. The following assertions are equivalent.*

(i) *${}_RR$ is s-supplemented.*
(ii) *${}_RR$ is amply s-supplemented.*
(iii) *Every finitely generated $R-$module is s-supplemented.*
(iv) *Every finitely generated $R-$module is amply s-supplemented.*

Proof. $(i) \Longleftrightarrow (ii)$ Clear from Corollary 3, since ${}_RR$ is projective.

$(i) \Longrightarrow (iii)$ Clear from Lemma 2.

$(iii) \Longrightarrow (iv)$ Let M be a finitely generated $R-$module. Then there exist a finite index set Λ and an $R-$module epimorphism $f : R^{(\Lambda)} \longrightarrow M$. Since every finitely generated $R-$module is s-supplemented, $R^{(\Lambda)}$ is s-supplemented. Since ${}_RR$ is projective, by [16, 18.1], $R^{(\Lambda)}$ is also projective. Then by Corollary 3, $R^{(\Lambda)}$ is amply s-supplemented. Since $f : R^{(\Lambda)} \longrightarrow M$ is an $R-$module epimorphism, by Corollary 2, M is also amply s-supplemented.

$(iv) \Longrightarrow (i)$ Clear. □

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