Miskolc Mathematical Notes
Vol. 25 (2024), No. 1, pp. 49–62

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4403

SOME RESULTS ON CHARACTERS OF A CLASS OF P-POLYNOMIAL TABLE ALGEBRAS

AMIR AMIRASLANI, MASOUMEH KOOHESTANI,
AND AMIR RAHNAMAI BARGHI

Received 09 September, 2022

Abstract. In this paper, we study the character values of homogeneous monotonic P-polynomial table algebras with finite dimension $d \geq 5$. To this end, we obtain a trigonometric polynomial to calculate the eigenvalues of the first intersection matrix of these table algebras using the $\mathbf{z}$-transform. Finally by applying some methods for tridiagonal matrices, the character values of these table algebras are given in terms of Chebyshev polynomials.

2010 *Mathematics Subject Classification:* 05C50; 15A18; 15A23

Keywords: character, P-polynomial table algebra, tridiagonal matrix, $\mathbf{z}$-transform

1. INTRODUCTION

The characters of table algebras are applied to study the properties of table algebras and can be used in association schemes and finite groups (see [7] and [16]). In particular, the eigenvalues of an association scheme which determine its algebraic structure can be obtained using the characters of table algebras (see [10]). The character values of certain table algebras have also been calculated in some articles such as [19] for lower dimensions.

In our previous work [12], we have calculated the character values of two classes of perfect P-polynomial table algebras given in [17] using the eigenstructure of some special tridiagonal matrices. Here, we intend to study the character values of homogeneous monotonic P-polynomial table algebras with finite dimension $d \geq 5$ whose first intersection matrix is a $(d+1)\times(d+1)$ tridiagonal matrix (cf. [4]) as follows:

$$\frac{1}{4}\begin{pmatrix} 0 & 4 & 0 & & & \\ 4k & 3k-6 & k+2 & \ddots & & \\ 0 & k+2 & 2k-4 & k+2 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & & k+2 & 2k-4 & k+2 \\ & & & & k+2 & 3k-2 \end{pmatrix}, \tag{1.1}$$

where k is the valency of the table algebras. It should be noted that the adjacency algebra of a distance-regular graph is a monotonic P-polynomial table algebra [2, Proposition III.1.2]. The adjacency algebra arising from a distance-regular graph whose nontrivial distance relations all are of the same valency is also homogeneous. However, there are other homogeneous monotonic P-polynomial table algebras which do not correspond to distance-regular graphs [4].

As we know, it is possible to calculate the character values of P-polynomial table algebras by obtaining the eigenvalues of their first intersection matrix (see [4, Remark 3.1]). Additionally, the first intersection matrix of P-polynomial table algebras is tridiagonal and the eigenstructure of tridiagonal matrices has been studied in many articles such as [13, 14] and [15]. However in this work, the eigenvalues of the tridiagonal matrix (1.1) are found through developing methods using the $\mathbf{z}$-transform which lead to a trigonometric polynomial. The roots of trigonometric polynomials can be obtained by some numerical methods such as the approach in [5] and through them, reasonable approximations of character values are found. That is of course not our goal in this work.

The structure of this paper is as follows. Section 2 is about important concepts and definitions that this work is based on. In Section 3, we study the eigenvalues of the first intersection matrix of homogeneous monotonic P-polynomial table algebras and obtain some results regarding their character values. Some concluding remarks are stated in Section 4.

Throughout this paper, $\mathbb{C}$ and $\mathbb{R}^+$ denote the complex numbers and the positive real numbers, respectively.

2. Preliminary definitions and concepts

In this section, we review some important concepts related to this work. In particular, we introduce table algebras and specifically P-polynomial table algebras in 2.1. An overview of the $\mathbf{z}$-transform and some of its properties is given in 2.2.

2.1. *Table algebras*

We first go over some concepts related to table algebras and P-polynomial table algebras. Note that some definitions and interpretations of table algebras are not exactly the same in all references. For example, table algebras are generally non-commutative in some papers (e.g., [18]). However, we follow the definitions in [4].

Let A be a finite-dimensional associative commutative algebra with a basis $\mathbf{B} = \{x_0 = 1_A, x_1, \cdots, x_d\}$. Then $(A, \mathbf{B})$ is a *table algebra* if the following conditions are satisfied:

(i) $x_i x_j = \sum_{m=0}^{d} \beta_{ijm} x_m$ with $\beta_{ijm} \in \mathbb{R}^+ \cup \{0\}$, for all i, j;
(ii) there is an algebra automorphism of A (denoted by $^-$) such that its order divides 2 and if $x_i \in \mathbf{B}$, then $\bar{x}_i \in \mathbf{B}$ and $\bar{i}$ is defined by $x_{\bar{i}} = \bar{x}_i$;
(iii) for all i, j, we have $\beta_{ij0} \neq 0$ if and only if $j = \bar{i}$, and $\beta_{i\bar{i}0} > 0$.

In the item (i), the β_{ijm} are called the *intersection numbers* of $(A, \mathbf{B})$ and also, the elements of $\mathbf{B}$ are called the *basis elements* of $(A, \mathbf{B})$. $(A, \mathbf{B})$ is called a *real table algebra* if $i = \bar{i}$ for all $i = 0, 1, \ldots, d$. Additionally, the *i-th intersection matrix* of $(A, \mathbf{B})$ is a matrix whose entries are the intersection numbers of $(A, \mathbf{B})$ in the form of:

$$B_i = \begin{pmatrix} \beta_{i00} & \beta_{i01} & \cdots & \beta_{i0d} \\ \beta_{i10} & \beta_{i11} & \cdots & \beta_{i1d} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{id0} & \beta_{id1} & \cdots & \beta_{idd} \end{pmatrix}_{(d+1)\times(d+1)} .$$

For every table algebra $(A, \mathbf{B})$, there exists a unique algebra homomorphism such as $f : A \to \mathbb{C}$ with $f(x_i) = f(x_{\bar{i}}) \in \mathbb{R}^+$, $i = 0, 1, \ldots, d$, (cf. [1, Lemma 2.9]). If $f(x_i) = \beta_{i\bar{i}0}$, $i = 0, 1, \ldots, d$, then $(A, \mathbf{B})$ is called *standard*. If $d \geq 2$ and for $i > 0$, $f(x_i)$ is constant, then $(A, \mathbf{B})$ is called *homogeneous*.

A *P-polynomial table algebra* $(A, \mathbf{B})$ is a real standard table algebra for which there exist the complex coefficient polynomials $\nu_i(x)$ with $\nu_i(x_1) = x_i$, $i = 2, \ldots, d$. For every P-polynomial table algebra $(A, \mathbf{B})$, there exist $b_{i-1}, a_i, c_{i+1} \in \mathbb{R}$ such that

$$x_1 x_i = b_{i-1} x_{i-1} + a_i x_i + c_{i+1} x_{i+1}, \tag{2.1}$$

with $b_i \neq 0$, $(i = 0, 1, \ldots, d-1)$, $c_i \neq 0$, $(i = 1, \ldots, d)$, and $b_{-1} = c_{d+1} = 0$. So, the first intersection matrix of a P-polynomial table algebra is a tridiagonal matrix as follows:

$$B_1 = \begin{pmatrix} a_0 & c_1 & & & \\ b_0 & a_1 & c_2 & & \\ & b_1 & a_2 & \ddots & \\ & & \ddots & \ddots & c_d \\ & & & b_{d-1} & a_d \end{pmatrix}_{(d+1)\times(d+1)} , \tag{2.2}$$

and the *valency* of the P-polynomial table algebra is $b_0 = \beta_{110}$. A monotonic table algebra is a P-polynomial table algebra for which $c_i \leq c_{i+1}$ $(i = 1, \ldots, d)$, and $b_i \geq b_{i+1}$ $(i = 0, \ldots, d-2)$.

Suppose that $(A, \mathbf{B})$ is a table algebra. It is well known that A is semisimple and the set of its primitive idempotents $\{e_0, e_1, \cdots, e_d\}$ forms another basis for A (see

[2, Page 93]). Consequently, each $x_i \in \mathbf{B}$ can be written as

$$x_i = \sum_{j=0}^{d} p_i(j) e_j,$$

where $p_i(j) \in \mathbb{C}$. On the other hand, if we consider $\{\chi_0, \chi_1, \ldots, \chi_d\}$ as the set of irreducible characters of the algebra A, we are interested in the values which each χ_i takes on the basis elements of A. These values are called the *character values* of $(A, \mathbf{B})$. The $p_i(j)$ are equal to the character values of $(A, \mathbf{B})$. More precisely, for each $i = 0, 1, \ldots, d$

$$p_i(j) = \chi_j(x_i), \qquad j = 0, 1, \ldots, d,$$

see [3, Page 11] for more details.

Also, the $p_i(j)$ are equal to the eigenvalues of the i-th intersection matrix. If $(A, \mathbf{B})$ is a P-polynomial table algebra, then we have

$$p_i(j) = \nu_i(p_1(j)), \qquad i = 0, 1, \ldots, d, \tag{2.3}$$

where $\nu_i(x)$ is a complex coefficient polynomial of degree i with $\nu_i(x_1) = x_i$.

2.2. **z**-*transform*

We now give an overview of **z**-transform. The concept of **z**-transform has the same role in discrete-time signals as Laplace transform does in continuous-time signals. For a discrete-time signal which is a sequence of real or complex numbers such as $x[n]$, its **z**-transform is defined as the power series

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}, \tag{2.4}$$

where n is an integer and z is a complex variable. The function $X(z)$ in (2.4) is called the two-sided or bilateral **z**-transform of $x[n]$. The one-sided or unilateral **z**-transform of $x[n]$ is defined by

$$X(z) = \sum_{n=0}^{+\infty} x[n] z^{-n} = x[0] + x[1] z^{-1} + x[2] z^{-2} + \cdots.$$

We use the notation $x[n] \leftrightarrow X(z)$ to show that $X(z)$ is the **z**-transform of $x[n]$.

The **z**-transform is a linear operation. This means that if we have $x[n] \leftrightarrow X(z)$ and $u[n] \leftrightarrow U(z)$, then

$$ax[n] + bu[n] \leftrightarrow aX(z) + bU(z), \tag{2.5}$$

where $a, b \in \mathbb{C}$.

Let $x[n] \leftrightarrow X(z)$ and q be a positive integer. Then we have

$$x[n-q] \leftrightarrow z^{-q} X(z), \tag{2.6}$$

and

$$x[n+q] \leftrightarrow z^q X(z) - x[0] z^q - x[1] z^{q-1} - \cdots - x[q-1] z. \tag{2.7}$$

The proof of the above properties and more facts about the **z**-transform can be found in [11, Chapter 7].

3. HOMOGENEOUS MONOTONIC P-POLYNOMIAL TABLE ALGEBRAS

Throughout this section, we study the characters of homogeneous monotonic P-polynomial table algebras with finite dimension $d \geq 5$. To do so, we concentrate on calculating the eigenvalues of the first intersection matrix of homogeneous monotonic P-polynomial table algebras which is given in (1.1).

Theorem 1. *Let $(A, \mathbf{B})$ be a homogeneous monotonic P-polynomial table algebra with $\mathbf{B} = \{x_0 = 1_A, x_1, \cdots, x_d\}$ and $d \geq 5$. Then the eigenvalues of the first intersection matrix of $(A, \mathbf{B})$ are given by*

$$\eta_i = \frac{k+2}{2}\cos(\theta_i) + \frac{k-2}{2}, \qquad i = 0, 1, \ldots, d,$$

where k is the valency of $(A, \mathbf{B})$ and the θ_i are the roots of the following equation:

$$\begin{aligned}&(k+2)\sin((d+2)\theta) - 4\sin((d+1)\theta) - 2k\sin(d\theta) + 4\sin((d-1)\theta)\\&+ (k-2)\sin((d-2)\theta) = 0.\end{aligned}$$

Proof. The first intersection matrix of $(A, \mathbf{B})$ is given in (1.1). Since we wish to use right eigenvectors (i.e. column vectors) here, the transpose of the first intersection matrix should be used, so that the automatic degree map eigenvalue k clearly corresponds to the right eigenvector $[1, \cdots, 1]^t$. The transpose of the first intersection matrix can be written as follows:

$$B_1^T = \frac{1}{4}\begin{pmatrix} -2k+4 & 4k & 0 & & & \\ 4 & k-2 & k+2 & \ddots & & \\ 0 & k+2 & 0 & k+2 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & & k+2 & 0 & k+2 \\ & & & & k+2 & k+2 \end{pmatrix} + \left(\frac{k-2}{2}\right) \mathrm{I}_{(d+1)\times(d+1)}.$$

Denote the above tridiagonal matrix by P. Then the eigenvalues of B_1 are

$$\eta_i = \frac{\lambda_i}{4} + \frac{k-2}{2}, \qquad i = 0, 1, \ldots, d, \tag{3.1}$$

where λ_i, $(i = 0, 1, \ldots, d)$, are the eigenvalues of P. Let λ be an eigenvalue of P and $u = [u_1, \cdots, u_{d+1}]^t$ be the eigenvector corresponding to λ. Then we can consider the eigenvector u as a sequence $\{u[i]\}_{i=0}^{\infty}$ with

$$u[i] = \begin{cases} u_i & \text{if } i = 1, \cdots, d+1, \\ 0 & \text{otherwise}, \end{cases}$$

and $u_1 \neq 0$. Since $Pu = \lambda u$, we have

$$\begin{cases} u[0] = 0 \\ (k+2)u[0] + (k+2)u[2] = \lambda u[1] + (2k-4)u[1] + (2-3k)u[2] \\ (k+2)u[1] + (k+2)u[3] = \lambda u[2] + (k-2)u[1] + (2-k)u[2] \\ (k+2)u[2] + (k+2)u[4] = \lambda u[3] \\ \vdots \\ (k+2)u[d-1] + (k+2)u[d+1] = \lambda u[d] \\ (k+2)u[d] + (k+2)u[d+2] = \lambda u[d+1] - (k+2)u[d+1] \\ u[d+2] = 0. \end{cases} \tag{3.2}$$

Consequently, we have the following equation:

$$(k+2)u[h+2] + (k+2)u[h] = \lambda u[h+1] + f[h+1], \quad h = 0, 1, \cdots \tag{3.3}$$

where

$$f[h] = \begin{cases} (2k-4)u[1] + (2-3k)u[2] & \text{if } h = 1, \\ (k-2)u[1] + (2-k)u[2] & \text{if } h = 2, \\ -(k+2)u[d+1] & \text{if } h = d+1, \\ 0 & \text{otherwise.} \end{cases}$$

The $\mathbf{z}$-transform of $f[h]$ is

$$\begin{aligned} F[z] &= \sum_{h=0}^{\infty} f[h] z^{-h} \\ &= \left((2k-4)u[1] + (2-3k)u[2]\right) z^{-1} \\ &\quad + \left((k-2)u[1] + (2-k)u[2]\right) z^{-2} - (k+2)u[d+1] z^{-(d+1)}. \end{aligned} \tag{3.4}$$

From (2.7), we calculate the $\mathbf{z}$-transform of (3.3) which is

$$(k+2)\left(z^2 U(z) - u[0]z^2 - u[1]z + U(z)\right) = \lambda z U(z) - \lambda u[0] z + zF(z) - f[0]z$$

and since $u[0] = f[0] = 0$, we can obtain $U(z)$ as follows:

$$\begin{aligned} U(z) &= \frac{1}{(k+2)z^2 - \lambda z + k + 2} \left(F[z] + (k+2)u[1]\right) z \\ &= \frac{1}{(k+2)z^{-2} - \lambda z^{-1} + k + 2} \left(F[z] + (k+2)u[1]\right) z^{-1}. \end{aligned} \tag{3.5}$$

Note that from Remark 2.3, the eigenvalues of B_1 are real and therefore the coefficients of $(k+2)z^{-2} - \lambda z^{-1} + k + 2$ are all real and two cases may arise as follows.

(i) If $(k+2)z^{-2} - \lambda z^{-1} + k + 2$ has two distinct roots, then the roots are

$$\gamma_{\pm} = \frac{\lambda \pm \sqrt{\omega}}{2(k+2)},$$

where $\omega = \lambda^2 - 4(k+2)^2 \neq 0$. We can write $\gamma_\pm = p \pm iq$, where $p, q \in \mathbb{C}$ and $q \neq 0$. From $\gamma_+\gamma_- = p^2 + q^2 = 1$ and $\gamma_+ + \gamma_- = 2p = \lambda/(k+2)$, it follows that

$$\gamma_\pm = \sqrt{p^2+q^2}\,(\cos(\theta) \pm i\sin(\theta)) = e^{\pm i\theta}, \qquad \cos(\theta) = \frac{\lambda}{2(k+2)}, \tag{3.6}$$

where θ is either a real or a pure imaginary number. Let

$$X(z) = \frac{1}{(k+2)z^{-2} - \lambda z^{-1} + (k+2)}.$$

From (2.6), we know that

$$z^{-q}X(z) \leftrightarrow x[j-q]H[j-q] \quad \text{where} \quad H[x] = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{3.7}$$

So that, by getting the inverse **z**-transform from (3.5) and using (3.4), we have

$$\begin{aligned} u[j] = {}& (k+2)u[1]x[j-1]H[j-1] + ((2k-4)u[1] + (2-3k)u[2])\,x[j-2]H[j-2] \\ & + ((k-2)u[1] + (2-k)u[2])\,x[j-3]H[j-3] \\ & - (k+2)u[d+1]x[j-d-2]H[j-d-2]. \end{aligned} \tag{3.8}$$

We shall now find $x[j]$ and plug it into (3.8). By partial fractions decomposition of $X(z)$, we can write

$$X(z) = \frac{1}{\sqrt{\omega}}\left(\frac{1}{\gamma_- - z^{-1}} - \frac{1}{\gamma_+ - z^{-1}}\right),$$

therefore,

$$x[j] = \frac{1}{\sqrt{\omega}}\left((\gamma_-)^{-(j+1)} - (\gamma_+)^{-(j+1)}\right) = \frac{2i}{\sqrt{\omega}}\sin((j+1)\theta). \tag{3.9}$$

Note that $\{a^{-(n+1)}\}_{n=0}^{\infty} \leftrightarrow 1/(a - z^{-1})$. From (3.8) and (3.9), we have

$$\begin{aligned} u[j] = \frac{2i}{\sqrt{\omega}} \Big[& (k+2)u[1]\sin(j\theta)H[j-1] \\ & + ((2k-4)u[1] + (2-3k)u[2])\sin((j-1)\theta)H[j-2] \\ & + ((k-2)u[1] + (2-k)u[2])\sin((j-2)\theta)H[j-3] \\ & - (k+2)u[d+1]\sin((j-d-1)\theta)H[j-d-2]\Big]. \end{aligned} \tag{3.10}$$

Setting $j = d+2$ in (3.10) yields

$$\begin{aligned} u[d+2] = \frac{2i}{\sqrt{\omega}} \Big[& (k+2)u[1]\sin((d+2)\theta) \\ & + ((2k-4)u[1] + (2-3k)u[2])\sin((d+1)\theta) \\ & + ((k-2)u[1] + (2-k)u[2])\sin(d\theta) - (k+2)u[d+1]\sin(\theta)\Big]. \end{aligned} \tag{3.11}$$

Similarly, when $j = d+1$,

$$u[d+1] = \frac{2i}{\sqrt{\omega}}\Big[(k+2)u[1]\sin((d+1)\theta) + ((2k-4)u[1] + (2-3k)u[2])\sin(d\theta) + ((k-2)u[1] + (2-k)u[2])\sin((d-1)\theta)\Big]. \quad (3.12)$$

Moreover, we know that

$$\gamma_+ - \gamma_- = \frac{\sqrt{\omega}}{(k+2)} = 2i\sin(\theta). \quad (3.13)$$

From (3.11), (3.12) and (3.13), we conclude that

$$u[d+2] = \frac{2i}{\sqrt{\omega}}\Big[(k+2)u[1]\sin((d+2)\theta) + ((k-6)u[1] + (2-3k)u[2])\sin((d+1)\theta) + ((2-k)u[1] + 2ku[2])\sin(d\theta) + ((2-k)u[1] + (k-2)u[2])\sin((d-1)\theta)\Big].$$

From (3.2), we know that

$$u[2] = \frac{\lambda + 2k - 4}{4k}u[1], \quad (3.14)$$

and also, by (3.6), we have $\lambda = 2(k+2)\cos(\theta)$, consequently we get

$$u[d+2] = \frac{iu[1](k+2)}{2k\sqrt{\omega}}\Big[(k+2)\sin((d+2)\theta) - 4\sin((d+1)\theta) - 2k\sin(d\theta) + 4\sin((d-1)\theta) + (k-2)\sin((d-2)\theta)\Big].$$

Since $u[d+2] = 0$, θ is the root of the following equation:

$$(k+2)\sin((d+2)\theta) - 4\sin((d+1)\theta) - 2k\sin(d\theta) + 4\sin((d-1)\theta) + (k-2)\sin((d-2)\theta) = 0. \quad (3.15)$$

(ii) Suppose that $(k+2)z^{-2} - \lambda z^{-1} + (k+2)$ has a repeated root. As such, $\omega = 0$ and $\lambda = \pm 2(k+2)$. Let $Y(z) = \dfrac{z^{-1}}{(k+2)z^{-2} - \lambda z^{-1} + (k+2)}$. From (3.4), (3.5) and (3.7), we get

$$u[j] = (k+2)u[1]y[j] + ((2k-4)u[1] + (2-3k)u[2])\,y[j-1]H[j-1] + ((k-2)u[1] + (2-k)u[2])\,y[j-2]H[j-2] - (k+2)u[d+1]y[j-d-1]H[j-d-1]. \quad (3.16)$$

We shall now find $y[j]$ and plug it into (3.16). We can write

$$Y(z) = \frac{z^{-1}}{(k+2)z^{-2} - \lambda z^{-1} + k+2} = \frac{z^{-1}}{(k+2)(z^{-2} - \frac{\lambda}{k+2}z^{-1} + 1)} = \frac{z^{-1}}{(k+2)(z^{-1} \pm 1)^2}.$$

That yields

$$y[j] = \frac{1}{k+2}(\pm 1)^{j+1} j. \tag{3.17}$$

Notice that the above equality is obtained due to $\{na^n\}_{n=0}^{\infty} \leftrightarrow az/(z-a)^2$ (cf. [11, Table 7.3]). From (3.16) and (3.17), we have

$$\begin{aligned} u[j] = \frac{1}{k+2} \Big[&(k+2)u[1](\pm 1)^{j+1} j \\ &+ ((2k-4)u[1] + (2-3k)u[2])\,(\pm 1)^{j}(j-1)H[j-1] \\ &+ ((k-2)u[1] + (2-k)u[2])\,(\pm 1)^{j-1}(j-2)H[j-2] \\ &- (k+2)u[d+1](\pm 1)^{j-d}(j-d-1)H[j-d-1]\Big]. \end{aligned} \tag{3.18}$$

If we set $j = d+2$ in (3.18), then we have

$$\begin{aligned} u[d+2] = \frac{1}{k+2} \Big[&(k+2)u[1](\pm 1)^{d+3}(d+2) \\ &+ ((2k-4)u[1] + (2-3k)u[2])\,(\pm 1)^{d+2}(d+1) \\ &+ ((k-2)u[1] + (2-k)u[2])\,(\pm 1)^{d+1} d - (k+2)u[d+1]\Big], \end{aligned}$$

and also, $u[d+1]$ is as follows

$$\begin{aligned} u[d+1] = \frac{1}{k+2} \Big[&(k+2)u[1](\pm 1)^{(d+2)}(d+1) \\ &+ ((2k-4)u[1] + (2-3k)u[2])\,(\pm 1)^{(d+1)} d \\ &+ ((k-2)u[1] + (2-k)u[2])\,(\pm 1)^{d}(d-1)\Big], \end{aligned}$$

so by using this and $u[d+2] = 0$, we obtain that

$$\begin{aligned} &(k+2)(\pm 1)u[1](d+2) + ((k-6)u[1] + (2-3k)u[2])\,(d+1) \\ &+ ((2-k)u[1] + 2ku[2])\,(\pm 1)d - ((k-2)u[1] + (2-k)u[2])\,(d-1) = 0. \end{aligned} \tag{3.19}$$

We now consider two equations (3.20) and (3.21) from (3.19) as follows. The first equation is

$$\begin{aligned} &(k+2)u[1](d+2) + ((k-6)u[1] + (2-3k)u[2])\,(d+1) \\ &+ ((2-k)u[1] + 2ku[2])\,d - ((k-2)u[1] + (2-k)u[2])\,(d-1) = 0. \end{aligned} \tag{3.20}$$

In this case, $\lambda = -2(k+2)$ and also from (3.14), we conclude that $u[2] = -2u[1]/k$. After some simplifications, we obtain from (3.20) that $d = -2$ which is a contradiction. The other equation is

$$\begin{aligned} &-(k+2)u[1](d+2) + ((k-6)u[1] + (2-3k)u[2])\,(d+1) \\ &- ((2-k)u[1] + 2ku[2])\,d - ((k-2)u[1] + (2-k)u[2])\,(d-1) = 0. \end{aligned} \tag{3.21}$$

In this case, $\lambda = 2a$ and from (3.14), we conclude that $u[2] = u[1]$. After some simplifications, we get $d = -1$ which is a contradiction.

All in all, we conclude that the θ_i can all be obtained from (3.15) and so, from (3.6) we have $\lambda_i = 2(k+2)\cos(\theta_i)$, where the λ_i are the eigenvalues of the matrix P. Consequently, by (3.1), the eigenvalues of B_1 are

$$\eta_i = \frac{k+2}{2} cos(\theta_i) + \frac{k-2}{2} \qquad (i = 0, 1, \ldots, d),$$

as expected. □

Importantly, Theorem 1 shows that the general form of the eigenvalues of the first intersection matrix of a homogeneous monotonic P-polynomial table algebra with valency k is $\cos(\theta_i)(k+2)/2 + (k-2)/2$ which means that the eigenvalues lie in the interval

$$[-2, k],$$

and when $k > 2$, this is a better boundary than $[-k, k]$ which is given in [4, Proposition 3.1].

In the following theorem, we calculate the characters of homogeneous monotonic P-polynomial table algebras with finite dimension $d \geq 5$.

Theorem 2. *Let $(A, \boldsymbol{B})$ be a homogeneous monotonic P-polynomial table algebra with $\mathbf{B} = \{x_0 = 1_A, x_1, \cdots, x_d\}$ and $d \geq 5$. Then the character values of $(A, \boldsymbol{B})$ are*

$$\begin{aligned}
p_0(j) &= 1, \\
p_1(j) &= \eta_j, \\
p_i(j) &= \left(\frac{\sqrt{k+2}}{2}\right)^{i-4} \left[\left(\eta_j^2 - \frac{3k-6}{4}\eta_j - k\right) U_{i-2}\left(\frac{2\eta_j - k + 2}{2\sqrt{k+2}}\right) \right. \\
&\qquad \left. - \left(\frac{\sqrt{k+2}}{2}\right)^3 \eta_j \, U_{i-3}\left(\frac{2\eta_j - k + 2}{2\sqrt{k+2}}\right) \right],
\end{aligned}$$

where $i = 2, \ldots, d$, $j = 0, 1, \ldots, d$ and the η_j are the eigenvalues of B_1 which are obtained in Theorem 1.

Proof. For each i, $i = 0, 1, \ldots, d$, the $p_i(j)$, $j = 0, 1, \ldots, d$, are equal to the eigenvalues of the i-th intersection matrix B_i. Since $B_0 = I_{d+1}$, we have $p_0(j) = 1$ for all j. Similarly, the $p_1(j)$ are equal to the eigenvalues of B_1 which are calculated in Theorem 1. To obtain the $p_i(j)$, $i = 2, \ldots, d$, we must calculate the complex coefficient polynomial $\nu_i(x)$, where $x_i = \nu_i(x_1)$. Obviously, $\nu_1(x) = x$, and from (1.1) and (2.1) we get

$$x_1 x_1 = k + \frac{3k-6}{4} x_1 + \frac{k+2}{4} x_2 \Rightarrow \nu_2(x) = \frac{4x^2 - (3k-6)x - 4k}{k+2}.$$

We claim that

$$\nu_i(x) = \frac{4}{k+2}\left(\left(x - \frac{k-2}{2}\right)\nu_{i-1}(x) - \frac{k+2}{4}\nu_{i-2}(x)\right), \tag{3.22}$$

where $i = 2, \ldots, d$. To prove this, we use induction on i. It is fairly straightforward using (2.1) to get

$$\nu_3(x) = \frac{4}{k+2}\left(\left(x - \frac{k-2}{2}\right)\nu_2(x) - \frac{k+2}{4}\nu_1(x)\right).$$

Now, we assume that (3.22) holds for $i < d$. From (2.1) and the induction hypothesis, it is concluded that

$$\nu_d(x) = \frac{4}{k+2}\left(\left(x - \frac{k-2}{2}\right)\nu_{d-1}(x) - \frac{k+2}{4}\nu_{d-2}(x)\right).$$

We now consider the following recursive relation:

$$\varphi_n(x) = \left(x - \frac{k-2}{2}\right)\varphi_{n-1}(x) - \frac{k+2}{4}\varphi_{n-2}(x), \qquad n > 2,$$

with $\varphi_1(x) = (k+2)x/4$ and $\varphi_2(x) = x^2 - (3k-6)x/4 - k$. By using [6, Lemma 1], we find that

$$\varphi_n(x) = \begin{vmatrix} \frac{k+2}{4}x & 1 & 0 & & & \\ k & \frac{4}{k+2}x - \frac{3k-6}{k+2} & 1 & \ddots & & \\ 0 & \frac{k+2}{4} & x - \frac{k-2}{2} & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & & \frac{k+2}{4} & x - \frac{k-2}{2} & 1 \\ & & & & \frac{k+2}{4} & x - \frac{k-2}{2} \end{vmatrix}_{n\times n}.$$

Laplace expansion gives

$$\varphi_n(x) = \left(x^2 - \frac{3k-6}{4}x - k\right)D_{n-2}(x) - \left(\frac{k+2}{4}\right)^2 x D_{n-3}(x), \tag{3.23}$$

where $D_n(x)$ is the characteristic polynomial of

$$\begin{pmatrix} \frac{k-2}{2} & 1 & 0 & & \\ \frac{k+2}{4} & \frac{k-2}{2} & 1 & \ddots & \\ 0 & \frac{k+2}{4} & \frac{k-2}{2} & \ddots & \\ & \ddots & \ddots & \ddots & 1 \\ & & & \frac{k+2}{4} & \frac{k-2}{2} \end{pmatrix}_{n\times n},$$

so from [8, Section 2], we can see that

$$D_n(x) = \left(\frac{\sqrt{k+2}}{2}\right)^n U_n\left(\frac{2x-k+2}{2\sqrt{k+2}}\right). \tag{3.24}$$

Finally, from (3.22), (3.23) and (3.24) we conclude that

$$\begin{aligned} \nu_i(x) &= \frac{4}{k+2}\varphi_i(x) \\ &= \left(\frac{\sqrt{k+2}}{2}\right)^{i-4}\Bigg[\left(x^2-\frac{3k-6}{4}x-k\right)U_{i-2}\left(\frac{2x-k+2}{2\sqrt{k+2}}\right) \\ &\qquad -\left(\frac{\sqrt{k+2}}{2}\right)^3 x\,U_{i-3}\left(\frac{2x-k+2}{2\sqrt{k+2}}\right)\Bigg], \end{aligned}$$

for $i=2,\dots,d$. Due to (2.3), the proof is now complete. □

Example 1. Let $(A,\mathbf{B})$ be a homogeneous monotonic P-polynomial table algebra of valency $k=2$ and diameter $d\geq 5$. Then from (1.1), the first intersection of $(A,\mathbf{B})$ is

$$B_1=\begin{pmatrix} 0 & 1 & & & & \\ 2 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 1 \end{pmatrix}_{(d+1)\times(d+1)}.$$

Now, we calculate the characters of $(A,\mathbf{B})$. From Theorem 1, we must find the roots of the following equation:

$$\sin((d+2)\theta)-\sin((d+1)\theta)-\sin(d\theta)+\sin((d-1)\theta)=0. \tag{3.25}$$

It implies that

$$\cos((d+1)\theta)\sin(\theta)-\cos(d\theta)\sin(\theta)=0.$$

We can assume that $\sin(\theta)\neq 0$, because otherwise, $\theta=n\pi$, $(n\in\mathbb{N})$, and $\lambda=2a\cos(n\pi)=\pm 2a$ which leads to $k=2$ being an eigenvalue of B_1, i.e. case (ii) in Theorem 1, but k can be obtained from the equation (3.25) which follows from the case (i) in Theorem 1. This gives

$$p_1(j)=\eta_j=2\cos\left(\frac{2j\pi}{2d+1}\right),\quad j=0,\dots,d.$$

The other characters of $(A,\mathbf{B})$ can also be found through Theorem 2. The $p_i(j)$, $i=2,\dots,d$, are

$$p_i(j)=\left(\eta_j^2-2\right)U_{i-2}\left(\frac{\eta_j}{2}\right)-\eta_j U_{i-3}\left(\frac{\eta_j}{2}\right)=2\cos\left(\frac{ij\pi}{2d+1}\right),$$

for $j=0,\dots,d$, where $U_i(x)$ and $T_i(x)$ are the i-th degree Chebyshev polynomials of second kind and first kind, respectively. The above equalities follow from the properties of Chebyshev polynomials which can be found in [9].

4. Concluding remarks

In this paper, we use the **z**-transform concept along with techniques from linear algebra and matrix theory to calculate the character values of homogeneous monotonic P-polynomial table algebras with finite dimension $d \geq 5$. Importantly, we calculate the eigenvalues of a special classes of tridiagonal matrices which may have applications in other fields. Next, we obtain the characters of homogeneous monotonic P-polynomial table algebras with finite dimension $d \geq 5$ in terms of Chebyshev polynomials. For $k > 2$, we plan to apply the results of Theorem 2 combined with some other techniques to find the character values in our future work.

References

[1] Z. Arad and H. I. Blau, "On table algebras and applications to finite group theory," *Journal of Algebra*, vol. 138, no. 1, pp. 137–185, 1991, doi: 10.1016/0021-8693(91)90195-E.

[2] E. Bannai and T. Ito, "Algebraic Combinatorics I: Association Schemes," in *Algebraic Combinatorics I: Association Schemes*. Menlo Park, CA: Benjamin/Cummings, 1984.

[3] A. R. Barghi and J. Bagherian, "Standard character condition for table algebras," *The Electronic Journal of Combinatorics*, 2010, article number R113, doi: 10.37236/385.

[4] H. I. Blau and R. J. Hein, "A class of P-polynomial table algebras with and without integer multiplicities," *Communications in Algebra*, vol. 42, no. 12, pp. 5387–5424, 2014, doi: 10.1080/00927872.2013.840448.

[5] J. P. Boyd, "A comparison of companion matrix methods to find roots of a trigonometric polynomial," *Journal of Computational Physics*, vol. 246, pp. 96–112, 2013, doi: 10.1016/j.jcp.2013.03.022.

[6] N. D. Cahill, J. R. D Errico, and J. P. Spence, "Complex factorizations of the Fibonacci and Lucas numbers," *Fibonacci Quarterly*, vol. 41, no. 1, pp. 13–19, 2003.

[7] G. Chen and B. Xu, "Structures of commutative table algebras determined by their character tables and applications to finite groups," *Communications in Algebra*, vol. 46, no. 8, pp. 3510–3519, 2018, doi: 10.1080/00927872.2017.1422259.

[8] C. M. da Fonseca, S. Kouachi, D. A. Mazilu, and I. Mazilu, "A multi-temperature kinetic Ising model and the eigenvalues of some perturbed Jacobi matrices," *Applied Mathematics and Computation*, vol. 259, pp. 205–211, 2015, doi: 10.1016/j.amc.2015.02.058.

[9] L. Fox and I. B. Parker, "Chebyshev polynomials in numerical analysis," Tech. Rep., 1968, doi: 10.2307/3613223.

[10] C. Godsile, "Association Schemes 1, Combinatorics, Optimization," in *Association Schemes 1, Combinatorics, Optimization*. University of Waterloo, 2010.

[11] E. W. Kamen and B. S. Heck, *Fundamentals of signals and systems using the Web and Matlab*. Pearson, 2006.

[12] M. Koohestani, A. Rahnamai Barghi, and A. Amiraslani, "The application of tridiagonal matrices in P-polynomial table algebras," *Iranian Journal of Science and Technology, Transactions A: Science*, vol. 44, no. 4, pp. 1125–1129, 2020, doi: 10.1007/s40995-020-00924-1.

[13] S. Kouachi, "Eigenvalues and eigenvectors of tridiagonal matrices," *The Electronic Journal of Linear Algebra*, vol. 15, pp. 115–133, 2006, doi: 10.13001/1081-3810.1223.

[14] I. Rimas, "Investigation of dynamics of mutually synchronized systems," *Telecommunications and Radio Engineering*, vol. 31, no. 2, pp. 68–73, 1977.

[15] A. R. Willms, "Analytic results for the eigenvalues of certain tridiagonal matrices," *SIAM journal on Matrix Analysis and Applications*, vol. 30, no. 2, pp. 639–656, 2008, doi: 10.1137/070695411.

[16] B. Xu, "Characters of table algebras and applications to association schemes," *Journal of Combinatorial Theory, Series A*, vol. 115, no. 8, pp. 1358–1373, 2008, doi: 10.1016/j.jcta.2008.02.005.

[17] B. Xu, "On perfect P-polynomial table algebras," *Communications in Algebra*, vol. 37, no. 1, pp. 120–153, 2009, doi: 10.1080/00927870802241485.

[18] B. Xu, "On P-polynomial table algebras and applications to association schemes," *Communications in Algebra*, vol. 40, no. 6, pp. 2171–2183, 2012, doi: 10.1080/00927872.2011.576736.

[19] B. Xu, "On a class of integral table algebras," *Journal of Algebra*, vol. 178, no. 3, pp. 760–781, 1995, doi: 10.1006/jabr.1995.1376.

Authors' addresses

Amir Amiraslani

(**Corresponding author**) STEM Department, Capilano University, 2055 Purcell Way, North Vancouver, BC V7J 3H5, Canada

Department of Mathematics, K. N. Toosi University of Technology, P. O. Box 16765-3381, Tehran, Iran

E-mail address: amirhosseinamiraslan@capilanou.ca

Masoumeh Koohestani

Department of Mathematics, K. N. Toosi University of Technology, P. O. Box 16765-3381, Tehran, Iran

E-mail address: m.kuhestani@email.kntu.ac.ir

Amir Rahnamai Barghi

Department of Mathematics, K. N. Toosi University of Technology, P. O. Box 16765-3381, Tehran, Iran

E-mail address: rahnama@kntu.ac.ir