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A NEW LOOK AT FINSLER SURFACES AND LANDSBERG'S PDE

SALAH G. ELGENDI

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Abstract. In this paper, we introduce a new look at Finsler surfaces. Landsberg surfaces are Finsler surfaces that are solutions of a system of non-linear partial differential equations. Considering the unicorn's Landsberg problem, we reduce this system to a single non-linear PDE which we call the Landsberg's PDE. By making use of the new look of Finsler surfaces, we solve the Landsberg's PDE and get a class of solutions. Moreover, we show that these solutions and their conformal transformations are Berwaldain.

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1. Introduction

A Finsler manifold (M,F) is said to be *Berwald* if the coefficients of Berwald connection depend only on the position arguments and this is equivalent to that the Berwald parallel translation P_c along a curve $c(t)$ is linear isometry between (T_pM,F_p) and (T_qM,F_q) where $c(t)$ joins the points $p,q\in M$. A Finsler manifold (M,F) is said be *Landsberg* if the horizontal covariant derivative of the metric tensor of F with respect to the Berwald connection vanishes and this is equivalent to the fact that the parallel translation P_c along c preserves the induced Riemannian metrics on the slit tangent spaces, i.e., $P_c : (T_pM\backslash\{0\},g_p) \longrightarrow (T_qM\backslash\{0\},g_q)$ is an isometry. There are many other characterizations for Berwald and Landsberg metrics.

It is known that every Berwald space is Landsberg, but the converse is a long-existing problem in Finsler geometry, which is still open. Matsumoto, one of the best geometers who had a very significant contribution to Finsler geometry in the last century, called the problem the most important unsolved problem in Finsler geometry. Many geometers in the Finslerian area of research looking for a regular Landsberg space which is not Berwald. Even they want to know if such spaces exists or not. From the applicable point of view, especially in Physics, G. Asanov [1] obtained class of metrics (singular or y-local) arising from Finslerian General Relativity. These metrics are non-Berwaldian Landsberg metrics. Later, Z. Shen [10], generalized this class and classified all Landsberg (α,β)-metrics which are not Berwaldian. Due to the many unsuccessful attempts made for finding non-Berwaldian Landsberg metrics, D. Bao [2] called them “unicorns”.

There are some papers devoted to the unicorn problem in dimension two. For example, R. Bryant claimed there exists the singular Landsberg Finsler surfaces which are not Berwaldian, moreover among them there is surfaces with vanishing flag curvature (cf. [2]). Later, Zhou [15] confirmed R. Bryant’s claim by giving examples of Landsberg surfaces which are not Berwaldain. Recently in [7] it was shown that the examples obtained by Zhou are in fact Berwaldian. By the way, the spherically symmetric Landsberg surfaces are studied in [5]. In [14], G. Thompson claimed that there are Landsberg Finsler metrics in dimension two which are not Berwaldian, but no concrete examples of such surfaces are given. Also, one can see the work of S. V. Sabau in dimension two, for example, see [9]. For concrete examples and further studies of the unicorns for higher dimensions we refer to [4, 6, 8].

In this paper, we rewrite the Finsler function on any two-dimensional manifold as follows:

$$F = \left|y^1\right| f(x,\varepsilon u), \quad u = \frac{y^2}{y^1}, \;\; \varepsilon := \operatorname{sgn}(y^1)$$

where $f(x,\varepsilon u) := F(x,\varepsilon,\varepsilon u)$ is a positive smooth function on $M\times\mathbb{R}$ and $|\cdot|$ is the absolute value. It should be noted that if we start by regular Finsler function F, then the Finsler function $F(x,y) = \left|y^1\right| f(x,\varepsilon u)$ is regular although the function u has a singularity at $y^1 = 0$. As an example (cf. [11, Example 1.2.2 Page 15]):

$$F(x,y) = \sqrt{(y^1)^2+(y^2)^2} + By^1 = |y^1|\left(\sqrt{1+u^2}+\varepsilon B\right).$$

In this example $f(x,\varepsilon u) = \sqrt{1+u^2}+\varepsilon B$.

We calculate the Berwald and Landsberg curvatures. We show that the Landsberg condition leads to a single PDE. By solving this PDE we show that all two-dimensional Landsberg metrics on the form

$$F = \left|y^1\right|\phi(t), \quad t := \rho(x^1,x^2)u, \quad u := y^2/y^1$$

and their conformal transformations are Berwaldian (cf. Theorems 1 and 2). As by-product, we get explicit formulae for these solutions, precisely, the classes of the Landsberg solutions given by

$$F = \sqrt{c_3\rho^2(y^2)^2-(c-2)\rho y^1y^2-2c_1(y^1)^2}\; e^{\frac{c}{\sqrt{c^2-4c_2}}\operatorname{arctanh}\left(\frac{2c_3\rho y^2-(c-2)y^1}{y^1\sqrt{c^2-4c_2}}\right)},$$

are Berwaldian where $c := 2c_1c_3+c_2+1$ and $c_2>0, a, b, c_1, c_3$ are constants (cf. Theorem 3).

In case of higher dimensions $n\geq 3$, the conformal transformation of a Minkowski metrics can yield a non-Berwaldian Landsberg metrics (cf. [4]). Assuming that a manifold (M,F) is Minkowski if the Finsler function F depends on the directional argument y only, in contrast of the higher dimensions, we prove that if the conformal transformation of any two dimensional Minkowski metric is Landsberg then it must be Berwaldian (cf. Theorem 4).

2. Preliminaries

Let M be an n-dimensional manifold and (TM,π_M,M) be its tangent bundle and $(\mathcal{T}M,\pi,M)$ the subbundle of nonzero tangent vectors. We denote by (x^i) local coordinates on the base manifold M and by (x^i,y^i) the induced coordinates on TM. The vector 1-form J on TM defined, locally, by $J=\frac{\partial}{\partial y^i}\otimes dx^i$ is called the natural almost-tangent structure of TM. The vertical vector field $\mathcal{C}=y^i\frac{\partial}{\partial y^i}$ on TM is called the canonical or the Liouville vector field.

A vector field $S\in\mathfrak{X}(\mathcal{T}M)$ is called a spray if $JS=\mathcal{C}$ and $[\mathcal{C},S]=S$. Locally, a spray can be expressed as follows

$$S=y^i\frac{\partial}{\partial x^i}-2G^i\frac{\partial}{\partial y^i},$$

where the *spray coefficients* $G^i=G^i(x,y)$ are 2-homogeneous functions in the $y=(y^1,\dots,y^n)$ variable.

A nonlinear connection is defined by an n-dimensional distribution $H:u\in\mathcal{T}M\to H_u\subset T_u(\mathcal{T}M)$ that is supplementary to the vertical distribution, which means that for all $u\in\mathcal{T}M$, we have

$$T_u(\mathcal{T}M)=H_u(\mathcal{T}M)\oplus V_u(\mathcal{T}M). \tag{2.1}$$

Every spray S induces a canonical nonlinear connection through the corresponding horizontal and vertical projectors,

$$h=\frac{1}{2}(Id+[J,S]),\ \ v=\frac{1}{2}(Id-[J,S]).$$

Equivalently, the canonical nonlinear connection induced by a spray can be expressed in terms of an almost product structure $\Gamma=[J,S]=h-v$. With respect to the induced nonlinear connection, a spray S is horizontal, which means that $S=hS$. Locally, the two projectors h and v can be expressed as follows

$$h=\frac{\delta}{\delta x^i}\otimes dx^i,\qquad v=\frac{\partial}{\partial y^i}\otimes\delta y^i,$$

$$\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-G^j_i(x,y)\frac{\partial}{\partial y^j},\quad \delta y^i=dy^i+G^j_i(x,y)dx^i,\quad G^j_i(x,y)=\frac{\partial G^j}{\partial y^i}.$$

Moreover, the coefficients of the Berwald connection is given by

$$G^h_{ij}=\frac{\partial G^h_j}{\partial y^i}.$$

$$R=\frac{1}{2}[h,h]=\frac{1}{2}R^i_{jk}\frac{\partial}{\partial y^i}\otimes dx^j\wedge dx^k,\qquad R^i_{jk}=\frac{\delta G^i_j}{\delta x^k}-\frac{\delta G^i_k}{\delta x^j}$$

is called the curvature of S. From the curvature tensor one can obtain the Riemann curvature [11] (or the Jacobi endomorphism, see [3]), which is defined by

$$\Phi = R^i_j\, dx^j \otimes \frac{\partial}{\partial y^i}, \qquad R^i_j = 2\frac{\partial G^i}{\partial x^j} - S(G^i_j) - G^i_k G^k_j. \tag{2.2}$$

The two curvature tensors are related by

$$\Phi = i_S R, \qquad 3R = [J, \Phi], \tag{2.3}$$

respectively.

From now on, for simplicity, we use the notations $\partial_i := \dfrac{\partial}{\partial x^i}$, $\dot{\partial}_i := \dfrac{\partial}{\partial y^i}$.

Definition 1. A Finsler manifold of dimension n is a pair (M,F), where M is a differentiable manifold of dimension n and F is a map

$$F : TM \longrightarrow \mathbb{R},$$

such that

- **(a)** F is smooth and strictly positive on $\mathcal{T}M$ and $F(x,y) = 0$ if and only if $y = 0$,
- **(b)** F is positively homogeneous of degree 1 in the directional argument y: $\mathcal{L}_C F = F$,
- **(c)** The metric tensor $g_{ij} = \dot{\partial}_i \dot{\partial}_j E$ has rank n on $\mathcal{T}M$, where $E := \frac{1}{2}F^2$ is the energy function.

Since the 2-form $dd_J E$ is non-degenerate, the Euler-Lagrange equation

$$i_S dd_J E = -dE$$

uniquely determines a spray S on TM. This spray is called the *geodesic spray* of the Finsler function.

Definition 2. A spray S on a manifold M is called *Finsler metrizable* if there exists a Finsler function F such that the geodesic spray of the Finsler manifold (M,F) is S.

It is known that a spray S is Finsler metrizable if and only if there exists a non-degenerate solution F for the system

$$d_h F = 0, \quad d_C F = F, \tag{2.4}$$

where h is the horizontal projector associated to S.

The local formula for the hv-curvature tensor G of Berwlad connection and the Landsbeg tensor L given, respectively, by

$$G = G^h_{ijk} dx^i \otimes dx^j \otimes dx^k \otimes \dot{\partial}_h \tag{2.5}$$

$$L = L_{ijk} dx^i \otimes dx^j \otimes dx^k, \tag{2.6}$$

where $G^h_{ijk} := \dot{\partial}_k G^h_{ij}$ and $L_{ijk} := -\frac{1}{2} F G^h_{ijk} \ell_h$.

Definition 3. A Finsler manifold (M,F) is said to be *Berwald* if and only if G^h_{ijk} vanishes identically.

Definition 4. A Finsler manifold (M,F) is called *Landsberg* if if and only if L_{ijk} vanishes identically.

3. A NEW LOOK AT FINSLER SURFACES AND THE LANDSBERG'S PDE

In what follows, ∂_1 (rep. ∂_2) stands for the partial differentiation with respect to x^1 (resp. x^2) and $\dot{\partial}_1$ (resp. $\dot{\partial}_2$) stands for the partial differentiation with respect to y^1 (resp. y^2).

The following lemmas are useful for subsequent use.

Lemma 1. *Let F be a Finsler function on a two-dimensional manifold M, then F can be written in the form*

$$F = \begin{cases} \left|y^1\right| f(x,\varepsilon u), & u = \frac{y^2}{y^1},\ y^1 \neq 0,\ \varepsilon := \operatorname{sgn}(y^1) \\ 0, & y^1 = y^2 = 0 \\ \left|y^2\right| f(x,\varepsilon v), & v = \frac{y^1}{y^2},\ y^2 \neq 0,\ \varepsilon := \operatorname{sgn}(y^2) \end{cases} \tag{3.1}$$

where $f(x,\varepsilon u) := F(x,\varepsilon,\varepsilon u)$ is a positive smooth function on $M \times \mathbb{R}$ and $|\cdot|$ is the absolute value. Moreover, for the expression $F = \left|y^1\right| f(x,\varepsilon u)$, the coefficients G^1 and G^2 of the geodesic spray are given by

$$G^1 = f_1(x,u)(y^1)^2, \quad G^2 = f_2(x,u)(y^1)^2, \tag{3.2}$$

where the functions f_1 and f_2 are smooth functions on $M \times \mathbb{R}$ and given as follows

$$f_1 = \frac{(\partial_1 f + u\partial_2 f)f'' - (\partial_1 f' + u\partial_2 f' - \partial_2 f)f'}{2ff''}, \tag{3.3}$$

$$f_2 = \frac{u(\partial_1 f + u\partial_2 f)f'' + (\partial_1 f' + u\partial_2 f' - \partial_2 f)(f - uf')}{2ff''}, \tag{3.4}$$

where f' (resp. f'') is the first (resp. the second) derivative of f with respect to u and so on.

Before we go to prove the above lemma, let's give the following remark.

Remark 1. It should be noted that if we start by regular Finsler function F, then the Finsler function $F(x,y) = \left|y^1\right| f(x,\varepsilon u)$ is regular although the function u has a singularity at $y^1 = 0$. As an example (cf. [11, Example 1.2.2 Page 15]):

$$F(x,y) = \sqrt{(y^1)^2 + (y^2)^2} + By^1 = |y^1| \left(\sqrt{1+u^2} + \varepsilon B\right).$$

In this example $f(x,\varepsilon u) = \sqrt{1+u^2} + \varepsilon B$.

Proof of Lemma 1. Since $F=0$ only on the zero section, then away from the zero section at each $x\in M$, at least one of the y's is non zero. Without loss of generality, we assume that $y^1\neq 0$. Then by using the fact that the Finsler function F is positively homogeneous of degree 1 in y, we have

$$F(x,y^1,y^2)=F\left(x,\left|y^1\right|\frac{y^1}{|y^1|},\left|y^1\right|\frac{y^2}{|y^1|}\right)=\left|y^1\right|F(x,\varepsilon,\varepsilon u)=\left|y^1\right|f(x,\varepsilon u),$$

where $f(x,\varepsilon u):=F(x,\varepsilon,\varepsilon u)$.

Now, assume that $F=\left|y^1\right|f(x^1,x^2,\varepsilon u)$. Since the coefficients G^i of the geodesic spray of F are homogeneous of degree 2 in y, then we can write

$$G^1=f_1(x^1,x^2,u)(y^1)^2,\quad G^2=f_2(x^1,x^2,u)(y^1)^2.$$

Keeping the facts that $\dfrac{\partial f}{\partial(\varepsilon u)}=\varepsilon\dfrac{\partial f}{\partial u}=\varepsilon f'$ and $\varepsilon^2=1$ in mind, then straightforward calculations lead to

$$\ell_1:=\dot\partial_1F=\varepsilon f+\left|y^1\right|f'(-y^2/(y^1)^2)=\varepsilon f-\varepsilon uf'=\varepsilon(f-uf'),$$

$$\ell_2:=\dot\partial_2F=\left|y^1\right|f'(1/y^1)=\frac{|y^1|}{y^1}f'=\varepsilon f'.$$

The coefficients N^i_j of the non-linear connection can calculated in the form

$$\begin{aligned}N^1_1&=\dot\partial_1G^1=2y^1f_1-y^2f_1',\quad N^1_2=\dot\partial_2G^1=y^1f_1',\\ N^2_1&=\dot\partial_1G^2=2y^1f_2-y^2f_2',\quad N^2_2=\dot\partial_2G^2=y^1f_2'.\end{aligned}\tag{3.5}$$

Now, we have to determine the coefficients G^1 and G^2 of the geodesic spray of F. Since S is the geodesic spray of F, then $d_hF=0$ that is $\partial_iF-N^h_i\dot\partial_hF=0$. Then we have the system

$$\left|y^1\right|\partial_1f-(2y^1f_1-y^2f_1')(\varepsilon f-\varepsilon uf')-(2y^1f_2-y^2f_2')\varepsilon f'=0,$$

$$\left|y^1\right|\partial_2f-y^1f_1'(\varepsilon f-\varepsilon uf')-y^1f_2'\varepsilon f'=0.$$

Since we assume that $y^1\neq 0$, then dividing the above system by $\left|y^1\right|$, we get

$$\partial_1f-2f_1(f-uf')+uf_1'(f-uf')-2f_2f'+uf_2'f'=0,\tag{3.6}$$

$$\partial_2f-f_1'(f-uf')-f_2'f'=0.\tag{3.7}$$

Multiplying (3.7) by u and then adding it to (3.6), we have

$$\partial_1f+u\partial_2f-2f_1(f-uf')-2f_2f'=0.\tag{3.8}$$

Differentiating (3.6) and (3.7) with respect to u yields the following

$$\partial_1f'+2uf_1f''-f_1'(f-uf')+u(f_1'(f-uf'))'-f_2'f'-2f_2f''+u(f_2'f')'=0\tag{3.9}$$

$$\partial_2f'-(f_1'(f-uf'))'-(f_2'f')'=0.\tag{3.10}$$

Multiplying (3.10) by u and then adding it to (3.9) and using (3.7), we get

$$\partial_1 f' + u\partial_2 f' - \partial_2 f + 2uf_1 f'' - 2f_2 f'' = 0. \tag{3.11}$$

Multiplying (3.8) by f'' and (3.11) by f' and then by subtraction, we obtain the required formula for f_1. Finally, substituting by f_1 into (3.11), we get the formula of f_2. □

Lemma 2. *The coefficients G^h_{jk} of Berwald connection are given by*

$$\begin{aligned} G^1_{11} &= 2f_1 - 2uf_1' + u^2 f_1'', \quad G^1_{12} = f_1' - uf_1'', \quad G^1_{22} = f_1'', \\ G^2_{11} &= 2f_2 - 2uf_2' + u^2 f_2'', \quad G^2_{12} = f_2' - uf_2'', \quad G^2_{22} = f_2''. \end{aligned} \tag{3.12}$$

Proof. The proof comes from the fact that $G^h_{jk} = \dot{\partial}_k N^h_j$ together with (3.5). □

Proposition 1. *The components R^i_{jk} of the curvature tensor are given by*

$$R^1_{12} = y^1(uf_1'^2 - f_1'f_2' - 2uf_1f_1'' + 2f_2f_1'' - u\partial_2 f_1' + 2\partial_2 f_1 - \partial_1 f_1'), \tag{3.13}$$

$$R^1_{12} = y^1(-2uf_1f_2'' + 2f_2f_2'' + uf_1'f_2' - 2f_2f_1' + 2f_1f_2' - f_2'^2 - u\partial_2 f_2' + 2\partial_2 f_2 - \partial_1 f_2'). \tag{3.14}$$

Proof. Using the the fact that $R^h_{jk} = \delta_k N^h_j - \delta_k N^h_j$, we have

$$R^1_{12} = \delta_2 N^1_1 - \delta_1 N^1_2 = \partial_2 N^1_1 - N^1_2 \dot{\partial}_1 N^1_1 - N^2_2 \dot{\partial}_2 N^1_1 - \partial_1 N^1_2 + N^1_1 \dot{\partial}_1 N^1_2 + N^2_1 \dot{\partial}_2 N^1_2,$$

$$R^2_{12} = \delta_2 N^2_1 - \delta_1 N^2_2 = \partial_2 N^2_1 - N^1_2 \dot{\partial}_1 N^2_1 - N^2_2 \dot{\partial}_2 N^2_1 - \partial_1 N^2_2 + N^1_1 \dot{\partial}_1 N^2_2 + N^2_1 \dot{\partial}_2 N^2_2.$$

Now by substituting from (3.5) and (2) into the above formulae we get the result. □

Example 1. Let $M = \mathbb{R}^2$ and F be a Finsler function given by

$$F = \sqrt{e^{x^2}(y^1)^2 + (y^2)^2}.$$

Then we have

$$f(x,u) = \sqrt{e^{x^2} + u^2}.$$

By substituting into (3.3) and (3.4), we get $f_1 = \frac{1}{2}u$, $f_2 = -\frac{1}{4}e^{x^2}$. Hence, the spray coefficients are given by

$$G^1 = (y^1)^2 \frac{1}{2}u = \frac{1}{2}y^1y^2, \quad G^2 = -\frac{1}{4}e^{x^2}(y^1)^2 = -\frac{1}{4}e^{x^2}(y^1)^2.$$

Making use of (3.13), (3.14), we have

$$R^1_{12} = y^1\left(\frac{1}{4}u\right) = \frac{1}{4}y^2, \quad R^2_{12} = -y^1\left(\frac{1}{4}e^{x^2}\right) = -\frac{1}{4}e^{x^2}y^1.$$

Now, for any surface (M,F), we can calculate the components of the Berwald and Landsberg curvatures as follows.

Lemma 3. *The components G^h_{ijk} of Berwald curvature are given by*

$$\begin{aligned} G^1_{111} &= -\frac{u^3}{y^1} f_1''', \quad G^2_{111} = -\frac{u^3}{y^1} f_2''', \quad G^1_{211} = \frac{u^2}{y^1} f_1''', \quad G^2_{211} = \frac{u^2}{y^1} f_2''', \\ G^1_{221} &= -\frac{u}{y^1} f_1''', \quad G^2_{221} = -\frac{u}{y^1} f_2''', \quad G^1_{222} = \frac{1}{y^1} f_1''', \quad G^2_{222} = \frac{1}{y^1} f_2'''. \end{aligned} \tag{3.15}$$

The components L_{ijk} of the Landsberg curvature are given by

$$\begin{aligned} L_{111} &= \frac{u^3 f}{2}(f_1'''\ell_1 + f_2'''\ell_2), \quad L_{112} = -\frac{u^2 f}{2}(f_1'''\ell_1 + f_2'''\ell_2), \\ L_{122} &= -\frac{uf}{2}(f_1'''\ell_1 + f_2'''\ell_2), \quad L_{222} = -\frac{f}{2}(f_1'''\ell_1 + f_2'''\ell_2). \end{aligned} \tag{3.16}$$

Proof. We do the calculations for one component and the rest can be calculated in a similar manner. By using (2) and the definition of the Berwald tensor $G^h_{ijk} = \dot\partial_k G^h_{ij}$, we can calculate, for example, G^1_{111} as follows

$$G^1_{111} = \dot\partial_1 G^1_{11} = (2f_1 - 2uf_1' + u^2 f_1'')' \frac{-y^2}{(y^1)^2} = -\frac{u^3}{y^1} f_1'''.$$

By the same way, one gets the component $G^2_{111} = -\frac{u^3}{y^1} f_2'''$. Hence, the first component L_{111} of the Landsberg curvature is given by

$$L_{111} = -\frac{F}{2}(G^1_{111}\ell_1 + G^2_{111}\ell_2) = \frac{u^3 f}{2}(f_1'''\ell_1 + f_2'''\ell_2).$$

□

The above lemma gives rise to the following proposition.

Proposition 2. *Any two dimensional Finsler manifold (M,F) in the form (3.1) is Landsbergian if and only if the following PDE*

$$f_1'''\ell_1 + f_2'''\ell_2 = 0 \tag{3.17}$$

is satisfied. The above PDE will be called the two-dimensional Landsberg's PDE.

Proof. Since in the Landsberg space all the components of the Landsberg tensor L_{ijk} vanishes identically, then by making use of (3.16) the result follows. □

4. Solutions to the Landsberg's PDE

We are going to solve the Landsberg's PDE in a special case, so we focus our attention to the Finsler functions on the form

$$F = |y^1|\,\phi(t), \quad t := \rho(x^1,x^2)u, \tag{4.1}$$

where we consider, without loss of generality, that $y^1 \neq 0$.

Remark 2. It should be noted that one of our goals is to solve the Landsberg's PDE (3.17), so when $\phi(t) = t$ in (4.1) then we use the separation of variables method to solve (3.17). Therefore, the assumption (4.1) is a more general method than the separation of variables.

Let's define the function Q as follows

$$Q := \frac{\phi_t}{\phi - t\phi_t},$$

where the subscript t refers to the differentiation with respect to t. Moreover, in this case, ϕ is given by

$$\phi = \exp\left(\int \frac{Q}{1+tQ} dt\right). \tag{4.2}$$

Lemma 4. *The spray coefficients of the Finsler functions given in* (4.1)*, are determined by the functions*

$$f_1 = -\frac{Q^2}{Q_t}\frac{\partial_1 \rho}{2\rho}, \tag{4.3}$$

$$f_2 = \frac{u\partial_1\rho + u^2\partial_2\rho}{2\rho} + \frac{Q}{Q_t}\frac{\partial_1\rho}{2\rho^2}. \tag{4.4}$$

Proof. Let F be given by (4.1) and $f(x,u) = \phi(t)$. We have the following formulas

$$f' = \rho\ \phi_t, \quad f'' = \rho^2\ \phi_{tt},$$

$$\partial_1 f = u\ \phi_t\ \partial_1\rho, \quad \partial_2 f = u\ \phi_t\ \partial_2\rho,$$

$$\partial_1 f' = \phi_t\ \partial_1\rho + \rho\ u\ \phi_{tt}\ \partial_1\rho, \quad \partial_2 f' = \phi_t\ \partial_2\rho + \rho\ u\ \phi_{tt}\ \partial_2\rho.$$

Substituting by the above equalities into (3.3) and (3.4) and by making use of the facts that $Q = \frac{\phi_t}{\phi - t\phi_t}$ and $Q_t = \frac{\phi\phi_{tt}}{(\phi - t\phi_t)^2}$, we get the required formulae for f_1 and f_2. $\square$

Focusing more on Q, we have the following lemma.

Lemma 5. *Let the Finsler function be given in the form* $F = |y^1|\phi(t)$*. Then, we have the following assertions*

(a) *If* $Q_{tt} = 0$*, then the manifold* (M,F) *is Berwaldian.*
(b) *If* $Q = \theta(x)$*, then the Finsler function is degenerate.*

Proof. To prove (a), if $Q_{tt} = 0$, then $Q = at + b$, where a and b are arbitrary constants. Moreover, we have the following

$$\frac{Q^2}{Q_t} = \frac{a^2t^2 + 2abt + b^2}{a}, \quad \frac{Q}{Q_t} = \frac{at+b}{a}.$$

Making use of the above quantities together with (4.3) and (4.4), we conclude that the coefficients of the spray are quadratic and hence the metric is Berwaldain.

To prove (b), one can see that, if $Q=\theta(x)$, then we have

$$\frac{Q}{1+tQ}=\frac{\theta(x)}{1+t\theta(x)}.$$

Therefore, by using (3.1) and (4.2), we have

$$F=|y^1|\exp(\ln(1+t\theta(x))=\varepsilon(y^1+\rho\theta(x)y^2).$$

This means that the Finsler function is linear and hence the metric tensor is degenerate. □

Now, we are in a position to announce the main result in this work.

Theorem 1. *All two-dimensional Landsberg metrics in the form* (4.1)*, that is,*

$$F=|y^1|\,f(x^1,x^2,u),\quad f(x^1,x^2,u)=\phi(t),\quad t:=\rho(x^1,x^2)u,\quad u=y^2/y^1,\quad y^1\neq 0.$$

are Berwaldian.

Proof. Let (M,F) be two-dimensional Landsbergian, then by Proposition 2, the Landsberg's PDE is satisfied, that is

$$f_1'''\ell_1+f_2'''\ell_2=0.$$

Now, assume that the solutions of the Landsberg's PDE is in the form

$$f(x^1,x^2,u)=\phi(t),\quad t=\rho(x)u.$$

Since $\ell_1=\varepsilon(f-uf')=\varepsilon(\phi-t\phi_t)$ and $\ell_2=\varepsilon f'=\varepsilon\rho\phi_t$, the above PDE can be rewritten in the form

$$f_1'''+\rho Q f_2'''=0. \tag{4.5}$$

Now, we are going to find analytic solution for (4.5). By making use of (4.3) and (4.4), we have

$$f_1'''=\frac{Q\partial_1\rho}{2\rho Q_t^4}\left(QQ_t^2Q_{tttt}-6QQ_tQ_{tt}Q_{ttt}+6QQ_{ttt}^3+4Q_t^3Q_{ttt}-6Q_t^2Q_{tt}^2\right),$$

$$f_2'''=-\frac{\partial_1\rho}{2\rho^2Q_t^4}\left(QQ_t^2Q_{tttt}-6QQ_tQ_{tt}Q_{ttt}+6QQ_{ttt}^3+2Q_t^3Q_{ttt}-3Q_t^2Q_{tt}^2\right).$$

Now, using Lemma 5 (a) together with the facts that the functions Q, ρ, $\partial_1\rho$ and Q_t should be non-zero, the PDE (4.5) transformed to the ODE

$$3Q_{tt}^2-2Q_tQ_{ttt}=0.$$

Making use of Lemma 5 (b), $Q_{tt}\neq 0$. Hence, the above ODE can be rewritten in the form

$$1+2\left(\frac{Q_t}{Q_{tt}}\right)_t=0.$$

Moreover, the above ODE has the solution

$$\frac{Q_t}{Q_{tt}}=-\frac{1}{2}t+c_1,$$

where c_1 is arbitrary constant. Furthermore, we can find Q_t, since

$$\frac{Q_{tt}}{Q_t} = \frac{2}{2c_1 - t}.$$

Which gives easily the formula of Q_t as follows

$$Q_t = \frac{c_2}{(2c_1 - t)^2}, \quad c_2 > 0. \tag{4.6}$$

By solving the above differential equation, we get

$$Q = \frac{c_2}{2c_1 - t} + c_3, \tag{4.7}$$

where c_1, c_2, c_3 are arbitrary constants such that $c_2 > 0$.

Now, substituting by Q and Q_t into f_1 and f_2 given in Lemma 4 and then substituting into the coefficients G^1 and G^2 given by (3.2), we conclude that the coefficients of the spray are quadratic and hence the metric is Berwaldain. □

Consider the conformal transformation of the metrics on the form (4.1), precisely,

$$\overline{F} = \sigma(x^1, x^2)F = |y^1|\sigma\phi(t) \tag{4.8}$$

Now, to get information about the transformed spray (the geodesic spray of $\overline{F}$), we have to calculate $\overline{f}_1$ and $\overline{f}_2$.

Lemma 6. *Under the conformal transformation* (4.8)*, the functions* f_1 *and* f_2 *transform as follows:*

$$\overline{f}_1 = f_1 + \frac{\partial_1\sigma + u\partial_2\sigma}{2\sigma} + \frac{\partial_2\sigma}{2\sigma\rho}\frac{Q}{Q_t} - \frac{\partial_1\sigma}{2\sigma}\frac{Q^2}{Q_t},$$

$$\overline{f}_2 = f_2 + \frac{u(\partial_1\sigma + u\partial_2\sigma)}{2\sigma} + \frac{\partial_1\sigma}{2\rho\sigma}\frac{Q}{Q_t} - \frac{\partial_2\sigma}{2\sigma\rho^2}\frac{1}{Q_t}.$$

Proof. Consider the conformal transformation (4.8), precisely,

$$\overline{F} = \sigma(x^1, x^2)F = |y^1|\sigma\phi(t), \quad t = \rho(x^1, x^2)u, \quad u = y^2/y^1.$$

Since $\overline{F} = |y^1|\overline{f}(x,u) = |y^1|\sigma(x^1, x^2)f(x,u)$, where

$$\overline{f} = \sigma f.$$

Now, we have to calculate $\overline{f}_1$ and $\overline{f}_2$. For this purpose, we have the following

$$\overline{f}' = \sigma f', \quad \overline{f}'' = \sigma f'',$$

$$\partial_1\overline{f} = \sigma\partial_1 f + f\partial_1\sigma, \quad \partial_2\overline{f} = \sigma\partial_2 f + f\partial_2\sigma,$$

$$\partial_1\overline{f}' = \sigma\partial_1 f' + f'\partial_1\sigma, \quad \partial_2\overline{f}' = \sigma\partial_2 f' + f'\partial_2\sigma.$$

By making use of the above relations together with the help of the quantities $Q = \frac{\phi_t}{\phi - t\phi_t}$, $Q_t = \frac{\phi\phi_{tt}}{(\phi - t\phi_t)^2}$, $f' = \rho\phi_t$ and $f'' = \rho^2\phi_{tt}$, then (3.3) and (3.4) lead to

$$\overline{f}_1 = f_1 + \frac{\partial_1\sigma + u\partial_2\sigma}{2\sigma} + \frac{\partial_2\sigma}{2\sigma\rho}\frac{Q}{Q_t} - \frac{\partial_1\sigma}{2\sigma}\frac{Q^2}{Q_t},$$
$$\overline{f}_2 = f_2 + \frac{u(\partial_1\sigma + u\partial_2\sigma)}{2\sigma} + \frac{\partial_1\sigma}{2\rho\sigma}\frac{Q}{Q_t} - \frac{\partial_2\sigma}{2\sigma\rho^2}\frac{1}{Q_t}.$$

As to be shown. □

Theorem 2. *The conformal transformations of the Finsler functions given in* (4.1) *by any positive smooth function* $\sigma(x^1, x^2)$ *are Berwaldain.*

Proof. By the formulae (4.6) and (4.7), we have

$$Q_t = \frac{c_2}{(2c_1 - t)^2}, \quad Q = \frac{c_2}{2c_1 - t} + c_3.$$

Taking the fact that the geodesic spray of the class (4.1) is quadratic into account. Then, substituting by Q and Q_t into the formulae of $\overline{f}_1$ and $\overline{f}_2$ which are given in Lemma 6, we conclude that the geodesic spray of $\overline{F}$ is quadratic and hence the space is Berwaldian. □

As by-product, we get the following explicit formulae for two dimensional Landsberg metrics which are Berwaldain.

Theorem 3. *The Landsberg metric in the form* (4.1) *is given by*

$$F = \sqrt{c_3\rho^2(y^2)^2 - (c-2)\rho y^1 y^2 - 2c_1(y^1)^2}\; e^{\frac{c}{\sqrt{c^2-4c_2}} \operatorname{arctanh}\left(\frac{2c_3\rho y^2 - (c-2)y^1}{y^1\sqrt{c^2-4c_2}}\right)}. \tag{4.9}$$

As a special case, let $\rho(x) = 1$ *so that* $\phi(t) = \phi(u)$*. Then by making use of Theorem* 2*, we have*

$$f(x^1, x^2, u) = \sigma(x^1, x^2)\phi(u)$$

and we get the class

$$F = \sigma(x)\sqrt{c_3(y^2)^2 - (c-2)y^1 y^2 - 2c_1(y^1)^2}\; e^{\frac{c}{\sqrt{c^2-4c_2}} \operatorname{arctanh}\left(\frac{2c_3 y^2 - (c-2)y^1}{y^1\sqrt{c^2-4c_2}}\right)}. \tag{4.10}$$

In addition, the metrics that satisfy that $Q_{tt} = 0$ *are given by*

$$F = \sqrt{a\rho^2(y^2)^2 + b\rho y^1 y^2 + (y^1)^2}\; e^{-\frac{b}{\sqrt{b^2-4a}}\, arctanh\left(\frac{2a\rho y^2 + by^1}{y^1\sqrt{b^2-4a}}\right)}. \tag{4.11}$$

Where $t = \rho(x^1, x^2)u$*,* $u = \frac{y^2}{y^1}$*,* $c := 2c_1c_3 + c_2 + 1$ *and* $c_2 > 0, a, b, c_1, c_3$ *are constants.*

Proof. To calculate the first class, by (4.7), one can show that

$$\frac{Q}{1+tQ}=\frac{-c_3t+2c_1c_3+c_2}{-c_3t^2+(2c_1c_3+c_2-1)t+2c_1}$$

which can be rewritten in the following useful form

$$\frac{Q}{1+tQ}=\frac{1}{2}\frac{2c_3t-c+2}{c_3t^2-(c-2)t-2c_1}+\frac{2cc_3}{(c^2-4c_2)-(2c_3t-c+2)^2}.$$

Hence, we have

$$\int\frac{Q}{1+tQ}dt=\frac{1}{2}\ln\left(c_3t^2-(c-2)t-2c_1\right)+\frac{c}{\sqrt{c^2-4c_2}}\operatorname{arctanh}\left(\frac{2c_3t-(c-2)}{\sqrt{c^2-4c_2}}\right).$$

Now, using (4.2), the function $\phi(t)$ can be calculated and so we get the Finsler function.

To find the third class of the Finsler functions, since $Q_{tt}=0$, then $Q=at+b$. Now, we have

$$\frac{Q}{1+tQ}=\frac{at+b}{at^2+bt+1}=\frac{2at+b}{2(at^2+bt+1)}-\frac{2ab}{b^2-4a-(2at+b)^2}.$$

Hence, by the help of (3.1) and (4.2), the required formula can be obtained. □

As a direct consequence, we can make use of the surprising result due to Z. Szabó [12] and, for more details, we refer to [13].

Corollary 1. *All two-dimensional Landsberg metrics on the form (4.1) and their conformal transformations are flat (locally Minkowski) or Riemannian.*

In case of higher dimensions $n\geq 3$, the conformal transformation of a Minkowski metrics can yield a non-Berwaldian Landsberg metrics (cf. [4]). Assume that a manifold (M,F) is Minkowski if the Finsler function F depends on the directional argument y only. In contrast of the higher dimensions, we have the following result.

Theorem 4. *If the conformal transformation of any two dimensional Minkowski metric is Landsberg, then it is Berwaldian.*

Proof. Let (M,F) be a Minkowski space, then one can write

$$F=\left|y^1\right|f(u).$$

In this case F can be considered a special case of the form (4.1) by setting $\rho(x)=1$. Now, the conformal transformation of F takes the form

$$\overline{F}=\sigma(x)F=\left|y^1\right|\sigma(x)f(u).$$

Since $\overline{F}$ is Landsbergian then $f(u)$ is given by (4.10), that is $f(u)=\phi(u)$. Consequently, $\overline{F}$ is Berwaldian. □

5. Concluding remarks

We end this work by the following remarks.

• If we consider the Landsberg surface in the form

$$F = \left|y^2\right| \varphi(\rho(x)v), \quad v := y^1/y^2,$$

then we get a similar formula of $\varphi(\rho(x)v)$ as the formula giving by (4.9). In fact this corresponds a change of coordinates by switching x^1 and x^2 and this implies switching y^1 and y^2. Moreover, if we need to fix a coordinate system and work on $F = \left|y^2\right| \varphi(\rho(x)v)$, then $u = \frac{1}{v}$. To get the function φ, we have to substitute by $u = \frac{1}{v}$ in (4.9), that is $\varphi(\rho(x)v) = \phi(\frac{\rho(x)}{v})$. That is, we obtain the same formula of the Finsler function F.

• If there is a singularity in the surfaces given by (4.9) or (4.11) at certain direction, then this singularity does not come from the assumption that $F = \left|y^1\right| \phi(\rho(x)u)$ rather it comes from being F a solution of the Landsberg's PDE. Moreover, the choice of the constants c_1, c_2, c_3 play an essential rule to determine whether the resultant metric is regular. Many, but not all, Berwaldian surfaces among them the Riemannian ones can be obtained by specific choice of these constants. For example the choice $c_1 = c_2 = 1$ and $c_3 = -1$ leads to $c = 0$ and hence we get the Riemannian metrics

$$F = \sqrt{a(x)(y^1)^2 + b(x)y^1y^2 + c(x)(y^2)^2},$$

for some functions $a(x)$, $b(x)$, $c(x)$.

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Author's address

Salah G. Elgendi
Current address: Islamic University of Madinah, Madinah, Kingdom of Saudia Arabia
Benha University, Faculty of Science, Department of Mathematics, 13518 Benha, Egypt
E-mail address: salah.ali@fsc.bu.edu.eg, salahelgendi@yahoo.com