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THE GROUP INVERTIBILITY OF MATRICES OVER BÉZOUT DOMAINS

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Abstract. Let R be a Bézout domain, and let $A,B,C \in R^{n\times n}$ with $ABA = ACA$. If AB and CA are group invertible, we prove that AB is similar to CA. Moreover, we have $(AB)^{\#}$ is similar to $(CA)^{\#}$. This generalize the main result of Cao and Li (Group inverses for matrices over a Bézout domain, *Electronic J. Linear Algebra*, **18** (2009), 600–612).

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1. Introduction

Let R be a associate ring with an identity 1. An element $a \in R$ is Drazin invertible if there exists an element $x \in R$ such that

$$ax = xa, \qquad xax = x, \qquad a^{k+1}x = a^k$$

for some nonnegative integer k. The preceding x is unique if it exists. The element x is called Drazin inverse of a, and denote x by a^D. The least nonnegative integer k satisfying these equations above is the Drazin index $ind(a)$ of a. When $ind(a) = 1$, we say that a is group invertible, that is

$$ax = xa, \qquad x = xax, \qquad a = axa.$$

Denote x by $a^{\#}$. We use $R^{\#}$ to stand for the set of all group invertible elements in R.

Two elements $a,b \in R$ are similar, written $a \sim b$, if there exists an invertible element s such that $a = s^{-1}bs$.

It is important for Similarity to solve linear equations. However, it is difficulty to consider similarity for usual ring. In view of this, we will investigate the similarity over Bézout domain. An integral domain is called a Bézout domain if every its finitely

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generated ideal is principal. Some authors have discussed the similarity of products of matrices on background of generalized inverse. For example, in [1, Theorem 3.6], Cao and Li proved that $(AB)^{\#}$ and $(BA)^{\#}$ exist implies that $AB \sim BA$ for any $n \times n$ matrices A and B over a Bézout domain R.

This paper is partially motivated by Flanders' a classic theorem which states that the elementary divisors of AB which do not have zero as a root coincide with those of BA(see [4, Theorem 2]). In [5, Theorem 2], for strongly π-regular ring R, Hartwig proved that R is unit-regular if and only if R is regular and $(ab)^D \sim (ba)^D$ for every $a, b \in R$.

Inspired by the extension of Jacobson's lemma under the condition $ABA = ACA$ [3, 7–9], we shall improve Flanders, Cao and Li's results. In Section 2, we present a new characterization of the similarity. Under the condition $ABA = ACA$, we prove that AB and CA are similar if AB and CA are both group invertible for any $n \times n$ matrices A, B and C over a Bézout domain R. An example is provided to illustrate that our result is a nontrivial generalization of [1, Theorem 3.6]. In Section 3, we turn to the similarity of the group inverses of products of matrices over Bézout domains. It is shown that $(AB)^{\#}$ and $(CA)^{\#}$ are similar if AB and CA are both group invertible for any $n \times n$ matrices A, B and C over a Bézout domain R.

Throughout this paper, R is a Bézout domain. $R^{m\times n}$ denotes the set of all $m \times n$ matrices over R. Let $A \in R^{m\times n}$, then $R_r(A)$ is the space spanned by the columns of A:

$$R_r(A) = \{Ax | x \in R^{n\times 1}\} \subseteq R^{m\times 1}.$$

Likewise, $R_l(A)$ is the space spanned by the rows of A:

$$R_l(A) = \{yA | y \in R^{1\times m}\} \subseteq R^{1\times n}.$$

The column rank (respectively, row rank) of A is defined as the dimension of $R_r(A)$ (respectively, $R_l(A)$). The inner rank of A is determined as the least r such that $A = BC$, where $B \in R^{m\times r}, C \in R^{r\times n}$. Over a Bézout domain, it is proved that the column rank, row rank and inner rank of any matrix A coincide with each other(see [6, Proposition 2.3.4] and [2, Page 245]), the common number is called the rank of A, written $rank(A)$. Define the rank of a zero matrix is 0. The rank of $n \times n$ non-singular matrix over R is n. An $n \times n$ matrix A over a Bézout domain R is invertible in case $AB = I_n = BA$ for an $n \times n$ matrix B. Obviously, B is unique if it exists, denote A^{-1}.

2. THE SIMILARITY OF PRODUCT OF MATRICES

The purpose of this section is concerned with the similarity of product of matrices over Bézout domains. We now derive

Theorem 1. *Let R be a Bézout domain, and let $A, B, C \in R^{n\times n}$ with $ABA = ACA$. If AB and CA are group invertible, then AB is similar to CA.*

Proof. For $A,B,C \in R^{n\times n}$, by [1, Lemma 3.5], there exist invertible matrices P and Q such that

$$A = P\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} Q \ \text{ and } \ B = Q^{-1}\begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 0 & 0 \\ C_1 & C_2 & C_3 \\ 0 & 0 & 0 \end{bmatrix} P^{-1},$$

also there exists an invertible matrix Q' together with the invertible matrix P above such that

$$A = P\begin{bmatrix} A_1' & 0 \\ 0 & 0 \end{bmatrix} Q' \ \text{ and } \ C = Q'^{-1}\begin{bmatrix} B_1' & B_2' & B_3' \\ 0 & 0 & 0 \\ C_1' & C_2' & C_3' \\ 0 & 0 & 0 \end{bmatrix} P^{-1},$$

where $A_1, A_1' \in R^{r\times r}$, $Q = \mathrm{diag}(Q_1,Q_2)N$, $Q' = \mathrm{diag}(Q_1',Q_2')N$, $A_1Q_1 = \Delta = A_1'Q_1' \in R^{r\times r}$, $rank(\Delta) = rank(A_1) = rank(A_1') = rank(A) = r$, Q_1,Q_2,Q_1',Q_2' and N are invertible, $B_1, B_1' \in R^{s\times s}$, $C_1, C_1' \in R^{t\times s}$. Then

$$AB = P\begin{bmatrix} A_1\begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} & A_1\begin{bmatrix} B_3 \\ 0 \end{bmatrix} \\ 0 & 0 \end{bmatrix} P^{-1},$$

$$CA = Q'^{-1}\begin{bmatrix} \begin{bmatrix} B_1' & B_2' \\ 0 & 0 \end{bmatrix} A_1' & 0 \\ \begin{bmatrix} C_1' & C_2' \\ 0 & 0 \end{bmatrix} A_1' & 0 \end{bmatrix} Q'.$$

By [1, Corollary 2.4 and 2.5], we get

$$\begin{aligned}(AB)^{\#}\text{exists} \Longleftrightarrow & \left(A_1\begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix}\right)^{\#} \text{exists, and} \\ & R_r\left(A_1\begin{bmatrix} B_3 \\ 0 \end{bmatrix}\right) \subseteq R_r\left(A_1\begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix}\right). \\ (CA)^{\#}\text{exists} \Longleftrightarrow & \left(\begin{bmatrix} B_1' & B_2' \\ 0 & 0 \end{bmatrix} A_1'\right)^{\#} \text{exists, and} \\ & R_l\left(\begin{bmatrix} C_1' & C_2' \\ 0 & 0 \end{bmatrix} A_1'\right) \subseteq R_l\left(\begin{bmatrix} B_1' & B_2' \\ 0 & 0 \end{bmatrix} A_1'\right).\end{aligned}$$

Therefore, there exist E,F such that

$$A_1\begin{bmatrix} B_3 \\ 0 \end{bmatrix} = A_1\begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} E, \quad \begin{bmatrix} C_1' & C_2' \\ 0 & 0 \end{bmatrix} A_1' = F\begin{bmatrix} B_1' & B_2' \\ 0 & 0 \end{bmatrix} A_1'.$$

Then

$$AB = P \begin{bmatrix} I & -E \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & E \\ 0 & I \end{bmatrix} P^{-1},$$

$$CA = Q'^{-1} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} B'_1 & B'_2 \\ 0 & 0 \end{bmatrix} A'_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -F & I \end{bmatrix} Q'.$$

Let

$$\begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} = R \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} S, \qquad \begin{bmatrix} B'_1 & B'_2 \\ 0 & 0 \end{bmatrix} = R' \begin{bmatrix} D' & 0 \\ 0 & 0 \end{bmatrix} S'$$

for some invertible matrices R, S, R', S', where $D \in R^{r_1 \times r_1}$, $D' \in R^{s_1 \times s_1}$, $rank(D) = r_1, rank(D') = s_1$. We obtain

$$A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} = A_1 R \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} S, \qquad \begin{bmatrix} B'_1 & B'_2 \\ 0 & 0 \end{bmatrix} A'_1 = R' \begin{bmatrix} D' & 0 \\ 0 & 0 \end{bmatrix} S' A'_1.$$

Let $SA_1R = C = \begin{bmatrix} C_4 & C_5 \\ C_6 & C_7 \end{bmatrix}, S'A'_1R' = C' = \begin{bmatrix} C'_4 & C'_5 \\ C'_6 & C'_7 \end{bmatrix}$. Then $rank(C) = rank(C') = r$, and

$$A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} = S^{-1} \begin{bmatrix} C_4D & 0 \\ C_6D & 0 \end{bmatrix} S,$$

$$\begin{bmatrix} B'_1 & B'_2 \\ 0 & 0 \end{bmatrix} A'_1 = R' \begin{bmatrix} D'C'_4 & D'C'_5 \\ 0 & 0 \end{bmatrix} R'^{-1}.$$

Since $\left(A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix}\right)^{\#}$ exists, $\begin{bmatrix} C_4D & 0 \\ C_6D & 0 \end{bmatrix}$ is group invertible. Furthermore, $(C_4D)^{\#}$ exists and $R_l(C_6D) \subset R_l(C_4D)$, then $C_6D = GC_4D$ for some matrix G. Hence

$$\begin{bmatrix} C_4D & 0 \\ C_6D & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ G & I \end{bmatrix} \begin{bmatrix} C_4D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix},$$

$$rank(C_4) = rank(C_4D) = r_1.$$

Therefore,

$$A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} C_4D & 0 \\ 0 & 0 \end{bmatrix}.$$

Analogously, $D'C'_4$ is group invertible, and

$$\begin{bmatrix} B'_1 & B'_2 \\ 0 & 0 \end{bmatrix} A'_1 \sim \begin{bmatrix} D'C'_4 & 0 \\ 0 & 0 \end{bmatrix},$$

$$rank(C'_4) = rank(D'C'_4) = s_1.$$

By $(C_4D)^{\#}$ exists and $rank(C_4) = rank(D) = r_1$, there exists X such that $C_4DXC_4D = C_4D$, hence $DXC_4D = D$ and $C_4DXC_4 = C_4$. Since $rank(C_4) = rank(D) = r_1$, we get

$DXC_4 = I = XC_4D$ and $C_4DX = I = DXC_4$, which means C_4 and D are invertible. Thus C_4D is invertible. Likewise, C_4', D' and $D'C_4'$ are also invertible.

On the other hand, we have

$$\begin{aligned}
ABA &= P\begin{bmatrix} I & -E \\ 0 & I \end{bmatrix}\begin{bmatrix} A_1\begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & E \\ 0 & I \end{bmatrix}P^{-1}P\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}Q \\
&= P\begin{bmatrix} A_1\begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix}A_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}N \\
&= P\begin{bmatrix} S^{-1}\begin{bmatrix} I & 0 \\ G & I \end{bmatrix}\begin{bmatrix} C_4D & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ -G & I \end{bmatrix}SA_1Q_1 & 0 \\ 0 & 0 \end{bmatrix}N, \\
ACA &= P\begin{bmatrix} A_1' & 0 \\ 0 & 0 \end{bmatrix}Q'Q'^{-1}\begin{bmatrix} I & 0 \\ F & I \end{bmatrix}\begin{bmatrix} \begin{bmatrix} B_1' & B_2' \\ 0 & 0 \end{bmatrix}A_1' & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ -F & I \end{bmatrix}Q' \\
&= P\begin{bmatrix} A_1'\begin{bmatrix} B_1' & B_2' \\ 0 & 0 \end{bmatrix}A_1' & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Q_1' & 0 \\ 0 & Q_2' \end{bmatrix}N \\
&= P\begin{bmatrix} A_1'R'\begin{bmatrix} D' & 0 \\ 0 & 0 \end{bmatrix}S'A_1'Q_1' & 0 \\ 0 & 0 \end{bmatrix}N \\
&= P\begin{bmatrix} S'^{-1}\begin{bmatrix} C_4'D' & 0 \\ C_6'D' & 0 \end{bmatrix}S'A_1'Q_1' & 0 \\ 0 & 0 \end{bmatrix}N \\
&= P\begin{bmatrix} S'^{-1}\begin{bmatrix} I & 0 \\ C_6'C_4'^{-1} & I \end{bmatrix}\begin{bmatrix} C_4'D' & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ -C_6'C_4'^{-1} & I \end{bmatrix}S'A_1'Q_1' & 0 \\ 0 & 0 \end{bmatrix}N.
\end{aligned}$$

In view of $ABA = ACA$ and the invertibility of P and N, we get

$$\begin{aligned}
S^{-1}&\begin{bmatrix} I & 0 \\ G & I \end{bmatrix}\begin{bmatrix} C_4D & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ -G & I \end{bmatrix}S \\
&= S'^{-1}\begin{bmatrix} I & 0 \\ C_6'C_4'^{-1} & I \end{bmatrix}\begin{bmatrix} C_4'D' & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ -C_6'C_4'^{-1} & I \end{bmatrix}S',
\end{aligned}$$

it follows that

$$\begin{bmatrix} C_4D & 0 \\ 0 & 0 \end{bmatrix}Y = Y\begin{bmatrix} C_4'D' & 0 \\ 0 & 0 \end{bmatrix},$$

in which $Y = \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix} SS'^{-1} \begin{bmatrix} I & 0 \\ C_6'C_4'^{-1} & I \end{bmatrix}$. Obviously, Y is invertible. Partition $Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$ conformally with $\begin{bmatrix} C_4D & 0 \\ 0 & 0 \end{bmatrix}$. We have the three identities:

$$C_4DY_{11} = Y_{11}C_4'D', \qquad C_4DY_{12} = 0, \qquad Y_{21}C_4'D' = 0.$$

Combining the last two identities with the invertibility of C_4D and $C_4'D'$ ensure that $Y_{12} = 0$ and $Y_{21} = 0$. Then $Y = \operatorname{diag}(Y_{11}, Y_{22})$ implies that Y_{11} is invertible. So $C_4D = Y_{11}C_4'D'Y_{11}^{-1}$, which reveals that

$$C_4D \sim C_4'D' = C_4'D'C_4'C_4'^{-1}.$$

Therefore,

$$C_4D \sim D'C_4'.$$

That is,

$$\begin{bmatrix} C_4D & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} D'C_4' & 0 \\ 0 & 0 \end{bmatrix}.$$

We have the conclusion $AB \sim CA$. □

Corollary 1. *Let $A, B, C \in R^{n\times n}$ with $ABA = ACA$. If $R_r(A) = R_r(ABA)$, then $AB \sim CA$.*

Proof. By $R_r(A) = R_r(ABA)$, $(AB)^{\#}$ exists. $R_r(A) = R_r(ACA)$ implies $(CA)^{\#}$ exists. Then $AB \sim CA$. □

Corollary 2. *Let $A, B, C \in R^{n\times n}$ with $ABA = ACA$. If $R_r(A) = R_r(AB)$, $R_r(B) = R_r(BA)$, then $AB \sim CA$.*

Proof. Since $R_r(A) = R_r(AB) = AR_r(B) = AR_r(BA) = R_r(ABA)$, by Corollary 1, $AB \sim CA$. □

Theorem 2. *Let $A, B, C \in R^{n\times n}$ with $ABA = ACA$. If $R_r(AB) = R_r(ABA)$, $R_r(CA) = R_r(CAB)$, then $AB \sim CA$.*

Proof. By the column space of a matrix, we have the formulas $R_r(AB) = R_r(ABA) = R_r(ACA) = A \cdot R_r(CA) = A \cdot R_r(CAB) = R_r(ACAB) = R_r(ABAB)$ and $R_r(CA) = R_r(CAB) = C \cdot R_r(AB) = C \cdot R_r(ABAB) = C \cdot R_r(ACAB) = R_r(CACAB) \subseteq R_r(CACA)$. Obviously, $R_r(CA) \supseteq R_r(CACA)$. by [1, Theorem 2.1], AB and CA are group invertible. Then the conclusion follows from Theorem 1. □

As an immediate consequence, we have

Corollary 3. *Let $A, B, C \in R^{n\times n}$ with $ABA = ACA$. If $R_r(A) = R_r(AC) = R_r(ABA)$, then $AC \sim BA$.*

Proof. Since $R_r(A) = R_r(AB) = AR_R(B) = AR_R(BA) = R_R(ABA)$, by Corollary 1, $AB \sim CA$. □

Example 1. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{2\times 2}$, then $ABA = ACA$. Since AB and CA are idempotent, $(AB)^{\#} = AB$ and $(CA)^{\#} = CA$. By virtue of Theorem 1, we have $AB \sim CA$. In fact, $AB = P(CA)P^{-1}$, where $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

3. Extensions

In this section, we turn to the similarity of group inverses. We have

Theorem 3. *Let R be a Bézout domain, and let $A,B,C \in R^{n\times n}$ with $ABA = ACA$. If AB and CA are group invertible, then $(AB)^{\#}$ is similar to $(CA)^{\#}$.*

Proof. As in the proof of Theorem 1,

$$A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} = S^{-1} \begin{bmatrix} I & 0 \\ G & I \end{bmatrix} \begin{bmatrix} C_4 D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix} S.$$

Similarly, there exists G' such that

$$\begin{bmatrix} B_1' & B_2' \\ 0 & 0 \end{bmatrix} A_1' = R' \begin{bmatrix} I & -G' \\ 0 & I \end{bmatrix} \begin{bmatrix} D'C_4' & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & G' \\ 0 & I \end{bmatrix} R'^{-1}.$$

So

$$AB = H \begin{bmatrix} C_4 D & 0 \\ 0 & 0 \end{bmatrix} H^{-1} \text{ and } CA = K \begin{bmatrix} D'C_4' & 0 \\ 0 & 0 \end{bmatrix} K^{-1},$$

where

$$H = P \begin{bmatrix} I & -E \\ 0 & I \end{bmatrix} \begin{bmatrix} S^{-1} \begin{bmatrix} I & 0 \\ G & I \end{bmatrix} & \\ & I \end{bmatrix}$$

and

$$K = Q'^{-1} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix} \begin{bmatrix} R' \begin{bmatrix} I & -G' \\ 0 & I \end{bmatrix} & \\ & I \end{bmatrix}.$$

It is easy to check that AB and CA are group invertible, and

$$(AB)^{\#} = H \begin{bmatrix} D^{-1}C_4^{-1} & 0 \\ 0 & 0 \end{bmatrix} H^{-1},$$

$$(CA)^{\#} = K \begin{bmatrix} C_4'^{-1}D'^{-1} & 0 \\ 0 & 0 \end{bmatrix} K^{-1}.$$

In view of the similarity of $\begin{bmatrix} C_4 D & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} D'C_4' & 0 \\ 0 & 0 \end{bmatrix}$ which has been proved in Theorem 1, we derive that $(AB)^{\#}$ and $(CA)^{\#}$ are similar. $\square$

Corollary 4. *Let $A,B,C \in \mathbb{C}^{n\times n}$ with $ABA = ACA$. If $(AC)^{\#}$ and $(BA)^{\#}$ exist, then $(AC)^{\#}$ is similar to $(BA)^{\#}$.*

Proof. This is obvious by Theorem 3. □

Corollary 5. *Let R be a Bézout domain, and let $A,B,C \in R^{n\times n}$ with $ABA = ACA$. If AB and CA are group invertible, then $(AB)(AB)^{\#}$ is similar to $(CA)(CA)^{\#}$.*

Proof. From the proof of Theorem 3,

$$\begin{aligned}AB(AB)^{\#} &= H\begin{bmatrix} C_4D & \\ & 0\end{bmatrix}H^{-1}H\begin{bmatrix} (C_4D)^{-1} & \\ & 0\end{bmatrix}H^{-1}\\ &= H\begin{bmatrix} I & \\ & 0\end{bmatrix}H^{-1},\\ CA(CA)^{\#} &= K\begin{bmatrix} D'C_4' & \\ & 0\end{bmatrix}K^{-1}K\begin{bmatrix} (D'C_4')^{-1} & \\ & 0\end{bmatrix}K^{-1}\\ &= K\begin{bmatrix} I & \\ & 0\end{bmatrix}K^{-1}.\end{aligned}$$

This implies that $(AB)(AB)^{\#}$ and $(CA)(CA)^{\#}$ are similar. □

Cline proved that ba is Drazin invertible if ab has Drazin inverse. In this case, $(ba)^D = b[(ab)^D]^2a$. We now derive.

Theorem 4. *Let R be a Bézout domain, and let $A,B,C \in R^{n\times n}$ with $ABA = ACA$. If AB has Drazin inverse, then there exists $k \in \mathbb{N}$ such that $(AB)^s$ is similar to $(CA)^s$ for any $s \geq k$.*

Proof. Suppose that AB has Drazin inverse with $ind(AB) = k$. By [9, Theorem 2.7], CA is Drazin invertible with $ind(CA) \leq k+1$, and $(CA)^D = C[(AB)^D]^2A$.

Set $s > k$. Since $AB(AB)^D$ is idempotent, we have

$$\begin{aligned}(AB)^s[(AB)^D]^s &= AB(AB)^D = [(AB)^D]^s(AB)^s,\\ ([(AB)^D]^s)^2(AB)^s &= [(AB)^D]^{s+1}AB = [(AB)^D]^s,\\ [(AB)^s]^2[(AB)^D]^s &= (AB)^{s+1}(AB)^D = (AB)^s.\end{aligned}$$

Accordingly, $(AB)^s$ is group invertible and $[(AB)^s]^{\#} = [(AB)^D]^s$. Similarly, $(CA)^s$ is group invertible and $[(CA)^s]^{\#} = [(CA)^D]^s$.

Let $B' = B(AB)^{s-1}$ and $C' = (CA)^{s-1}C$. Then $AB'A = AC'A$, where AB' and $C'A$ are group invertible. Therefore, the result of theorem follows by Theorem 1. □

Corollary 6. *Let $A,B \in M_n(\mathbb{C})$. Then there exists $k \in \mathbb{N}$ such that $(AB)^s$ is similar to $(BA)^s$ for any $s \geq k$.*

Proof. This is obvious by choosing $B = C$ in Theorem 4. □

Corollary 7. *Let R be a Bézout domain, and let $A,B,C \in R^{n\times n}$ with $ABA = ACA$. If AB and CA are group invertible, then $(AB)^2(AB)^D$ is similar to $(CA)^2(CA)^D$.*

Proof. Since AB and CA are group invertible, $(AB)^D = (AB)^\#, (CA)^D = (CA)^\#$. By the proof of Corollary 5,

$$(AB)^2(AB)^D = H \begin{bmatrix} C_4 D & \\ & 0 \end{bmatrix} H^{-1},$$

$$(CA)^2(CA)^D = K \begin{bmatrix} D'C_4' & \\ & 0 \end{bmatrix} K^{-1}.$$

According to the proof of Theorem 1, $\begin{bmatrix} C_4 D & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} D'C_4' & 0 \\ 0 & 0 \end{bmatrix}$ are similar. Then the similarity of $(AB)^2(AB)^D$ and $(CA)^2(CA)^D$ follows. □

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