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COUNTEREXAMPLE TO A CONJECTURE ABOUT DIHEDRAL QUANDLE

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Abstract. It was conjectured that the augmentation ideal of a dihedral quandle of even order $n > 2$ satisfies $|\Delta^k(\mathrm{R}_n)/\Delta^{k+1}(\mathrm{R}_n)| = n$ for all $k \geq 2$. In this article we provide a counterexample against this conjecture.

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1. INTRODUCTION

A *quandle* is a pair $(A, \cdot)$ such that '$\cdot$' is a binary operation satisfying the following two conditions:

(1) the map $S_a : A \longrightarrow A$, defined as $S_a(b) = b \cdot a$ is an automorphism for all $a \in A$,
(2) for all $a \in A$, we have $S_a(a) = a$.

To have a better understanding of the structure, a theory parallel to group rings was introduced by Bardakov, Passi and Singh in [1]. Let $\mathbb{Z}_n$ denote the cyclic group of order n. Then defining $a \cdot b = 2b - a$ gives a quandle structure on $A = \mathbb{Z}_n$. This is known as *dihedral quandle*. For other examples see [1]. The quandle ring of a quandle A is defined as follows. Let R be a commutative ring. Consider

$$R[A] := \left\{ \sum_i r_i a_i : r_i \in R, a_i \in A,\ r_i = 0 \text{ for all but finitely many } i \right\}.$$

Then this is an additive group in usual way. Define multiplication as

$$\left(\sum_i r_i a_i\right) \cdot \left(\sum_j s_j a_j\right) := \sum_{i,j} r_i s_j (a_i \cdot a_j).$$

The *augmentation ideal* of $R[A]$, $\Delta_R(A)$ is defined as the kernel of the augmentation map

$$\varepsilon : R[A] \to R, \sum_i r_i a_i \mapsto \sum_i r_i.$$

The powers $\Delta_R^k(A)$ are defined as $(\Delta_R(A))^k$. When $R = \mathbb{Z}$, we will be omitting the subscript R. The following proposition gives a basis for $\Delta_R(X)$.

Proposition 1. *[1, Proposition 3.2, Page 6] A basis of $\Delta_R(X)$ as an R-module is given by $\{a - a_0 : a \in A \setminus \{a_0\}\}$, where $a_0 \in A$ is a fixed element.*

The following has been conjectured in [1, Conjecture 6.5, Page 20].

Conjecture 1. *Let $\mathrm{R}_n = \{a_0, a_1, \cdots, a_{n-1}\}$ denote the dihedral quandle of order n. Then we have the following statements.*

(1) *For an odd integer $n > 1$, $\Delta^k(\mathrm{R}_n)/\Delta^{k+1}(\mathrm{R}_n) \cong \mathbb{Z}_n$ for all $k \geq 1$.*
(2) *For an even integer $n > 2$, $\left|\Delta^k(\mathrm{R}_n)/\Delta^{k+1}(\mathrm{R}_n)\right| = n$ for $k \geq 2$.*

The first statement has been confirmed by Elhamdadi, Fernando and Tsvelikhovskiy in [2, Theorem 6.2, Page 182]. The second statement holds true for $n = 4$, see [1]. Here we have given a counterexample in Theorem 1 to show that the conjecture is not true in general.

2. Counterexample

Theorem 1. *Let R_8 be the dihedral quandle of order 8. Then*

$$\left|\Delta^2(\mathrm{R}_8)/\Delta^3(\mathrm{R}_8)\right| = 16.$$

From Proposition 1, we get that $\{e_i = a_i - a_0 : i = 1, 2, \cdots, n-1\}$ is a basis for $\Delta(\mathrm{R}_n)$. We will be using this notation in the subsequent computation.

Lemma 1. *Let R_{2n} denote the dihedral quandle of order $2n$ $(n \geq 2)$. Then $e_i \cdot e_n = 0$ for all $i = 1, 2, \cdots, 2n-1$.*

Proof. Observe that

$$e_i \cdot e_n = (a_i - a_0) \cdot (a_n - a_0) = a_{2n-i} - a_{2n-i} - a_0 + a_0 = 0.$$

□

Lemma 2. *Let R_{2n} denote the dihedral quandle of order $2n$ $(n \geq 2)$. Then $e_i \cdot e_j = e_i \cdot e_{n+j}$ for all $j = 1, 2, \cdots, n-1$ and for all $i = 1, 2, \cdots, 2n-1$.*

Proof. Note that

$$\begin{aligned} e_i \cdot e_{n+j} &= a_i a_{n+j} - a_i a_0 - a_0 a_{k+j} + a_0 \\ &= a_i a_j - a_i a_0 - a_0 a_j + a_0 = e_i \cdot e_j. \end{aligned}$$

□

We will use Lemma 1 and Lemma 2 to simplify the multiplication tables.

Proof of Theorem 1. Recall that a basis of $\Delta(\mathrm{R}_8)$ is given by $\mathcal{B}_1 = \{e_1, e_2, \cdots, e_7\}$. The multiplication table for the $e_i \cdot e_j$ is given as follows:

	e_1	e_2	e_3
e_1	$e_1 - e_2 - e_7$	$e_3 - e_4 - e_7$	$e_5 - e_6 - e_7$
e_2	$-e_2 - e_6$	$e_2 - e_4 - e_6$	$-2e_6$
e_3	$-e_2 - e_5 + e_7$	$e_1 - e_4 - e_5$	$e_3 - e_5 - e_6$
e_4	$-e_2 - e_4 + e_6$	$-2e_4$	$e_2 - e_4 - e_6$
e_5	$-e_2 - e_3 + e_5$	$-e_3 - e_4 + e_7$	$e_1 - e_3 - e_6$
e_6	$-2e_2 + e_4$	$-e_2 - e_4 + e_6$	$-e_2 - e_6$
e_7	$-e_1 - e_2 + e_3$	$-e_1 - e_4 + e_5$	$-e_1 - e_6 + e_7$

Since $\Delta^2(\mathrm{R}_8)$ is generated by $e_i \cdot e_j$ as a $\mathbb{Z}$-module, using row reduction over $\mathbb{Z}$ one can show that a $\mathbb{Z}$-basis is given by

$$\mathcal{B}_2 = \{u_1 = e_1 - e_2 - e_7, u_2 = e_2 + e_6, u_3 = e_3 - e_4 - e_7, \\ u_4 = e_4 + 2e_6, u_5 = e_5 - e_6 - e_7, u_6 = 4e_6\}.$$

We now want to express a $\mathbb{Z}$-basis of $\Delta^3(\mathrm{R}_8)$ in terms of $\mathcal{B}_2$. First we calculate the products $u_i \cdot e_j$. This is presented in the following table.

	e_1	e_2	e_3
u_1	$2e_1 + e_2 - e_3 + e_6 - e_7$	$e_1 - e_2 + e_3 + e_4 - e_5 + e_6 - e_7$	$e_1 - e_4 + e_5 + 2e_6 - 2e_7$
u_2	$-3e_2 + e_4 - e_6$	$-2e_4$	$-e_2 + e_4 - 3e_6$
u_3	$e_1 + e_2 - e_3 + e_4 - e_5 - e_6 + e_7$	$2e_1 + 2e_4 - 2e_5$	$e_1 - e_2 + e_3 + e_4 - e_5 + e_6 - e_7$
u_4	$-5e_2 - e_4 + e_6$	$-2e_2 - 4e_4 + 2e_6$	$-e_2 - e_4 - 3e_6$
u_5	$e_1 + 2e_2 - 2e_3 - e_4 + e_5$	$e_1 + e_2 - e_3 + e_4 - e_5 - e_6 + e_7$	$2e_1 + e_2 - e_3 + e_6 - e_7$
u_6	$-8e_2 + 4e_4$	$-4e_2 - 4e_4 + 4e_6$	$-4e_2 - 4e_6$

Hence, a $\mathbb{Z}$-basis for $\Delta^3(\mathrm{R}_8)$ is given by

$$\mathcal{B}_3 = \{v_1 = e_1 - e_2 + e_3 + e_4 - e_5 + e_6 - e_7, v_2 = e_2 - e_3 - 2e_4 + 2e_5 + e_6 - e_7, \\ v_3 = -e_3 - e_4 + 2e_5 - 2e_6 - e_7, v_4 = -2e_4, v_5 = -4e_5 - 4e_6 + 4e_7, v_6 = 8e_6\}.$$

Now we will present the elements of $\mathcal{B}_3$ in terms of $\mathcal{B}_2$. We have the following presentation.

$$\begin{array}{rclrrrrrr} v_1 & = & u_1 & & & +2u_4 & -u_5 & -u_6 \\ v_2 & = & & u_2 & -u_3 & -u_4 & +2u_5 & +u_6 \\ v_3 & = & & & -u_3 & -2u_4 & +2u_5 & +u_6 \\ v_4 & = & & & & 2u_4 & & -u_6 \\ v_5 & = & & & & & -4u_5 & \\ v_6 & = & & & & & & 2u_6. \end{array}$$

Note that we can alter the basis $\mathcal{B}_2$ of $\Delta^2(\mathrm{R}_8)$ as follows:

$$\{u_1+2u_4-u_5-u_6, u_2-u_3-u_4+2u_5+u_6, u_3+2u_4-2u_5-u_6, u_4, u_5, u_6\}.$$

Hence,

$$\begin{aligned}\frac{\Delta^2(\mathrm{R}_8)}{\Delta^3(\mathrm{R}_8)} &\cong \frac{\mathbb{Z}v_1\oplus\mathbb{Z}v_2\oplus\mathbb{Z}v_3\oplus\mathbb{Z}u_4\oplus\mathbb{Z}u_5\oplus\mathbb{Z}u_6}{\mathbb{Z}v_1\oplus\mathbb{Z}v_2\oplus\mathbb{Z}v_3\oplus\mathbb{Z}(2u_4-u_6)\oplus\mathbb{Z}(-4u_5)\oplus\mathbb{Z}(2u_6)}\\ &\cong \mathbb{Z}_4\oplus\frac{\mathbb{Z}u_4\oplus\mathbb{Z}u_6}{\mathbb{Z}(2u_4-u_6)\oplus\mathbb{Z}(2u_6)} \cong \mathbb{Z}_4\oplus\frac{\mathbb{Z}u_4\oplus\mathbb{Z}u_6}{\mathbb{Z}u_4\oplus\mathbb{Z}(4u_6)} \cong \mathbb{Z}_4\oplus\mathbb{Z}_4.\end{aligned}$$

□

3. Further Remarks

We have calculated that for $k=2,3,4$ and $n=6,8,10$, the isomorphism

$$\frac{\Delta^k(\mathrm{R}_{2n})}{\Delta^{k+1}(\mathrm{R}_{2n})} \cong \mathbb{Z}_n\oplus\mathbb{Z}_n.$$

holds. Hence, we propose the following improved version of the main conjecture given in [1].

Conjecture 2. *Let R_{2n} denotes the dihedral quandle of order $2n$ for $n \geq 2$. Then for $k>1$,*

$$\frac{\Delta^k(\mathrm{R}_{2n})}{\Delta^{k+1}(\mathrm{R}_{2n})} \cong \mathbb{Z}_n\oplus\mathbb{Z}_n.$$

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