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FRACTIONAL INTEGRAL INEQUALITIES FOR PREINVEX FUNCTIONS VIA ATANGANA-BALEANU INTEGRAL OPERATORS

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Abstract. Different types of functions have been used in many various areas of mathematics until today, and today they are used to obtain new equality and inequalities in fractional analysis, which is one of the important areas of mathematics. In this study, we obtained integral inequalities for preinvex functions with the help of Atangana Baleanu fractional integral operator.

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1. INTRODUCTION

In this section, let us recall well known definitions and concepts as follows.

Firstly, we recall the definition of invex set that has a clear geometric interpretation as follows.

Definition 1 (see [9]). Let $\mu : \mathbb{X} \times \mathbb{X} \neq \varnothing \to \mathbb{R}$ be a real valued function, then $\mathbb{X}$ is said to be invex with respect to $\mu(.,.)$ if $k_1 + t\mu(k_2, k_1) \in \mathbb{X}$, $\forall k_1, k_2 \in \mathbb{X}$ and $t \in [0,1]$.

Note that, every convex set is also invex with respect to $\mu(k_2, k_1) = k_2 - k_1$, but every invex set is not necessarily convex (see [9]).

The concept of function, whose emergence in the world of mathematics dates back to ancient times, is seen as one of the distinguishing features between classical and modern mathematics. The concept of function, which was defined in various ways and developed by mathematicians, was first introduced in the 17th century, when the basic objects of mathematics were taken as geometric curves. Functions are also used in science branches such as physics, biology, engineering, apart from mathematics. There are many types of functions and one of these types of functions is preinvex functions. This interesting class of functions is defined as follows.

Definition 2 (see, e.g, [34]). Let $\mathbb{X} \neq \varnothing \subseteq \mathbb{R}$ be an invex set with respect to $\mu : \mathbb{X} \times \mathbb{X} \neq \varnothing \to \mathbb{R}$. Then the function $f : \mathbb{X} \to \mathbb{R}$ is said to be preinvex with respect to μ if

$$f\big(k_1 + t\mu(k_2, k_1)\big) \leq (1-t) f(k_1) + t f(k_2), \ \forall k_1, k_2 \in \mathbb{X},\ t \in [0,1]. \tag{1.1}$$

We say that f is preincave if $(-f)$ is preinvex. If we take $\mu(k_2, k_1) = k_2 - k_1$ in (1.1), preinvex functions reduce to convex functions in the classical sense. Then, it is obvious that every convex function is a preinvex function but every preinvex function is not convex. For example, the function $f(x) = -|x|$ is not a convex function, but it is a preinvex function with respect to μ, where

$$\mu(k_2, k_1) = \begin{cases} k_2 - k_1, & \text{if } k_1 \leq 0, k_2 \leq 0 \text{ and } k_1 \geq 0, k_2 \geq 0 \\ k_1 - k_2, & \text{otherwise.} \end{cases}$$

Mohan and Neogy [22] introduced a condition defined as follows:

Condition 1. Let $A \subseteq \mathbb{R}$ be an invex set with respect to $\mu : A \times A \to \mathbb{R}$. We say that the function μ satisfies Condition 1, if for any $x, y \in A$ and any $t \in [0,1]$,

$$\begin{aligned} \mu\big(y, y + t\mu(x,y)\big) &= -t\mu(x,y) \\ \mu\big(x, y + t\mu(x,y)\big) &= (1-t)\mu(x,y). \end{aligned} \tag{1.2}$$

Note that for every $x, y \in A$ and every $t_1, t_2 \in [0,1]$ from Condition 1, we have

$$\mu\big(y + t_2\mu(x,y), y + t_1\mu(x,y)\big) = (t_2 - t_1)\mu(x,y). \tag{1.3}$$

We will use the condition in our main results.

Convex functions take place in many areas of mathematics and these functions have attracted the attention of mathematicians working in the field of inequality theory. Therefore, studies in the theory of inequality have provided to the emergence of many new inequalities. One of the most important of these inequalities is the Hermite-Hadamard inequality obtained by Charles Hermite and Jacques Hadamard and many generalizations of this inequality have been obtained. This inequality, which has attracted the attention of many mathematicians is below.

Assume that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function defined on the interval I of $\mathbb{R}$ where $a < b$. The following statement

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.4}$$

holds and known as Hermite-Hadamard inequality. Both inequalities hold in the reversed direction if f is concave.

To provide detail information on convexity, some different type inequalities and more, see the papers [5, 7, 11, 19–21, 30–32, 35, 36].

Many new version of Hermite-Hadamard inequality is proved for different functions and inequalities. One of these proved by Noor [23] for preinvex functions as follows.

Theorem 1. *Let $f : I = [k_1, k_1 + \mu(k_2, k_1)] \to (0, \infty)$ be a preinvex function on the interval of real numbers I° and $k_1, k_2 \in I^\circ$ with $k_1 < k_1 + \mu(k_2, k_1)$. Then the following inequality holds.*

$$f\left(\frac{2k_1 + \mu(k_2, k_1)}{2}\right) \leq \frac{1}{\mu(k_2, k_1)} \int_{k_1}^{k_1+\mu(k_2,k_1)} f(x)dx \leq \frac{f(k_1) + f(k_2)}{2}.$$

Fractional calculus is a generalization of ordinary calculus with more than 300 years of history. Fractional calculus was started in 1965 by Leibniz and L'Hospital as a result of a correspondence which continued several months and the question Leibniz asked to L'Hospital of "Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?". And then, this question attracted the interest of many well known mathematicians. The concepts of fractional calculus were examined further over the course of the 18th and 19th centuries. In today's world, the applications of fractional calculus are very wide at fields as viscoelasticity, rheology, acoustics, optics, chemical and control theory, statistical physics, robotics, electrical and mechanical engineering, bio engineering, etc. Many mathematicians with the development of fractional calculus have defined many fractional derivative and integral operators to find solutions to real-world problems. Some of them are as follows.

Firstly, we recall the Caputo-Fabrizio fractional integral operators.

Definition 3 ([3]). Let $f \in H^1(0, b)$, $b > a$, $\alpha \in [0, 1]$ then, the definition of the left and right side of Caputo-Fabrizio fractional integral is:

$$\left({}_{a}^{CF}I^{\alpha}\right)(t) = \frac{1-\alpha}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)} \int_a^t f(y)dy,$$

and

$$\left({}^{CF}I_b^{\alpha}\right)(t) = \frac{1-\alpha}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)} \int_t^b f(y)dy,$$

where $M(\alpha)$ is normalization function.

In the sequel of the paper, we will denote normalization function as $B(\alpha)$ with $B(0) = B(1) = 1$.

Muhammad Tariq et.al [33] provided an integral inequality of Hermite-Hadamard type for preinvex functions via Caputo-Fabrizio fractional integral inequality as follows.

Theorem 2. *Let $f : I = [k_1, k_1 + \mu(k_2, k_1)] \to (0, \infty)$ be a preinvex function on I° and $f \in L[k_1, k_1 + \mu(k_2, k_1)]$. If $\alpha \in [0, 1]$, then the following inequality holds*

$$f\left(\frac{2k_1 + \mu(k_2, k_1)}{2}\right) \leq \frac{B(\alpha}{\alpha\mu(k_2, k_1)} \Big[{}_{k_1}^{CF}I^{\alpha}\{f(k)\} + {}^{CF}I^{\alpha}_{k_1+\mu(k_2,k_1)}\{f(k)\} - \frac{2(1-\alpha)}{B(\alpha)} f(k) \Big]$$

$$\leq \frac{f(k_1)+f(k_2)}{2},$$

where $k \in [k_1, k_1+\mu(k_2,k_1)]$.

Atangana and Baleanu produced a new derivative operators using Mittag-Leffler function in Caputo-Fabrizio derivative operator as follows.

Definition 4 ([10])**.** Let $f \in H^1(a,b)$, $b > a$, $\alpha \in [0,1)$ then, the definition of the new fractional derivative is given:

$$ {}^{ABC}_{\ a}D_t^{\alpha}[f(t)] = \frac{B(\alpha)}{1-\alpha}\int_a^t f'(x)E_{\alpha}\left[-\alpha\frac{(t-x)^{\alpha}}{(1-\alpha)}\right]dx. \tag{1.5}$$

Definition 5 ([10])**.** Let $f \in H^1(a,b)$, $b > a$, $\alpha \in [0,1)$. Then the definition of the new fractional derivative is given by

$$ {}^{ABR}_{\ a}D_t^{\alpha}[f(t)] = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t f(x)E_{\alpha}\left[-\alpha\frac{(t-x)^{\alpha}}{(1-\alpha)}\right]dx. \tag{1.6}$$

Equations (1.5) and (1.6) have a non-local kernel. Also in equation (1.5), when the function is constant, we get zero.

The related fractional integral operator has been defined by Atangana-Baleanu as follows.

Definition 6 ([10])**.** The fractional integral associate to the new fractional derivative with non-local kernel of a function $f \in H^1(a,b)$ as defined:

$$ {}^{AB}_{\ a}I^{\alpha}\{f(t)\} = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t f(y)(t-y)^{\alpha-1}dy,$$

where $b > a, \alpha \in (0,1]$.

In [2], Abdeljawad and Baleanu introduced the right hand side of integral operator as follows. The right fractional new integral with ML kernel of order $\alpha \in (0,1]$ is defined by

$$ {}^{AB}I_b^{\alpha}\{f(t)\} = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_t^b f(y)(y-t)^{\alpha-1}dy.$$

For more information related to different kinds of fractional operators, we recommend the following papers to the readers [1, 3, 4, 6, 12–15, 24–29].

The purpose of this paper is to provide some new inequalities for preinvex functions that includes the Atangana-Baleanu integral operators. Some special cases are also considered.

2. Hermite-Hadamard type inequalities for preinvex functions via Atangana-Baleanu fractional integral operators

Theorem 3. *Let $I \subseteq \mathbb{R}$ be an open invex subset with respect to $\mu : I \times I \neq \varnothing \to \mathbb{R}$ and $k_1, k_2 \in I$ with $k_1 < k_1 + \mu(k_2, k_1)$. If $f : [k_1, k_1 + \mu(k_2, k_1)] \to \mathbb{R}$ is a preinvex function, $f \in L[k_1, k_1 + \mu(k_2, k_1)]$ and μ satisfies Condition 1, the following inequalities for Atangana-Baleanu fractional integral operators hold*

$$\begin{aligned} f\left(\frac{2k_1+\mu(k_2,k_1)}{2}\right) &\leq \frac{B(\alpha)\Gamma(\alpha)}{2[\mu(k_2,k_1)]^\alpha}\left[{}^{AB}_{k_1}I^\alpha\{f(k_1+\mu(k_2,k_1))\} + {}^{AB}I^\alpha_{k_1+\mu(k_2,k_1)}\{f(k_1)\}\right] \\ &\quad - \frac{(1-\alpha)\Gamma(\alpha)}{2[\mu(k_2,k_1)]^\alpha}\left[f(k_1)+f(k_1+\mu(k_2,k_1))\right] \\ &\leq \frac{f(k_1)+f(k_2)}{2} \end{aligned}$$

where $\alpha \in (0,1]$, $B(\alpha) > 0$ is normalization function and $\Gamma(.)$ is the Gamma function.

Proof. Since f is preinvex function on $[k_1, k_1+\mu(k_2,k_1)]$, we can write (see, e.g., [17, 18]):

$$2f\left(\frac{2k_1+\mu(k_2,k_1)}{2}\right) \leq f(k_1+t\mu(k_2,k_1)) + f(k_1+(1-t)\mu(k_2,k_1)) \tag{2.1}$$

Multiplying both sides of (2.1) by $\frac{\alpha}{B(\alpha)\Gamma(\alpha)}t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned} &\frac{2}{B(\alpha)\Gamma(\alpha)} f\left(\frac{2k_1+\mu(k_2,k_1)}{2}\right) \\ &\leq \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^1 t^{\alpha-1} f(k_1+t\mu(k_2,k_1))dt \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^1 t^{\alpha-1} f(k_1+(1-t)\mu(k_2,k_1))dt \\ &= \frac{\alpha}{B(\alpha)\Gamma(\alpha)[\mu(k_2,k_1)]^\alpha}\int_{k_1}^{k_1+\mu(k_2,k_1)} (x-k_1)^{\alpha-1} f(x)dx \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)[\mu(k_2,k_1)]^\alpha}\int_{k_1}^{k_1+\mu(k_2,k_1)} (k_1+\mu(k_2,k_1)-y)^{\alpha-1} f(y)dy. \end{aligned}$$

Then we can write

$$\begin{aligned} &\frac{2}{B(\alpha)\Gamma(\alpha)} f\left(\frac{2k_1+\mu(k_2,k_1)}{2}\right) \\ &\leq \frac{1}{[\mu(k_2,k_1)]^\alpha}\left[\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{k_1}^{k_1+\mu(k_2,k_1)} (x-k_1)^{\alpha-1} f(x)dx + \frac{(1-\alpha)}{B(\alpha)}f(k_1)\right] \\ &\quad - \frac{(1-\alpha)}{B(\alpha)[\mu(k_2,k_1)]^\alpha} f(k_1) \end{aligned}$$

$$+\frac{1}{[\mu(k_2,k_1)]^\alpha}\left[\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{k_1}^{k_1+\mu(k_2,k_1)}\left(k_1+\mu(k_2,k_1)-y\right)^{\alpha-1}f(y)dy\right.$$
$$\left.+\frac{(1-\alpha)}{B(\alpha)}f\left(k_1+\mu(k_2,k_1)\right)\right]-\frac{(1-\alpha)}{B(\alpha)\,[\mu(k_2,k_1)]^\alpha}f\left(k_1+\mu(k_2,k_1)\right).$$

So, using Atangana-Baleanu fractional integral operators, we get

$$\frac{2}{B(\alpha)\Gamma(\alpha)}f\left(\frac{2k_1+\mu(k_2,k_1)}{2}\right)$$
$$\leq\frac{1}{[\mu(k_2,k_1)]^\alpha}\left[{}^{AB}_{k_1}I^{\alpha}\left\{f\left(k_1+\mu(k_2,k_1)\right)\right\}+{}^{AB}I^{\alpha}_{k_1+\mu(k_2,k_1)}\left\{f(k_1)\right\}\right]$$
$$-\frac{(1-\alpha)}{B(\alpha)\,[\mu(k_2,k_1)]^\alpha}\left[f(k_1)+f\left(k_1+\mu(k_2,k_1)\right)\right].$$

and the first inequality is proved.

For the proof of the second inequality in (2.1), we first note that if f is a preinvex function, then we can write

$$f\left(k_1+t\mu(k_2,k_1)\right)\leq(1-t)f(k_1)+tf(k_2)$$

and

$$f\left(k_1+(1-t)\mu(k_2,k_1)\right)\leq tf(k_1)+(1-t)f(k_2).$$

By adding these inequalities side by side, we have

$$f\left(k_1+t\mu(k_2,k_1)\right)+f\left(k_1+(1-t)\mu(k_2,k_1)\right)\leq f(k_1)+f(k_2). \tag{2.2}$$

Then, multiplying both sides of (2.2) by $\frac{\alpha}{B(\alpha)\Gamma(\alpha)}t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^1 t^{\alpha-1}f\left(k_1+t\mu(k_2,k_1)\right)dt+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^1 t^{\alpha-1}f\left(k_1+(1-t)\mu(k_2,k_1)\right)dt$$
$$\leq\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\left[f(k_1)+f(k_2)\right]\int_0^1 t^{\alpha-1}dt.$$

Then, we can write

$$\frac{1}{[\mu(k_2,k_1)]^\alpha}\left[{}^{AB}_{k_1}I^{\alpha}\left\{f\left(k_1+\mu(k_2,k_1)\right)\right\}+{}^{AB}I^{\alpha}_{k_1+\mu(k_2,k_1)}\left\{f(k_1)\right\}\right]$$
$$-\frac{(1-\alpha)}{B(\alpha)\,[\mu(k_2,k_1)]^\alpha}\left[f(k_1)+f\left(k_1+\mu(k_2,k_1)\right)\right]$$
$$\leq\frac{f(k_1)+f(k_2)}{B(\alpha)\Gamma(\alpha)}.$$

So, the proof of this theorem is completed. □

Remark 1. Setting $\mu(k_2,k_1) = k_2 - k_1$ in Theorem 3 gives the same result as in [16], Proposition 2.1, inequality (13).

Remark 2. Setting $\alpha = 1$ in Theorem 3 gives the same result as in Theorem 1.

Theorem 4. *Let $I \subseteq \mathbb{R}$ be an open invex subset with respect to $\mu : I \times I \neq \varnothing \to \mathbb{R}$ and $k_1, k_2 \in I$ with $k_1 < k_1 + \mu(k_2,k_1)$. If $f, g : [k_1, k_1 + \mu(k_2,k_1)] \to \mathbb{R}$ is preinvex functions, $f, g \in L[k_1, k_1 + \mu(k_2,k_1)]$, then the following inequalities for Atangana-Baleanu fractional integral operators hold:*

$$
\begin{aligned}
&\frac{1}{[\mu(k_2,k_1)]^{\alpha}} \left[{}^{AB}_{k_1}I^{\alpha} \left\{ fg\big(k_1 + \mu(k_2,k_1)\big) \right\} + {}^{AB}I^{\alpha}_{k_1+\mu(k_2,k_1)} \left\{ fg(k_1) \right\} \right] \\
&\leq \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \left[[f(k_1)g(k_1) + f(k_2)g(k_2)] \left(\frac{2}{\alpha(\alpha+1)(\alpha+2)} + \frac{1}{\alpha+2} \right) \right. \\
&\quad \left. + 2\frac{[f(k_1)g(k_2) + f(k_2)g(k_1)]}{(\alpha+1)(\alpha+2)} \right] \\
&\quad + \frac{(1-\alpha)}{B(\alpha)\,[\mu(k_2,k_1)]^{\alpha}} \left[f(k_1)g(k_1) + f\big(k_1 + \mu(k_2,k_1)\big) g\big(k_1 + \mu(k_2,k_1)\big) \right],
\end{aligned}
$$

where $\alpha \in (0,1]$, $B(\alpha) > 0$ is a normalization function and $\Gamma(.)$ is the Gamma function.

Proof. Since f and g are preinvex functions on $[k_1, k_1 + \mu(k_2,k_1)]$, we get

$$f\big(k_1 + t\mu(k_2,k_1)\big) \leq (1-t)f(k_1) + tf(k_2)$$

and

$$g\big(k_1 + t\mu(k_2,k_1)\big) \leq (1-t)g(k_1) + tg(k_2).$$

By multiplying both inequalities side by side, we get

$$
\begin{aligned}
&f\big(k_1 + t\mu(k_2,k_1)\big) g\big(k_1 + t\mu(k_2,k_1)\big) \\
&\leq (1-t)^2 f(k_1)g(k_1) + t^2 f(k_2)g(k_2) + t(1-t)\,[f(k_1)g(k_2) + f(k_2)g(k_1)]\,.
\end{aligned} \tag{2.3}
$$

By multiplying both sides of (2.3) with $(1-t)^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$
\begin{aligned}
&\int_0^1 (1-t)^{\alpha-1} f\big(k_1 + t\mu(k_2,k_1)\big) g\big(k_1 + t\mu(k_2,k_1)\big) dt \\
&\leq \int_0^1 (1-t)^{\alpha-1} \Big[(1-t)^2 f(k_1)g(k_1) + t^2 f(k_2)g(k_2) + t(1-t)\,[f(k_1)g(k_2) + f(k_2)g(k_1)] \Big] dt \\
&= \frac{f(k_1)g(k_1)}{\alpha+2} + 2\frac{f(k_2)g(k_2)}{\alpha(\alpha+1)(\alpha+2)} + \frac{[f(k_1)g(k_2) + f(k_2)g(k_1)]}{(\alpha+1)(\alpha+2)}.
\end{aligned} \tag{2.4}
$$

By changing the variable $k_1+t\mu(k_2,k_1)=x$, we can write the inequality in (2.4) as

$$\begin{aligned}&\frac{1}{[\mu(k_2,k_1)]^{\alpha}}\int_{k_1}^{k_1+\mu(k_2,k_1)}\left(k_1+\mu(k_2,k_1)-x\right)^{\alpha-1}f(x)g(x)dx \\ &\leq \frac{f(k_1)g(k_1)}{\alpha+2}+2\frac{f(k_2)g(k_2)}{\alpha(\alpha+1)(\alpha+2)}+\frac{[f(k_1)g(k_2)+f(k_2)g(k_1)]}{(\alpha+1)(\alpha+2)}.\end{aligned} \tag{2.5}$$

Multiplying both sides of (2.5) by $\frac{\alpha}{B(\alpha)\Gamma(\alpha)}$ and then adding the term

$$\frac{(1-\alpha)}{B(\alpha)\,[\mu(k_2,k_1)]^{\alpha}}f\left(k_1+\mu(k_2,k_1)\right)g\left(k_1+\mu(k_2,k_1)\right)$$

to both sides of (2.5) and finally using Atangana-Baleanu fractional integral operators, we get

$$\begin{aligned}&\frac{1}{[\mu(k_2,k_1)]^{\alpha}}\left[{}^{AB}_{k_1}I^{\alpha}\left\{fg\left(k_1+\mu(k_2,k_1)\right)\right\}\right] \\ &\leq \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\left[\frac{f(k_1)g(k_1)}{\alpha+2}+2\frac{f(k_2)g(k_2)}{\alpha(\alpha+1)(\alpha+2)}+\frac{[f(k_1)g(k_2)+f(k_2)g(k_1)]}{(\alpha+1)(\alpha+2)}\right] \\ &\quad+\frac{(1-\alpha)}{B(\alpha)\,[\mu(k_2,k_1)]^{\alpha}}f\left(k_1+\mu(k_2,k_1)\right)g\left(k_1+\mu(k_2,k_1)\right).\end{aligned} \tag{2.6}$$

Similarly, by multiplying both sides of (2.3) with $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1 t^{\alpha-1}f\left(k_1+t\mu(k_2,k_1)\right)g\left(k_1+t\mu(k_2,k_1)\right)dt \\ &\leq \int_0^1 t^{\alpha-1}\left[(1-t)^2 f(k_1)g(k_1)+t^2 f(k_2)g(k_2)+t(1-t)\,[f(k_1)g(k_2)+f(k_2)g(k_1)]\right]dt \\ &=2\frac{f(k_1)g(k_1)}{\alpha(\alpha+1)(\alpha+2)}+\frac{f(k_2)g(k_2)}{\alpha+2}+\frac{[f(k_1)g(k_2)+f(k_2)g(k_1)]}{(\alpha+1)(\alpha+2)}.\end{aligned}$$

By making similar calculations to those in the proof of (2.6), we obtain

$$\begin{aligned}&\frac{1}{[\mu(k_2,k_1)]^{\alpha}}\left[{}^{AB}I^{\alpha}_{k_1+\mu(k_2,k_1)}\left\{fg\left(k_1\right)\right\}\right] \\ &\leq \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\left[2\frac{f(k_1)g(k_1)}{\alpha(\alpha+1)(\alpha+2)}+2\frac{f(k_2)g(k_2)}{\alpha+2}+\frac{[f(k_1)g(k_2)+f(k_2)g(k_1)]}{(\alpha+1)(\alpha+2)}\right] \\ &\quad+\frac{(1-\alpha)}{B(\alpha)\,[\mu(k_2,k_1)]^{\alpha}}f(k_1)g(k_1).\end{aligned} \tag{2.7}$$

Adding (2.6) and (2.7) side by side, we get

$$\frac{1}{[\mu(k_2,k_1)]^{\alpha}}\left[{}^{AB}_{k_1}I^{\alpha}\left\{fg\left(k_1+\mu(k_2,k_1)\right)\right\}+{}^{AB}I^{\alpha}_{k_1+\mu(k_2,k_1)}\left\{fg\left(k_1\right)\right\}\right]$$

$$\leq \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\left[[f(k_1)g(k_1)+f(k_2)g(k_2)]\left(\frac{2}{\alpha(\alpha+1)(\alpha+2)}+\frac{1}{\alpha+2}\right)\right.$$
$$\left.+2\frac{[f(k_1)g(k_2)+f(k_2)g(k_1)]}{(\alpha+1)(\alpha+2)}\right]$$
$$+\frac{(1-\alpha)}{B(\alpha)\left[\mu(k_2,k_1)\right]^{\alpha}}\left[f(k_1)g(k_1)+f\big(k_1+\mu(k_2,k_1)\big)g\big(k_1+\mu(k_2,k_1)\big)\right]$$

and the proof is completed. □

3. Some further results for preinvex functions via Atangana-Baleanu fractional integral operators

Lemma 1 ([8]). *Let $I\subseteq\mathbb{R}$ be an open invex subset with respect to $\mu: I\times I\neq\varnothing\to\mathbb{R}$ and $k_1,k_2\in I$ with $k_1<k_1+\mu(k_2,k_1)$. Suppose that $f: I\to\mathbb{R}$ be a differentiable function. If $f'\in L[k_1,k_1+\mu(k_2,k_1)]$, the following identity for Atangana-Baleanu fractional integral operators holds:*

$$\frac{B(\alpha)\Gamma(\alpha)}{\left[\mu(k_2,k_1)\right]^{\alpha+1}}\left[{}^{AB}_{k_1}I^{\alpha}\left\{f\big(k_1+\mu(k_2,k_1)\big)\right\}+{}^{AB}I^{\alpha}_{k_1+\mu(k_2,k_1)}\left\{f(k_1)\right\}\right] \tag{3.1}$$
$$-\left(\frac{\left[\mu(k_2,k_1)\right]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{\left[\mu(k_2,k_1)\right]^{\alpha+1}}\right)\left[f(k_1)+f\big(k_1+\mu(k_2,k_1)\big)\right]$$
$$=\int_0^1(1-t)^{\alpha}f'\big(k_1+t\mu(k_2,k_1)\big)dt-\int_0^1 t^{\alpha}f'\big(k_1+t\mu(k_2,k_1)\big)dt,$$

where $\alpha\in(0,1], t\in[0,1]$, $B(\alpha)$ is a normalization function and $\Gamma(.)$ is the Gamma function.

Proof. By using integration, we have

$$\int_0^1(1-t)^{\alpha}f'\big(k_1+t\mu(k_2,k_1)\big)dt \tag{3.2}$$
$$=\frac{(1-t)^{\alpha}f\big(k_1+t\mu(k_2,k_1)\big)}{\mu(k_2,k_1)}\Bigg|_0^1+\frac{\alpha}{\mu(k_2,k_1)}\int_0^1 f\big(k_1+t\mu(k_2,k_1)\big)(1-t)^{\alpha-1}dt$$
$$=-\frac{f(k_1)}{\mu(k_2,k_1)}+\frac{\alpha}{\mu(k_2,k_1)}\int_0^1(1-t)^{\alpha-1}f\big(k_1+t\mu(k_2,k_1)\big)dt$$
$$=-\frac{f(k_1)}{\mu(k_2,k_1)}+\frac{\alpha}{\left[\mu(k_2,k_1)\right]^{\alpha+1}}\int_{k_1}^{k_1+\mu(k_2,k_1)}\big(k_1+\mu(k_2,k_1)-x\big)^{\alpha-1}f(x)dx.$$

If we multiply both sides of (3.2) by $\frac{1}{B(\alpha)\Gamma(\alpha)}$, we get

$$\frac{1}{B(\alpha)\Gamma(\alpha)}\int_0^1(1-t)^{\alpha}f'\big(k_1+t\mu(k_2,k_1)\big)dt$$

$$= -\frac{f(k_1)}{B(\alpha)\Gamma(\alpha)\mu(k_2,k_1)} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)\left[\mu(k_2,k_1)\right]^{\alpha+1}} \int_{k_1}^{k_1+\mu(k_2,k_1)} \left(k_1+\mu(k_2,k_1)-x\right)^{\alpha-1} f(x)dx.$$

Then, we can write

$$\frac{1}{B(\alpha)\Gamma(\alpha)} \int_0^1 (1-t)^\alpha f'\left(k_1+t\mu(k_2,k_1)\right)dt$$
$$= -\frac{f(k_1)}{B(\alpha)\Gamma(\alpha)\mu(k_2,k_1)} + \frac{1}{\left[\mu(k_2,k_1)\right]^{\alpha+1}} \left[\frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{k_1}^{k_1+\mu(k_2,k_1)} \left(k_1+\mu(k_2,k_1)-x\right)^{\alpha-1} f(x)dx + \frac{(1-\alpha)}{B(\alpha)} f\left(k_1+\mu(k_2,k_1)\right)\right] - \frac{(1-\alpha)}{B(\alpha)\left[\mu(k_2,k_1)\right]^{\alpha+1}} f\left(k_1+\mu(k_2,k_1)\right).$$

Using Atangana-Baleanu fractional integral operators, we have

$$\frac{1}{B(\alpha)\Gamma(\alpha)} \int_0^1 (1-t)^\alpha f'\left(k_1+t\mu(k_2,k_1)\right)dt \tag{3.3}$$
$$= -\frac{f(k_1)}{B(\alpha)\Gamma(\alpha)\mu(k_2,k_1)} + \frac{1}{\left[\mu(k_2,k_1)\right]^{\alpha+1}} \left[{}_{k_1}^{AB}I^\alpha \left\{ f\left(k_1+\mu(k_2,k_1)\right)\right\}\right] - \frac{(1-\alpha)}{B(\alpha)\left[\mu(k_2,k_1)\right]^{\alpha+1}} f\left(k_1+\mu(k_2,k_1)\right).$$

Similarly, using integration, we get

$$\int_0^1 t^\alpha f'\left(k_1+t\mu(k_2,k_1)\right)dt \tag{3.4}$$
$$= \frac{t^\alpha f\left(k_1+t\mu(k_2,k_1)\right)}{\mu(k_2,k_1)}\Bigg|_0^1 - \frac{\alpha}{\mu(k_2,k_1)} \int_0^1 f\left(k_1+t\mu(k_2,k_1)\right)t^{\alpha-1}dt$$
$$= \frac{f\left(k_1+\mu(k_2,k_1)\right)}{\mu(k_2,k_1)} - \frac{\alpha}{\mu(k_2,k_1)} \int_0^1 t^{\alpha-1} f\left(k_1+t\mu(k_2,k_1)\right)dt$$
$$= \frac{f\left(k_1+\mu(k_2,k_1)\right)}{\mu(k_2,k_1)} - \frac{\alpha}{\left[\mu(k_2,k_1)\right]^{\alpha+1}} \int_{k_1}^{k_1+\mu(k_2,k_1)} (u-k_1)^{\alpha-1} f(u)du.$$

If we multiply both sides of (3.4) by $-\frac{1}{B(\alpha)\Gamma(\alpha)}$, we have

$$-\frac{1}{B(\alpha)\Gamma(\alpha)} \int_0^1 t^\alpha f'\left(k_1+t\mu(k_2,k_1)\right)dt$$

$$= -\frac{f\left(k_1+\mu(k_2,k_1)\right)}{B(\alpha)\Gamma(\alpha)\mu(k_2,k_1)} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)\left[\mu(k_2,k_1)\right]^{\alpha+1}} \int_{k_1}^{k_1+\mu(k_2,k_1)} (u-k_1)^{\alpha-1} f(u)du.$$

Then we can write

$$\begin{aligned}
&-\frac{1}{B(\alpha)\Gamma(\alpha)} \int_0^1 t^{\alpha} f'\left(k_1+t\mu(k_2,k_1)\right)dt \\
&= -\frac{f\left(k_1+\mu(k_2,k_1)\right)}{B(\alpha)\Gamma(\alpha)\mu(k_2,k_1)} + \frac{1}{\left[\mu(k_2,k_1)\right]^{\alpha+1}} \left[\frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{k_1}^{k_1+\mu(k_2,k_1)} (u-k_1)^{\alpha-1} f(u)du \right. \\
&\quad \left. + \frac{(1-\alpha)}{B(\alpha)} f(k_1)\right] - \frac{(1-\alpha)}{B(\alpha)\left[\mu(k_2,k_1)\right]^{\alpha+1}} f(k_1).
\end{aligned}$$

Using Atangana-Baleanu fractional integral operators, we have

$$\begin{aligned}
&-\frac{1}{B(\alpha)\Gamma(\alpha)} \int_0^1 t^{\alpha} f'\left(k_1+t\mu(k_2,k_1)\right)dt \\
&= -\frac{f\left(k_1+\mu(k_2,k_1)\right)}{B(\alpha)\Gamma(\alpha)\mu(k_2,k_1)} + \frac{1}{\left[\mu(k_2,k_1)\right]^{\alpha+1}} \left[{}^{AB}I^{\alpha}_{k_1+\mu(k_2,k_1)} \{f(k_1)\}\right] \\
&\quad - \frac{(1-\alpha)}{B(\alpha)\left[\mu(k_2,k_1)\right]^{\alpha+1}} f(k_1).
\end{aligned} \tag{3.5}$$

By adding identities (3.3) and (3.5), we obtain desired result. So, the proof is completed. □

Remark 3. Setting $\mu(k_2,k_1) = k_2 - k_1$ in Lemma 1 gives the same result as in [16], Theorem 3.1, equality (29).

Theorem 5. *Let $I \subseteq \mathbb{R}$ be an open invex subset with respect to $\mu : I \times I \neq \varnothing \to \mathbb{R}$ and $k_1, k_2 \in I$ with $k_1 < k_1 + \mu(k_2,k_1)$. Suppose that $f : I \to \mathbb{R}$ be a differentiable function and $f' \in L[k_1, k_1+\mu(k_2,k_1)]$. If $|f'|$ is a preinvex function, we have the following inequality for Atangana-Baleanu fractional integral operators:*

$$\begin{aligned}
&\left| \frac{B(\alpha)\Gamma(\alpha)}{\left[\mu(k_2,k_1)\right]^{\alpha+1}} \left[{}^{AB}_{k_1}I^{\alpha} \left\{f\left(k_1+\mu(k_2,k_1)\right)\right\} + {}^{AB}I^{\alpha}_{k_1+\mu(k_2,k_1)} \{f(k_1)\} \right] \right. \\
&\quad \left. - \left(\frac{\left[\mu(k_2,k_1)\right]^{\alpha} + (1-\alpha)\Gamma(\alpha)}{\left[\mu(k_2,k_1)\right]^{\alpha+1}}\right) \left[f(k_1) + f\left(k_1+\mu(k_2,k_1)\right)\right] \right| \\
&\le \frac{|f'(k_1)| + |f'(k_2)|}{\alpha+1},
\end{aligned}$$

where $\alpha \in (0,1]$, $B(\alpha)$ is a normalization function and $\Gamma(.)$ is the Gamma function.

Proof. By using the identity that is given in Lemma 1 and properties of modulus, we can write

$$\begin{aligned}&\left|\frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}}\left[{}_{k_1}^{AB}I^{\alpha}\left\{f\left(k_1+\mu(k_2,k_1)\right)\right\}+{}^{AB}I^{\alpha}_{k_1+\mu(k_2,k_1)}\left\{f(k_1)\right\}\right]\right.\\&\left.\quad-\left(\frac{[\mu(k_2,k_1)]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}}\right)\left[f(k_1)+f\left(k_1+\mu(k_2,k_1)\right)\right]\right|\\&=\left|\int_0^1(1-t)^{\alpha}f'\left(k_1+t\mu(k_2,k_1)\right)dt-\int_0^1 t^{\alpha}f'\left(k_1+t\mu(k_2,k_1)\right)dt\right|\\&\leq\int_0^1(1-t)^{\alpha}\left|f'\left(k_1+t\mu(k_2,k_1)\right)\right|dt+\int_0^1 t^{\alpha}\left|f'\left(k_1+t\mu(k_2,k_1)\right)\right|dt.\end{aligned}$$

Since $|f'|$ is a preinvex function, we obtain

$$\begin{aligned}&\left|\frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}}\left[{}_{k_1}^{AB}I^{\alpha}\left\{f\left(k_1+\mu(k_2,k_1)\right)\right\}+{}^{AB}I^{\alpha}_{k_1+\mu(k_2,k_1)}\left\{f(k_1)\right\}\right]\right.\\&\left.\quad-\left(\frac{[\mu(k_2,k_1)]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}}\right)\left[f(k_1)+f\left(k_1+\mu(k_2,k_1)\right)\right]\right|\\&\leq\int_0^1(1-t)^{\alpha}\left[(1-t)\left|f'(k_1)\right|+t\left|f'(k_2)\right|\right]dt+\int_0^1 t^{\alpha}\left[(1-t)\left|f'(k_1)\right|+t\left|f'(k_2)\right|\right]dt\\&=\frac{|f'(k_1)|+|f'(k_2)|}{\alpha+1}.\end{aligned}$$

So, the proof is completed. □

Corollary 1. *In Theorem 5, if we choose $\mu(k_2,k_1)=k_2-k_1$ we obtain*

$$\begin{aligned}&\left|\frac{B(\alpha)\Gamma(\alpha)}{(k_2-k_1)^{\alpha+1}}\left[{}_{k_1}^{AB}I^{\alpha}\left\{f(k_2)\right\}+{}^{AB}I^{\alpha}_{k_2}\left\{f(k_1)\right\}\right]\right.\\&\left.\quad-\left(\frac{(k_2-k_1)^{\alpha}+(1-\alpha)\Gamma(\alpha)}{(k_2-k_1)^{\alpha+1}}\right)\left[f(k_1)+f(k_2)\right]\right|\\&\leq\frac{|f'(k_1)|+|f'(k_2)|}{\alpha+1}.\end{aligned}$$

Theorem 6. *Let $I\subseteq\mathbb{R}$ be an open invex subset with respect to $\mu:I\times I\neq\varnothing\to\mathbb{R}$ and $k_1,k_2\in I$ with $k_1<k_1+\mu(k_2,k_1)$. Suppose that $f:I\to\mathbb{R}$ be a differentiable function and $f'\in L[k_1,k_1+\mu(k_2,k_1)]$. If $|f'|^q$ is a preinvex function, we have the following inequality for Atangana-Baleanu fractional integral operators:*

$$\left|\frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}}\left[{}_{k_1}^{AB}I^{\alpha}\left\{f\left(k_1+\mu(k_2,k_1)\right)\right\}+{}^{AB}I^{\alpha}_{k_1+\mu(k_2,k_1)}\left\{f(k_1)\right\}\right]\right.$$

$$-\left(\frac{[\mu(k_2,k_1)]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}}\right)\left[f(k_1)+f\big(k_1+\mu(k_2,k_1)\big)\right]\Bigg|$$
$$\leq 2\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(k_1)|^q+|f'(k_2)|^q}{2}\right)^{\frac{1}{q}},$$

where $p^{-1}+q^{-1}=1$, $q>1$, $\alpha\in(0,1]$, $B(\alpha)$ *is a normalization function and* $\Gamma(.)$ *is the Gamma function.*

Proof. By using Lemma 1, we get

$$\Bigg|\frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}}\left[{}^{AB}_{k_1}I^{\alpha}\left\{f\big(k_1+\mu(k_2,k_1)\big)\right\}+{}^{AB}I^{\alpha}_{k_1+\mu(k_2,k_1)}\left\{f(k_1)\right\}\right]$$
$$-\left(\frac{[\mu(k_2,k_1)]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}}\right)\left[f(k_1)+f\big(k_1+\mu(k_2,k_1)\big)\right]\Bigg|$$
$$\leq\int_0^1(1-t)^{\alpha}\left|f'\big(k_1+t\mu(k_2,k_1)\big)\right|dt+\int_0^1 t^{\alpha}\left|f'\big(k_1+t\mu(k_2,k_1)\big)\right|dt.$$

By applying Hölder inequality, we get

$$\Bigg|\frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}}\left[{}^{AB}_{k_1}I^{\alpha}\left\{f\big(k_1+\mu(k_2,k_1)\big)\right\}+{}^{AB}I^{\alpha}_{k_1+\mu(k_2,k_1)}\left\{f(k_1)\right\}\right]$$
$$-\left(\frac{[\mu(k_2,k_1)]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}}\right)\left[f(k_1)+f\big(k_1+\mu(k_2,k_1)\big)\right]\Bigg|$$
$$\leq\left(\int_0^1(1-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\big(k_1+t\mu(k_2,k_1)\big)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\big(k_1+t\mu(k_2,k_1)\big)\right|^q dt\right)^{\frac{1}{q}}.$$

By using the preinvexity of $|f'|^q$, we obtain

$$\Bigg|\frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}}\left[{}^{AB}_{k_1}I^{\alpha}\left\{f\big(k_1+\mu(k_2,k_1)\big)\right\}+{}^{AB}I^{\alpha}_{k_1+\mu(k_2,k_1)}\left\{f(k_1)\right\}\right]$$
$$-\left(\frac{[\mu(k_2,k_1)]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}}\right)\left[f(k_1)+f\big(k_1+\mu(k_2,k_1)\big)\right]\Bigg|$$
$$\leq\left(\int_0^1(1-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left[(1-t)\left|f'(k_1)\right|^q+t\left|f'(k_2)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left[(1-t)\left|f'(k_1)\right|^q+t\left|f'(k_2)\right|^q\right]dt\right)^{\frac{1}{q}}.$$

By calculating the integrals given in the above inequality, we get the desired result. □

Corollary 2. *In Theorem 6, if we choose* $\mu(k_2,k_1) = k_2 - k_1$ *we obtain*

$$\left| \frac{B(\alpha)\Gamma(\alpha)}{(k_2-k_1)^{\alpha+1}} \left[{}^{AB}_{k_1}I^{\alpha}\{f(k_2)\} + {}^{AB}I^{\alpha}_{k_2}\{f(k_1)\} \right] - \left(\frac{(k_2-k_1)^{\alpha} + (1-\alpha)\Gamma(\alpha)}{(k_2-k_1)^{\alpha+1}} \right) [f(k_1)+f(k_2)] \right| \leq 2\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \left(\frac{|f'(k_1)|^q + |f'(k_2)|^q}{2} \right)^{\frac{1}{q}}.$$

Theorem 7. *Let* $I \subseteq \mathbb{R}$ *be an open invex subset with respect to* $\mu : I \times I \neq \varnothing \to \mathbb{R}$ *and* $k_1, k_2 \in I$ *with* $k_1 < k_1 + \mu(k_2,k_1)$*. Suppose that* $f : I \to \mathbb{R}$ *be a differentiable function and* $f' \in L[k_1, k_1+\mu(k_2,k_1)]$*. If* $|f'|^q$ *is a preinvex function, we have the following inequality for Atangana-Baleanu fractional integral operators:*

$$\left| \frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}} \left[{}^{AB}_{k_1}I^{\alpha}\{f(k_1+\mu(k_2,k_1))\} + {}^{AB}I^{\alpha}_{k_1+\mu(k_2,k_1)}\{f(k_1)\} \right] - \left(\frac{[\mu(k_2,k_1)]^{\alpha} + (1-\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}} \right) [f(k_1)+f(k_1+\mu(k_2,k_1))] \right|$$
$$\leq \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \left[\left(\frac{|f'(k_1)|^q}{\alpha+2} + \frac{|f'(k_2)|^q}{(\alpha+1)(\alpha+2)} \right)^{\frac{1}{q}} + \left(\frac{|f'(k_1)|^q}{(\alpha+1)(\alpha+2)} + \frac{|f'(k_2)|^q}{\alpha+2} \right)^{\frac{1}{q}} \right],$$

where $\alpha \in (0,1]$*,* $q \geq 1$*,* $B(\alpha)$ *is a normalization function and* $\Gamma(.)$ *is the Gamma function.*

Proof. By using Lemma 1 and applying power mean inequality, we have

$$\left| \frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}} \left[{}^{AB}_{k_1}I^{\alpha}\{f(k_1+\mu(k_2,k_1))\} + {}^{AB}I^{\alpha}_{k_1+\mu(k_2,k_1)}\{f(k_1)\} \right] - \left(\frac{[\mu(k_2,k_1)]^{\alpha} + (1-\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}} \right) [f(k_1)+f(k_1+\mu(k_2,k_1))] \right|$$
$$\leq \int_0^1 (1-t)^{\alpha} \left| f'(k_1+t\mu(k_2,k_1)) \right| dt + \int_0^1 t^{\alpha} \left| f'(k_1+t\mu(k_2,k_1)) \right| dt$$
$$\leq \left(\int_0^1 (1-t)^{\alpha} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^{\alpha} \left| f'(k_1+t\mu(k_2,k_1)) \right|^q dt \right)^{\frac{1}{q}} + \left(\int_0^1 t^{\alpha} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\alpha} \left| f'(k_1+t\mu(k_2,k_1)) \right|^q dt \right)^{\frac{1}{q}}.$$

By using the preinvexity of $|f'|^q$, we have

$$\begin{aligned}
&\Bigg| \frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}} \left[{}_{k_1}^{AB}I^{\alpha}\{f(k_1+\mu(k_2,k_1))\} + {}^{AB}I^{\alpha}_{k_1+\mu(k_2,k_1)}\{f(k_1)\}\right] \\
&\quad - \left(\frac{[\mu(k_2,k_1)]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}}\right)\left[f(k_1)+f(k_1+\mu(k_2,k_1))\right]\Bigg| \\
&\le \left(\int_0^1 (1-t)^{\alpha}dt\right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^{\alpha}\left[(1-t)|f'(k_1)|^q + t|f'(k_2)|^q\right]dt\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\alpha}\left[(1-t)|f'(k_1)|^q + t|f'(k_2)|^q\right]dt\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \left[\left(\frac{|f'(k_1)|^q}{\alpha+2}+\frac{|f'(k_2)|^q}{(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}} + \left(\frac{|f'(k_1)|^q}{(\alpha+1)(\alpha+2)}+\frac{|f'(k_2)|^q}{\alpha+2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

So, the proof is completed. □

Corollary 3. *In Theorem 7, if we choose $\mu(k_2,k_1)=k_2-k_1$ we obtain*

$$\begin{aligned}
&\Bigg| \frac{B(\alpha)\Gamma(\alpha)}{(k_2-k_1)^{\alpha+1}} \left[{}_{k_1}^{AB}I^{\alpha}\{f(k_2)\} + {}^{AB}I^{\alpha}_{k_2}\{f(k_1)\}\right] \\
&\quad - \left(\frac{(k_2-k_1)^{\alpha}+(1-\alpha)\Gamma(\alpha)}{(k_2-k_1)^{\alpha+1}}\right)\left[f(k_1)+f(k_2)\right]\Bigg| \\
&\le \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \left[\left(\frac{|f'(k_1)|^q}{\alpha+2}+\frac{|f'(k_2)|^q}{(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}} + \left(\frac{|f'(k_1)|^q}{(\alpha+1)(\alpha+2)}+\frac{|f'(k_2)|^q}{\alpha+2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Theorem 8. *Let $I\subseteq\mathbb{R}$ be an open invex subset with respect to $\mu: I\times I\neq\varnothing\to\mathbb{R}$ and $k_1,k_2\in I$ with $k_1<k_1+\mu(k_2,k_1)$. Suppose that $f: I\to\mathbb{R}$ be a differentiable function and $f'\in L[k_1,k_1+\mu(k_2,k_1)]$. If $|f'|^q$ is a preinvex function, we have the following inequality for Atangana-Baleanu fractional integral operators:*

$$\begin{aligned}
&\Bigg| \frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}} \left[{}_{k_1}^{AB}I^{\alpha}\{f(k_1+\mu(k_2,k_1))\} + {}^{AB}I^{\alpha}_{k_1+\mu(k_2,k_1)}\{f(k_1)\}\right] \\
&\quad - \left(\frac{[\mu(k_2,k_1)]^{\alpha}+(1-\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}}\right)\left[f(k_1)+f(k_1+\mu(k_2,k_1))\right]\Bigg| \\
&\le \frac{2}{p(\alpha p+1)} + \frac{|f'(k_1)|^q+|f'(k_2)|^q}{q},
\end{aligned}$$

where $p^{-1}+q^{-1}=1$, $q>1$, $\alpha\in(0,1]$, $B(\alpha)$ is a normalization function and $\Gamma(.)$ is the Gamma function.

Proof. By using identity that is given in Lemma 1 and applying the Young inequality as $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$, we get

$$\begin{aligned}
&\left| \frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}} \left[{}^{AB}_{k_1}I^{\alpha}\left\{ f\left(k_1+\mu(k_2,k_1)\right)\right\} + {}^{AB}I^{\alpha}_{k_1+\mu(k_2,k_1)}\left\{ f(k_1)\right\}\right] \right.\\
&\left. - \left(\frac{[\mu(k_2,k_1)]^{\alpha} + (1-\alpha)\Gamma(\alpha)}{[\mu(k_2,k_1)]^{\alpha+1}} \right) \left[f(k_1) + f\left(k_1+\mu(k_2,k_1)\right)\right] \right| \\
&\leq \int_0^1 (1-t)^{\alpha} \left| f'\left(k_1+t\mu(k_2,k_1)\right)\right| dt + \int_0^1 t^{\alpha} \left| f'\left(k_1+t\mu(k_2,k_1)\right)\right| dt. \\
&\leq \frac{1}{p}\int_0^1 (1-t)^{\alpha p} dt + \frac{1}{q}\int_0^1 \left| f'\left(k_1+t\mu(k_2,k_1)\right)\right|^q dt \\
&\quad + \frac{1}{p}\int_0^1 t^{\alpha p} dt + \frac{1}{q}\int_0^1 \left| f'\left(k_1+t\mu(k_2,k_1)\right)\right|^q dt.
\end{aligned}$$

By using the preinvexity of $|f'|^q$ and by a simple computation, we have the desired result. □

Corollary 4. *In Theorem 8, if we choose* $\mu(k_2,k_1) = k_2 - k_1$ *we obtain*

$$\begin{aligned}
&\left| \frac{B(\alpha)\Gamma(\alpha)}{(k_2-k_1)^{\alpha+1}} \left[{}^{AB}_{k_1}I^{\alpha}\left\{ f(k_2)\right\} + {}^{AB}I^{\alpha}_{k_2}\left\{ f(k_1)\right\}\right] \right.\\
&\left. - \left(\frac{(k_2-k_1)^{\alpha} + (1-\alpha)\Gamma(\alpha)}{(k_2-k_1)^{\alpha+1}} \right) \left[f(k_1) + f(k_2)\right] \right| \\
&\leq \frac{2}{p(\alpha p+1)} + \frac{|f'(k_1)|^q + |f'(k_2)|^q}{q}.
\end{aligned}$$

4. Conclusion

In this study, firstly, a new inequalities of Hermite-Hadamard type for preinvex functions via Atangana-Belanu fractional integral operators was provided. And then an identity including Atangana-Baleanu integral operators has been proved and some integral inequalities are established by using preinvex functions, Hölder inequality, power-mean inequality, Young inequality with the help of this identity. Some results in this study are the generalizations and refinements of the existing results. Researchers can produce new equalities such as the integral identity in this study and obtain similar inequalities of these identity-based inequalities.

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