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Ring-like algebras associated with symmetric difference

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RING-LIKE ALGEBRAS ASSOCIATED WITH SYMMETRIC DIFFERENCE

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Abstract. We generalize the one-to-one correspondence between Boolean algebras and Boolean rings to so-called difference lattices and commutative strong difference ring-like algebras. Moreover, we show that difference ring-like algebras induce some sort of symmetric difference in corresponding posets.

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1. INTRODUCTION AND BASIC DEFINITIONS

The well-known one-to-one correspondence between Boolean algebras and Boolean rings was generalized in many ways from Boolean algebras to more general structures (cf. e.g. [1],[2],[3],[4],[5] and [7],[9],[8],[10],[11]). This correspondence is realized via the so-called symmetric difference. A certain modification of this approach for orthocomplemented lattices is developed e.g. in [11]. We already generalized the concept of symmetric difference for posets with an antitone involution in [6]. The aim of this paper is to generalize this correspondence to certain bounded lattices with an antitone involution. Moreover, we generalize Boolean rings in such a way that they induce some sort of symmetric difference on a certain bounded poset with an antitone involution.

We start our investigations with some basic definitions.

Definition 1. An **antitone involution** on a poset (P, \leq) is a unary operation $'$ on P satisfying

$$x \leq y \text{ implies } x' \geq y'$$

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$$(x')' = x$$

for all $x, y \in P$. A **bounded poset with an antitone involution** is an ordered quintuple $(P, \leq, ', 0, 1)$ where $(P, \leq, 0, 1)$ is a bounded poset and $'$ is an antitone involution on (P, \leq) . A **bounded lattice with an antitone involution** is an algebra $(L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and $'$ is an antitone involution on the poset (L, \leq) corresponding to (L, \vee, \wedge) .

Remark 1. An algebra $(L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ is a bounded lattice with an antitone involution if and only if $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and the de Morgan laws

$$(x \vee y)' = x' \wedge y'$$

$$(x \wedge y)' = x' \vee y'$$

hold for all $x, y \in L$.

2. DIFFERENCE RING-LIKE ALGEBRAS

In this section we introduce certain generalizations of Boolean rings and show that they induce posets with an antitone involution where the addition plays the role of some sort of symmetric difference.

First we define a generalized symmetric difference on a bounded poset with an antitone involution.

Definition 2. A **generalized symmetric difference** on a bounded poset $(P, \leq, ', 0, 1)$ with an antitone involution is a binary operation $+$ on P satisfying the following axioms:

$$(x + y) + z = x + (y + z)$$

$$x + y = y + x$$

$$x + 0 = x$$

$$x + x = 0$$

$$x + x' = 1$$

Remark 2. The operation $+$ satisfies $x + 1 = x'$ and $x + y = x' + y'$ since $x + 1 = x + (x + x') = (x + x) + x' = 0 + x' = x' + 0 = x'$ and $x + y = (x + y) + 0 = (x + y) + (1 + 1) = (x + 1) + (y + 1) = x' + y'$.

Now we define difference ring-like algebras.

Definition 3. A **difference ring-like algebra** is an algebra $(R, +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ satisfying the following axioms:

$$(x + y) + z = x + (y + z)$$

$$x + y = y + x$$

$$\begin{aligned}
 x + 0 &= x \\
 x + x &= 0 \\
 (xy)z &= x(yz) \\
 xx &= x \\
 x1 &= x \\
 1x &= x \\
 (x + 1)(xy + 1) &= x + 1 \tag{2.1} \\
 (xy + 1)(x + 1) &= x + 1 \tag{2.2}
 \end{aligned}$$

$(R, +, \cdot, 0, 1)$ is called a **commutative difference ring-like algebra** whenever the operation \cdot is commutative.

Remark 3. There exist non-commutative difference ring-like algebras as the following example shows:

Example 1. If $R = \{0, a, b, 1\}$ and $+$ and \cdot are defined by

$+$	0	a	b	1	\cdot	0	a	b	1
0	0	a	b	1	0	0	0	0	0
a	a	0	1	b	a	0	a	a	a
b	b	1	0	a	b	0	b	b	b
1	1	b	a	0	1	0	a	b	1

then $(R, +, \cdot, 0, 1)$ is a non-commutative difference ring-like algebra since $ab = a \neq b = ba$.

Now we can prove that every difference ring-like algebra induces in a natural way a generalized symmetric difference on a certain bounded poset with an antitone involution.

Theorem 1. *If $(R, +, \cdot, 0, 1)$ is a difference ring-like algebra then there exists a bounded poset $\mathcal{P} = (R, \leq, ', 0, 1)$ with an antitone involution such that $+$ is a generalized symmetric difference on \mathcal{P} .*

Proof. Define $x \leq y$ if $xy = yx = x$ ($x, y \in R$) and $x' := x + 1$ for all $x \in R$. Then (R, \leq) is a poset with greatest element 1. Now let $a, b, c \in R$. If $a \leq b$ then $ab = ba = a$, $(b + 1)(a + 1) = (b + 1)(ba + 1) = b + 1$ according to (2.1) and $(a + 1)(b + 1) = (ba + 1)(b + 1) = b + 1$ according to (2.2), i. e. $b + 1 \leq a + 1$. Moreover, $(a + 1) + 1 = a + (1 + 1) = a + 0 = a$. Hence $'$ is an antitone involution on (R, \leq) and $1' = 1 + 1 = 0$ its least element. We have proved that $\mathcal{P} := (R, \leq, ', 0, 1)$ is a bounded poset with an antitone involution. Finally, we have

$$\begin{aligned}
 (a + b) + c &= a + (b + c), \\
 a + b &= b + a,
 \end{aligned}$$

$$\begin{aligned} a + 0 &= a, \\ a + a &= 0 \text{ and} \\ a + 1 &= a' \text{ by definition.} \end{aligned}$$

From here we infer

$$a + a' = a + (a + 1) = (a + a) + 1 = 0 + 1 = 1 + 0 = 1.$$

□

Definition 4. The bounded poset $(R, \leq, 0, 1)$ constructed in the proof of Theorem 1 will be called the **poset induced** by the difference ring-like algebra $(R, +, \cdot, 0, 1)$.

Remark 4. Every Boolean ring is a commutative difference ring-like algebra. That the converse is not true can be seen from the following example:

Example 2. If $R = \{0, a, b, 1\}$ and $+$ and \cdot are defined by

$+$	0	a	b	1	\cdot	0	a	b	1
0	0	a	b	1	0	0	0	0	0
a	a	0	1	b	a	0	a	a	a
b	b	1	0	a	b	0	a	b	b
1	1	b	a	0	1	0	a	b	1

then $(R, +, \cdot, 0, 1)$ is a commutative difference ring-like algebra that is not a Boolean ring since

$$(a + b)b = 1 \cdot b = b \neq 1 = a + b = ab + bb.$$

Moreover, the induced poset is the chain $0 < a < b < 1$ and the involution $'$ is the permutation $(01)(ab)$.

Next we show that every commutative difference ring-like algebra already induces a generalized symmetric difference on a bounded lattice with an antitone involution.

Theorem 2. *If $(R, +, \cdot, 0, 1)$ is a commutative difference ring-like algebra and*

$$\begin{aligned} x \vee y &:= (x + 1)(y + 1) + 1 \\ x \wedge y &:= xy \text{ and} \\ x' &:= x + 1 \end{aligned}$$

for all $x, y \in R$ then $\mathcal{L} := (R, \vee, \wedge, ', 0, 1)$ is a bounded lattice with an antitone involution and $+$ is a generalized symmetric difference on \mathcal{L} .

Proof. The proof follows the same lines as that of Theorem 1. However, $x \leq y$ is now equivalent to $xy = x$ ($x, y \in R$) and (R, \cdot) is a meet-semilattice and the corresponding poset of which is (R, \leq) . Since $(R, \leq, ', 0, 1)$ is a bounded poset with an antitone involution and (R, \wedge) with $x \wedge y := xy$ for all $x, y \in R$ the corresponding meet-semilattice, $(R, \vee, \wedge, ', 0, 1)$ with $x \vee y := (x + 1)(y + 1) + 1$ for all $x, y \in R$ is a bounded lattice with an antitone involution. The rest follows from the proof of Theorem 1. □

3. DIFFERENCE LATTICES AND COMMUTATIVE STRONG DIFFERENCE RING-LIKE ALGEBRAS

In this section we want to derive a one-to-one correspondence between certain lattices with some kind of symmetric difference and certain ring-like algebras generalizing the well-known one-to-one correspondence between Boolean algebras and Boolean rings. We generalize the approach by Matoušek and Pták (cf. [11]).

First we define the lattice structures.

Definition 5. A **difference** on a bounded lattice $(L, \vee, \wedge, ', 0, 1)$ with an antitone involution is a binary operation Δ on L satisfying the following axioms:

$$(x \Delta y) \Delta z = x \Delta (y \Delta z) \tag{3.1}$$

$$x \Delta 1 = x' \tag{3.2}$$

$$1 \Delta x = x' \tag{3.3}$$

$$x \Delta x = 0 \tag{3.4}$$

$$x \Delta y \leq x \vee y \tag{3.5}$$

A **difference lattice** is an algebra $(L, \vee, \wedge, \Delta, ', 0, 1)$ of type $(2, 2, 1, 0, 0, 2)$ such that $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$ is a bounded lattice with an antitone involution and Δ is a difference on \mathcal{L} .

Remark 5. If $(B, \vee, \wedge, ', 0, 1)$ is a Boolean algebra and Δ denotes the usual symmetric difference given by $x \Delta y = (x' \wedge y) \vee (x \wedge y')$ then $(B, \vee, \wedge, \Delta, ', 0, 1)$ is a difference lattice with an antitone involution. That the converse is not true is shown by the following example:

Example 3. If $L = \{0, a, b, c, d, e, f, 1\}$ and $\vee, \wedge, '$ and Δ are defined by the tables

\vee	0	a	b	c	d	e	f	1	\wedge	0	a	b	c	d	e	f	1	'	
0	0	a	b	c	d	e	f	1	0	0	0	0	0	0	0	0	0	0	1
a	a	a	1	1	1	1	1	1	a	0	a	0	0	0	0	0	a	f	
b	b	1	b	1	1	1	1	1	b	0	0	b	0	0	0	0	b	e	
c	c	1	1	c	1	1	1	1	c	0	0	0	c	0	0	0	c	d	
d	d	1	1	1	d	1	1	1	d	0	0	0	0	d	0	0	d	c	
e	e	1	1	1	1	e	1	1	e	0	0	0	0	0	e	0	e	b	
f	f	1	1	1	1	1	f	1	f	0	0	0	0	0	0	f	f	a	
1	1	1	1	1	1	1	1	1	1	0	a	b	c	d	e	f	1	0	

Δ	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	0	c	b	e	d	1	f
b	b	c	0	a	f	1	d	e
c	c	b	a	0	1	f	e	d
d	d	e	f	1	0	a	b	c
e	e	d	1	f	a	0	c	b
f	f	1	d	e	b	c	0	a
1	1	f	e	d	c	b	a	0

then $(L, \vee, \wedge, \Delta, ', 0, 1)$ is a difference lattice such that $(L, \vee, \wedge, ', 0, 1)$ is not a Boolean algebra since

$$(a \vee b) \wedge c = 1 \wedge c = c \neq 0 = 0 \vee 0 = (a \wedge c) \vee (b \wedge c).$$

Next we define the ring-like counterpart.

Definition 6. A difference ring-like algebra $(R, +, \cdot, 0, 1)$ is called **strong** if it satisfies the axiom

$$(x + y)(xy + 1) = x + y.$$

Remark 6. Every Boolean ring is a commutative strong difference ring-like algebra. That the converse is not true is shown by the following example:

Example 4. Let $R = \{0, a, b, c, d, e, f, 1\}$ and define $+$ and \cdot as Δ and \wedge in Example 3, respectively. Then $(R, +, \cdot, 0, 1)$ is a commutative strong difference ring-like algebra that is not a Boolean ring since $(a + f)b = 1 \cdot b = b \neq 0 = 0 + 0 = ab + fb$.

Now we can prove that every difference lattice induces a commutative strong difference ring-like algebra in a natural way.

Theorem 3. If $\mathcal{L} = (L, \vee, \wedge, \Delta, ', 0, 1)$ is a difference lattice and one defines

$$\begin{aligned} x + y &:= x \Delta y \text{ and} \\ xy &:= x \wedge y \end{aligned}$$

for all $x, y \in L$ then $\mathbf{R}(\mathcal{L}) := (L, +, \cdot, 0, 1)$ is a commutative strong difference ring-like algebra.

Proof. Let $\mathcal{L} = (L, \vee, \wedge, \Delta, ', 0, 1)$ be a difference lattice, define

$$\begin{aligned} x + y &:= x \Delta y \text{ and} \\ xy &:= x \wedge y \end{aligned}$$

for all $x, y \in L$ and let $a, b, c \in L$. Then

$$(a + b) + c = (a \Delta b) \Delta c = a \Delta (b \Delta c) = a + (b + c),$$

$$\begin{aligned}
 a + 1 &= a \Delta 1 = a', \\
 1 + a &= 1 \Delta a = a', \\
 a + a &= a \Delta a = 0, \\
 a + 0 &= a + (1 + 1) = (a + 1) + 1 = (a')' = a, \\
 0 + a &= (1 + 1) + a = 1 + (1 + a) = (a')' = a, \\
 a + b &= (a + b) + 0 = (a + b) + ((b + a) + (b + a)) = \\
 &= ((a + (b + b)) + a) + (b + a) = ((a + 0) + a) + (b + a) = \\
 &= (a + a) + (b + a) = 0 + (b + a) = b + a, \\
 (ab)c &= (a \wedge b) \wedge c = a \wedge (b \wedge c) = a(bc), \\
 ab &= a \wedge b = b \wedge a = ba, \\
 aa &= a \wedge a = a, \\
 a1 &= a \wedge 1 = a, \\
 (a + 1)(ab + 1) &= a' \wedge (a \wedge b)' = a' = a + 1, \\
 a' + b' &= (a + 1) + (1 + b) = (a + (1 + 1)) + b = (a + 0) + b = \\
 &= a + b \text{ and} \\
 (a + b)(ab + 1) &= (a + b) \wedge (a \wedge b)' = (a' + b') \wedge (a' \vee b') = \\
 &= (a' \Delta b') \wedge (a' \vee b') = a' \Delta b' = a' + b' = a + b \text{ according to} \\
 &\quad (3.5).
 \end{aligned}$$

Hence $\mathbf{R}(\mathcal{L}) = (L, +, \cdot, 0, 1)$ is a commutative strong difference ring-like algebra. \square

We can also prove the converse, namely that every commutative strong difference ring-like algebra induces in a natural way a difference lattice.

Theorem 4. *If $\mathcal{R} = (R, +, \cdot, 0, 1)$ is a commutative strong difference ring-like algebra and*

$$\begin{aligned}
 x \vee y &:= (x + 1)(y + 1) + 1, \\
 x \wedge y &:= xy, \\
 x' &:= x + 1 \text{ and} \\
 x \Delta y &:= x + y
 \end{aligned}$$

for all $x, y \in R$ then $\mathbf{L}(\mathcal{R}) := (R, \vee, \wedge, \Delta, ', 0, 1)$ is a difference lattice.

Proof. Let $\mathcal{R} = (R, +, \cdot, 0, 1)$ be a commutative strong difference ring-like algebra, define

$$x \vee y := (x + 1)(y + 1) + 1,$$

$$\begin{aligned}x \wedge y &:= xy, \\x' &:= x + 1 \text{ and} \\x \Delta y &:= x + y\end{aligned}$$

for all $x, y \in R$ and let $a, b, c \in R$. According to Theorem 2, $(R, \vee, \wedge, ', 0, 1)$ is a bounded lattice with an antitone involution. Moreover, we have

$$\begin{aligned}(a \Delta b) \Delta c &= (a + b) + c = a + (b + c) = a \Delta (b \Delta c), \\a \Delta 1 &= a + 1 = a', \\1 \Delta a &= 1 + a = a + 1 = a', \\a \Delta a &= a + a = 0 \text{ and} \\a \Delta b &= a + b = (a + b) + 0 = (a + b) + (1 + 1) = (a + 1) + (b + 1) = \\&= ((a + 1) + (b + 1))((a + 1)(b + 1) + 1) = \\&= ((a + 1) + (b + 1)) \wedge (a \vee b) \leq a \vee b.\end{aligned}$$

Hence $\mathbf{L}(\mathcal{R}) = (R, \vee, \wedge, \Delta, ', 0, 1)$ is a difference lattice. \square

We can now show that the correspondence between difference lattices and commutative strong difference ring-like algebras described in Theorems 3 and 4 is one-to-one.

Theorem 5.

- (i) $\mathbf{L}(\mathbf{R}(\mathcal{L})) = \mathcal{L}$ for every difference lattice \mathcal{L} .
- (ii) $\mathbf{R}(\mathbf{L}(\mathcal{R})) = \mathcal{R}$ for every commutative strong difference ring-like algebra \mathcal{R} .

Proof.

- (i) If $\mathcal{L} = (L, \vee, \wedge, \Delta, ', 0, 1)$ is a difference lattice, $\mathbf{R}(\mathcal{L}) = (L, +, \cdot, 0, 1)$, $\mathbf{L}(\mathbf{R}(\mathcal{L})) = (L, \cup, \cap, \circ, *, 0, 1)$ and $a, b \in L$ then

$$\begin{aligned}a \cup b &= (a + 1)(b + 1) + 1 = ((a \Delta 1) \wedge (b \Delta 1)) \Delta 1 = (a' \wedge b')' = a \vee b, \\a \cap b &= ab = a \wedge b, \\a \circ b &= a + b = a \Delta b \text{ and} \\a^* &= a + 1 = a \Delta 1 = a'.\end{aligned}$$

- (ii) If $\mathcal{R} = (R, +, \cdot, 0, 1)$ is a commutative strong difference ring-like algebra, $\mathbf{L}(\mathcal{R}) = (R, \vee, \wedge, \Delta, ', 0, 1)$, $\mathbf{R}(\mathbf{L}(\mathcal{R})) = (R, \oplus, \odot, 0, 1)$ and $a, b \in R$ then

$$\begin{aligned}a \oplus b &= a \Delta b = a + b \text{ and} \\a \odot b &= a \wedge b = ab.\end{aligned}$$

\square

4. FURTHER EXAMPLES

In this section we generalize the examples given before.

Example 5. If $n \geq 2$, $R := \{0, 1\}^n$, $0 := (0, \dots, 0)$, $1 := (1, \dots, 1)$, $+$ the componentwise addition modulo 2, $0x = x0 := 0$ and $x1 = 1x := x$ for all $x \in R$ and $xy := x$ for all $x, y \in R \setminus \{0, 1\}$ then $(R, +, \cdot, 0, 1)$ is a non-commutative difference ring-like algebra since

$$(0, \dots, 0, 1)(1, \dots, 1, 0) = (0, \dots, 0, 1) \neq (1, \dots, 1, 0) = (1, \dots, 1, 0)(0, \dots, 0, 1).$$

For $n = 2$ we obtain Example 1.

Example 6. If $n \geq 2$, $R := \{0, 1\}^n$, $0 := (0, \dots, 0)$, $1 := (1, \dots, 1)$, $+$ the componentwise addition modulo 2 and \cdot the minimum with respect to the lexicographical order then $(R, +, \cdot, 0, 1)$ is a commutative difference ring-like algebra that is not a Boolean ring since

$$\begin{aligned} & ((0, \dots, 0, 1) + (1, \dots, 1, 0))(1, \dots, 1, 0) = \\ & = 1 \cdot (1, \dots, 1, 0) = (1, \dots, 1, 0) \neq 1 = (0, \dots, 0, 1) + (1, \dots, 1, 0) = \\ & = (0, \dots, 0, 1)(1, \dots, 1, 0) + (1, \dots, 1, 0)(1, \dots, 1, 0). \end{aligned}$$

For $n = 2$ we obtain Example 2.

Example 7. If $n \geq 3$, $L := \{0, 1\}^n$, $0 := (0, \dots, 0)$, $1 := (1, \dots, 1)$, $x \vee x = x \vee e0 = 0 \vee x := x$ for all $x \in L$, $x \vee y := 1$ for all $x, y \in L \setminus \{0\}$ with $x \neq y$, $x \wedge x = x \wedge 1 = 1 \wedge x := x$ for all $x \in L$, $x \wedge y := 0$ for all $x, y \in L \setminus \{1\}$ with $x \neq y$, Δ denotes the componentwise addition modulo 2 and $x' := x \Delta 1$ for all $x \in L$ then $(L, \vee, \wedge, \Delta, ', 0, 1)$ is a difference lattice such that $(L, \vee, \wedge, ', 0, 1)$ is not a Boolean algebra since

$$\begin{aligned} & ((0, \dots, 0, 1) \vee (0, \dots, 0, 1, 0)) \wedge (0, \dots, 0, 1, 1) = \\ & = 1 \wedge (0, \dots, 0, 1, 1) = (0, \dots, 0, 1, 1) \neq 0 = 0 \vee 0 = \\ & = ((0, \dots, 0, 1) \wedge (0, \dots, 0, 1, 1)) \vee ((0, \dots, 0, 1, 0) \wedge (0, \dots, 0, 1, 1)). \end{aligned}$$

For $n = 3$ we obtain Example 3.

Example 8. If $n \geq 3$, $R := \{0, 1\}^n$, $0 := (0, \dots, 0)$, $1 := (1, \dots, 1)$ and $+$ and \cdot are defined as Δ and \wedge in Example 7, respectively, then $(R, +, \cdot, 0, 1)$ is a commutative strong difference ring-like algebra that is not a Boolean ring since

$$\begin{aligned} & ((0, \dots, 0, 1) + (1, \dots, 1, 0))(0, \dots, 0, 1, 0) = \\ & = 1 \cdot (0, \dots, 0, 1, 0) = (0, \dots, 0, 1, 0) \neq 0 = 0 + 0 = \\ & = (0, \dots, 0, 1)(0, \dots, 0, 1, 0) + (1, \dots, 1, 0)(0, \dots, 0, 1, 0). \end{aligned}$$

This commutative strong difference ring-like algebra corresponds to the difference lattice defined in Example 7. For $n = 3$ we obtain Example 4.

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