Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 563–576
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4328

NEW APPROACHES FOR SUBORDINATION AND SUPERORDINATION OF MULTIVALENT FUNCTIONS ASSOCIATED WITH A FAMILY OF LINEAR OPERATORS

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Received 14 July, 2022

Abstract. The aim of this paper is to study some subordination and superordination implication properties for multivalent functions in the open unit disk associated with a family of linear operators. Moreover, we apply the results and techniques presented here to a class of integral operators.

2010 *Mathematics Subject Classification:* 30C45; 30C80

Keywords: subordination, superordination, analytic function, Hadamard product, Carlson-Shaffer operator, Liu-Srivastava operator, integral operator

1. INTRODUCTION

Let $\mathcal{H} = \mathcal{H}(\triangle)$ be the class of analytic functions defined over the unit disk

$$\triangle = \{z \in \mathbb{C} : |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, \dots\}$, let

$$\mathcal{H}[c,n] = \{f \in \mathcal{H} : f(z) = c + c_n z^n + c_{n+1} z^{n+1} + \dots\}.$$

Let f and F belong to the class $\mathcal{H}$. Then we say that the function f is subordinate to F, or F is superordinate to f, denoted by $f \prec F$ or $f(z) \prec F(z)$ $(z \in \triangle)$, if there exists a Schwarz function w in $\triangle$, with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \triangle$, satisfying

$$f(z) = F(w(z)) \quad (z \in \triangle).$$

If the function F is univalent in $\triangle$, then the following relation holds (cf. [6]):

$$f \prec F \iff f(0) = F(0) \text{ and } f(\triangle) \subset F(\triangle).$$

The third author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2019R1I1A3A0105086).

Definition 1 ([6, page 16])**.** Let $\phi\colon \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in $\triangle$. If $\mathfrak{p}$ is analytic in $\triangle$ and satisfies the differential subordination

$$\phi(\mathfrak{p}(z), z\mathfrak{p}'(z)) \prec h(z) \quad (z \in \triangle), \tag{1.1}$$

then $\mathfrak{p}$ is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant if $\mathfrak{p} \prec q$ for all $\mathfrak{p}$ satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant.

Definition 2 ([7, page 815-817], Definition 1)**.** Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let h be analytic in $\triangle$. If $\mathfrak{p}$ and $\varphi(\mathfrak{p}(z), z\mathfrak{p}'(z))$ are univalent in $\triangle$ and satisfy the differential superordination

$$h(z) \prec \varphi(\mathfrak{p}(z), z\mathfrak{p}'(z)) \quad (z \in \triangle), \tag{1.2}$$

then $\mathfrak{p}$ is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec \mathfrak{p}$ for all $\mathfrak{p}$ satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant.

Definition 3 ([6, page 21], Definition 2.2b)**.** Denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\triangle} \backslash E(f)$, where

$$E(f) = \left\{ \zeta \in \partial\triangle : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\triangle \backslash E(f)$.

We also denote $\mathcal{M}_\beta^*$ by the class of univalent functions $q \in \mathcal{H}$ with $q(0) = 1$ satisfying the following condition:

$$\Re \left[(1-\beta)\frac{zq'(z)}{q(z)} + \beta\left(1 + \frac{zq''(z)}{q'(z)}\right) \right] > 0 \quad (\beta \in \mathbb{R};\ z \in \triangle).$$

Then we also note that $\mathcal{M}_1^*$ is the class of convex (not necessarily normalized) functions in $\triangle$ (cf. [8, 10]).

We denote by $\mathcal{A}_p$ the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} c_{k+p} z^{k+p} \quad (p \in \mathbb{N})$$

which are analytic and p-valent in $\triangle$. Now we define the function $\phi_p(a, b; z)$ by

$$\phi_p(a, b; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} z^{k+p} \quad (b \neq 0, -1, -2, \dots),$$

where $(\nu)_k$ is the Pochhammer symbol(or the shifted factorial) defined by

$$(\nu)_k = \begin{cases} 1 & \text{if } k = 0 \\ \nu(\nu+1)\dots(\nu+k-1) & \text{if } k \in \mathbb{N}. \end{cases}$$

Let $f \in \mathcal{A}_p$. Denote by $\mathcal{L}_p(a,b)\colon \mathcal{A}_p \to \mathcal{A}_p$ the operator defined by

$$\mathcal{L}_p(a,b)f(z) = \phi_p(a,b;z) * f(z) \quad (z \in \triangle), \tag{1.3}$$

where the symbol $(*)$ represents the Hadamard product (or convolution). We view that

$$\mathcal{L}_p(p+1,p)f(z) = zf'(z)/p \text{ and } \mathcal{L}_p(n+p,1)f(z) = D^{n+p-1}f(z),$$

where n is any real number with $n > -p$, and the operator D^n for $p = 1$ is the well-known Ruscheweyh derivative [14] (also, see [3]) for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Further, from the definition of the operator $\mathcal{L}_p(a,b)$ it is easily checked that

$$z(\mathcal{L}_p(a,b)f(z))' = a\mathcal{L}_p(a+1,b)f(z) - (a-p)\mathcal{L}_p(a,b)f(z). \tag{1.4}$$

The operator $\mathcal{L}_p(a,b)$ was introduced and studied by Saitoh [15], which is an extension of the recognized Carlson-Shaffer operator $\mathcal{L}_1(a,b)$. And also the operator $\mathcal{L}_p(a,b)$ has been extensively used by many researchers studying geometric function theory (see, for examples, [2, 18–20]). Moreover, we remark that Liu and Srivastava [5] considered a general class of linear convolution operators, popularly known as the Liu-Srivastava operator, which is much more general than the operator $\mathcal{L}_p(a,b)$ which we have studied in this paper.

By using the concepts of subordination, Miller *et al.* [9] investigated some interesting subordination-preserving results involving certain integral operators for analytic functions in $\triangle$. Also Owa and Srivastava [12] obtained the subordination properties for a family of integral operators. Moreover, Miller and Mocanu [7] introduced the concepts of differential superordinations, which is regarded as the dual problem of differential subordinations (see also [1]). We also note that several researchers studied many interesting results involving various operators in connection with differential subordinations and their dual problems (for details, see [16, 17, 21–25]). In this paper, motivated by the works stated above, we investigate the subordination and superordination implication properties of multivalent functions for the operator $\mathcal{L}_p(a,b)$ defined by (1.3). We also consider interesting applications to the integral operator.

The following lemmas will be required to derive our present investigation.

Lemma 1 ([6, page 24], Lemma 2.2d)**.** *Let $p \in Q$ with $\mathfrak{p}(0) = a$ and let*

$$q(z) = a + a_n z^n + \dots$$

be analytic in $\triangle$ with

$$q(z) \not\equiv a \text{ and } n \in \mathbb{N}.$$

If q is not subordinate to $\mathfrak{p}$, then there exist points

$$z_0 = r_0 e^{i\theta} \in \mathbb{U} \text{ and } \zeta_0 \in \partial\triangle \backslash E(f),$$

for which

$$q(\mathbb{U}_{r_0}) \subset \mathfrak{p}(\triangle),\ q(z_0) = \mathfrak{p}(\zeta_0) \text{ and } z_0 q'(z_0) = m\zeta_0 \mathfrak{p}'(\zeta_0) \quad (m \geq n).$$

A function $L(z,t)$ defined on $\triangle \times [0,\infty)$ is the subordination chain (or Löwner chain) if $L(\cdot,t)$ is analytic and univalent in $\triangle$ for all $t \in [0,\infty)$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \triangle$ and

$$L(z,s) \prec L(z,t) \quad (z \in \triangle;\ 0 \leq s < t).$$

Lemma 2 ([13, page 159], Theorem 6.2)**.** *The function*

$$L(z,t) = a_1(t)z + \dots$$

with

$$a_1(t) \neq 0 \text{ and } \lim_{t\to\infty} |a_1(t)| = \infty.$$

Suppose that $L(\cdot,t)$ ia analytic in $\triangle$ for all $t \geq 0$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \triangle$. If $L(z,t)$ satisfies the following two conditions

$$\Re\left\{\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} > 0 \quad (z \in \triangle;\ 0 \leq t < \infty)$$

and

$$|L(z,t)| \leq K_0|a_1(t)| \quad (|z| < r_0 < 1;\ 0 \leq t < \infty)$$

for some positive constants K_0 and r_0, then $L(z,t)$ is a subordination chain.

Lemma 3 ([7, page 822], Theorem 7)**.** *Let $q \in \mathcal{H}[a,1]$ and let $\varphi\colon \mathbb{C}^2 \to \mathbb{C}$. Also set*

$$\varphi(q(z), zq'(z)) \equiv h(z) \quad (z \in \triangle).$$

If

$$L(z,t) = \varphi(q(z), tzq'(z))$$

is a subordination chain and $\mathfrak{p} \in \mathcal{H}[a,1] \cap Q$, then

$$h(z) \prec \varphi(\mathfrak{p}(z), z\mathfrak{p}'(z)) \quad (z \in \triangle).$$

implies that

$$q(z) \prec \mathfrak{p}(z)\ (z \in \triangle).$$

Furthermore, if

$$\varphi(q(z), zq'(z)) = h(z)$$

has a univalent solution $q \in Q$, then q is the best subordinant.

2. Main Results

Firstly, we start by deriving the subordination implication stated below entailing the operator $\mathcal{L}_p(a,c)$ defined by (1.3).

Theorem 1. *Let $f,g \in \mathcal{A}_p$. Suppose also that*

$$\frac{\mathcal{L}_p(a,b)g(z)}{z^p} \in \mathcal{M}^*_\beta \quad (0 \leq \beta \leq 1;\ a > 0;\ z \in \triangle). \tag{2.1}$$

Then the following subordination relation:

$$\begin{aligned}\left[\frac{\mathcal{L}_p(a,b)f(z)}{z^p}\right]^{1-\beta}&\left[\frac{\mathcal{L}_p(a+1,b)f(z)}{z^p}\right]^{\beta}\\ &\prec \left[\frac{\mathcal{L}_p(a,b)g(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a+1,b)g(z)}{z^p}\right]^{\beta} \quad (z \in \triangle)\end{aligned} \tag{2.2}$$

implies that

$$\frac{\mathcal{L}_p(a,b)f(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)g(z)}{z^p} \quad (z \in \triangle).$$

Moreover, the function $\frac{\mathcal{L}_p(a,b)g(z)}{z^p}$ is the best dominant.

Proof. Let us define two functions F and G by

$$F(z) := \frac{\mathcal{L}_p(a,b)f(z)}{z^p} \text{ and } G(z) := \frac{\mathcal{L}_p(a,b)g(z)}{z^p} \quad (f,g \in \mathcal{A}_p;\ z \in \triangle). \tag{2.3}$$

By utilizing the equation (1.4) to (2.3), we get

$$\frac{\mathcal{L}_p(a+1,b)g(z)}{z^p} = G(z) + \frac{1}{a}zG'(z). \tag{2.4}$$

Hence, combining (2.3) and (2.4), we have

$$\left[\frac{\mathcal{L}_p(a,b)g(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a+1,b)g(z)}{z^p}\right]^{\beta} = G(z)\left[1+\frac{1}{a}\frac{zG'(z)}{G(z)}\right]^{\beta}. \tag{2.5}$$

Thus, from (2.5), we need to prove the following subordination implication:

$$\begin{aligned}F(z)\left[1+\frac{1}{a}\frac{zF'(z)}{F(z)}\right]^{\beta} &\prec G(z)\left[1+\frac{1}{a}\frac{zG'(z)}{G(z)}\right]^{\beta} \quad (z \in \triangle)\\ &\Longrightarrow F(z) \prec G(z) \quad (z \in \triangle).\end{aligned} \tag{2.6}$$

Since $G \in \mathcal{M}^*(\beta)$, without loss of generality, we can assume that G satisfies the conditions of Theorem 1 on the closed disk $\overline{\triangle}$ and

$$G'(\zeta) \neq 0 \quad (\zeta \in \partial\triangle).$$

If not, then we replace F and G by

$$F_r(z) = F(rz) \text{ and } G_r(z) = G(rz),$$

respectively, where $0<r<1$ and then G_r is univalent on $\overline{\triangle}$. Since

$$F_r(z)\left[1+\frac{1}{a}\frac{zF_r'(z)}{F_r(z)}\right]^{\beta}\prec G_r(z)\left[1+\frac{1}{a}\frac{zG_r'(z)}{G_r(z)}\right]^{\beta}\quad(z\in\triangle),$$

where

$$F_r(z)=F(rz)\quad(0<r<1;\ z\in\triangle),$$

we would then prove that

$$F_r(z)\prec G_r(z)\quad(0<r<1;\ z\in\triangle),$$

and by letting $r\to1^-$, we obtain

$$F(z)\prec G(z)\quad(z\in\triangle).$$

If we suppose that the implication (2.6) is not true, that is,

$$F(z)\not\prec G(z)\quad(z\in\triangle),$$

then, from Lemma 1, there exist points

$$z_0\in\triangle \text{ and } \zeta_0\in\partial\triangle$$

such that

$$F(z_0)=G(\zeta_0) \text{ and } z_0F'(z_0)=m\zeta_0G'(\zeta_0)\quad(m\geq1).\tag{2.7}$$

To prove the implication (2.6), we define the function

$$L\colon\triangle\times[0,\infty)\longrightarrow\mathbb{C}$$

by

$$L(z,t)=G(z)\left[1+\frac{1+t}{a}\frac{zG'(z)}{G(z)}\right]^{\beta}=a_1(t)z+\ldots,$$

and we will show that $L(z,t)$ is a subordination chain. At first, we note that $L(z,t)$ is analytic in $|z|<r<1$, for sufficient small $r>0$ and for all $t\geq0$. We also have that $L(z,t)$ is continuously differentiable on $[0,\infty)$ for each $|z|<r<1$. A simple calculation shows that

$$a_1(t)=\frac{\partial L(0,t)}{\partial z}=G'(0)\left[1+\frac{(1+t)\beta}{a}\right].$$

Hence we obtain

$$a_1(t)\neq0\ (t\geq0)$$

and also we can see that

$$\lim_{t\to\infty}|a_1(t)|=\infty.$$

While, by a direct computation, we have

$$\Re\left\{\frac{z\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\}=\frac{a}{\beta}+\frac{1+t}{\beta}\Re\left[(1-\beta)\frac{zG'(z)}{G(z)}+\beta\left(1+\frac{zG''(z)}{G'(z)}\right)\right].\tag{2.8}$$

By using the assumption of Theorem 1 to (2.8), we obtain

$$\Re\left\{\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} > 0 \quad (z \in \triangle;\ 0 \leq t < \infty),$$

which completes the proof of the first condition of Lemma 3. Moreover, we have

$$\begin{aligned}
\left|\frac{L(z,t)}{a_1(t)}\right|^{1/\beta} &= \left|\frac{G(z)}{G'(0)}\right|^{1/\beta} \frac{\left|1+\frac{(1+t)}{a}\frac{zG'(z)}{G(z)}\right|}{\left|1+\frac{(1+t)\beta}{a}\right|^{1/\beta}} \\
&\leq \frac{1}{\beta}\left|\frac{G(z)}{G'(0)}\right|^{1/\beta}\left[\left|\frac{zG'(z)}{G(z)}\right| + \frac{\left|\beta-\frac{zG'(z)}{G(z)}\right|}{\left|1+\frac{(1+t)\beta}{a}\right|}\right]\frac{1}{\left|1+\frac{(1+t)\beta}{a}\right|^{1/\beta-1}} \\
&\leq \frac{1}{\beta G'(0)}\left|\frac{G(z)}{G'(0)}\right|^{1/\beta-1}\left[|zG'(z)| + \frac{\beta|G(z)|+|zG'(z)|}{\left|1+\frac{(1+t)\beta}{a}\right|}\right]\frac{1}{\left|1+\frac{(1+t)\beta}{a}\right|^{1/\beta-1}}
\end{aligned} \tag{2.9}$$

Since G is univalent in $\triangle$, the function G can be written by

$$G(z) = G(0) + G'(0)K(z) \quad (z \in \triangle),$$

where K is a normalized univalent function in $\triangle$. Hence we get the following sharp growth and distortion results [13]:

$$\frac{r}{(1+r)^2} \leq |K(z)| \leq \frac{r}{(1-r)^2} \quad (|z| = r < 1) \tag{2.10}$$

and

$$\frac{1-r}{(1+r)^3} \leq |K'(z)| \leq \frac{1+r}{(1-r)^3} \quad (|z| = r < 1) \tag{2.11}$$

Hence, by using the equations (2.10) and (2.11) to (2.9), we can obtain easily an upper bound for the right-hand side of (2.9). Thus the function $L(z,t)$ satisfies the second condition of Lemma 3 and so $L(z,t)$ is a subordination chain. In particular, we note from the definition of subordination chain that

$$L(z,0) \prec L(z,t) \quad (z \in \triangle;\ t \geq 0).$$

Now, by utilizing the definition of $L(z,t)$ and the relation (2.7), we obtain

$$\begin{aligned}
L(\zeta_0,t) &= G(\zeta_0)\left[1+\frac{1+t}{a}\frac{\zeta_0 G'(\zeta_0)}{G(\zeta_0)}\right]^{\beta} \\
&= F(z_0)\left[1+\frac{1}{a}\frac{z_0F'(z_0)}{F(z_0)}\right]^{\beta} \\
&= [\mathcal{L}_p(a,b)f(z_0)]^{1-\beta}[\mathcal{L}_p(a+1,b)f(z_0)]^{\beta} \in L(\triangle,0),
\end{aligned}$$

by virtue of the condition (2.2). This contradicts the above observation that

$$L(\zeta_0,t) \notin L(\triangle,0).$$

Therefore, the subordination condition (2.2) must imply the subordination given by (2.6). Taking $F = G$, we know that the function G is the best dominant. This evidently completes the proof of Theorem 1. □

Next, we give another subordination property by using the equation (1.4) in Theorem 2 below.

Theorem 2. *Let $f, g \in \mathcal{A}_p$ and suppose that*

$$\frac{\mathcal{L}_p(a,b)g(z)}{z^p} \in \mathcal{M}_1^* \quad (a > 0;\ z \in \triangle). \tag{2.12}$$

Then the following subordination relation:

$$\beta\frac{\mathcal{L}_p(a+1,b)f(z)}{z^p} + (1-\beta)\frac{\mathcal{L}_p(a,b)f(z)}{z^p}$$
$$\prec \beta\frac{\mathcal{L}_p(a+1,b)g(z)}{z^p} + (1-\beta)\frac{\mathcal{L}_p(a,b)g(z)}{z^p} \quad (0 \le \beta \le 1;\ z \in \triangle)$$

implies that

$$\frac{\mathcal{L}_p(a,b)f(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)g(z)}{z^p} \quad (z \in \triangle).$$

Moreover, the function $\frac{\mathcal{L}_p(a,b)g(z)}{z^p}$ is the best dominant.

Proof. Let us define the functions F and G as (2.3) and by using the equation (1.4) to (2.3), we have (2.4). Hence, combining (2.3) and (2.4), we obtain

$$(1-\beta)\frac{\mathcal{L}_p(a+1,b)g(z)}{z^p} + \beta\frac{\mathcal{L}_p(a,b)g(z)}{z^p} = G(z)\left(1 + \frac{1-\beta}{a}\frac{zG'(z)}{G(z)}\right) \tag{2.13}$$

Thus, from (2.13), we need to prove the following subordination implication:

$$F(z)\left(1 + \frac{1-\beta}{a}\frac{zF'(z)}{F(z)}\right) \prec G(z)\left(1 + \frac{1-\beta}{a}\frac{zG'(z)}{G(z)}\right) \quad (z \in \triangle)$$
$$\Longrightarrow F(z) \prec G(z) \quad (z \in \triangle). \tag{2.14}$$

Without loss of generality as in the proof of Theorem 1, we can suppose that G satisfies the conditions of Theorem 1 on the closed disk $\overline{\triangle}$ and

$$G'(\zeta) \neq 0 \quad (\zeta \in \partial\triangle).$$

To derive the implication (2.14), we consider the function

$$L\colon \triangle \times [0,\infty) \longrightarrow \mathbb{C}$$

by

$$L(z,t) = G(z)\left(1 + \frac{(1-\beta)(1+t)}{a}\frac{zG'(z)}{G(z)}\right) = a_1(t)z + \dots,$$

and we want to prove that $L(z,t)$ is a subordination chain. But, by using a similar method given in the proof of Theorem 1 we can prove the remaining part of Theorem 2 and so we omit the detailed proof. □

We next consider dual problems of Theorem 1, in the point of view that the subordinations can be replaced by superordinations.

Theorem 3. *Let $f,g \in \mathcal{A}_p$. Suppose that the condition (2.1) is satisfied, the function*

$$\left[\frac{\mathcal{L}_p(a,b)f(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a+1,b)f(z)}{z^p}\right]^{\beta}$$

is univalent and $\mathcal{L}_p(a,b)f(z)/z^p \in \mathcal{H}[1,1]\cap Q$. Then the following subordination relation:

$$\left[\frac{\mathcal{L}_p(a,b)g(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a+1,b)g(z)}{z^p}\right]^{\beta} \prec \left[\frac{\mathcal{L}_p(a,b)f(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a+1,b)f(z)}{z^p}\right]^{\beta} \quad (z\in\triangle) \tag{2.15}$$

implies that

$$\frac{\mathcal{L}_p(a,b)g(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)f(z)}{z^p} \quad (z\in\triangle).$$

Moreover, the function $\frac{\mathcal{L}_p(a,b)g(z)}{z^p}$ is the best subordinant.

Proof. Let us define the functions F and G by (2.3), respectively. By using (2.3), we have

$$\left[\frac{\mathcal{L}_p(a,b)g(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a+1,b)g(z)}{z^p}\right]^{\beta} = G(z)\left[1+\frac{1}{a}\frac{zG'(z)}{G(z)}\right]^{\beta} =: \varphi(G(z),zG'(z)). \tag{2.16}$$

Here, we note that the function G is univalent in $\triangle$ by the condition (2.1).

Next, we show that the subordination condition (2.15) implies that

$$F(z) \prec G(z) \quad (z\in\triangle). \tag{2.17}$$

Now considering the function $L(z,t)$ defined by

$$L(z,t) := G(z)\left[1+\frac{t}{a}\frac{zG'(z)}{G(z)}\right]^{\beta} \quad (z\in\triangle;\ 0\le t<\infty).$$

we can prove easily that $L(z,t)$ is a subordination chain as done in the proof of Theorem 1. Therefore according to Lemma 2, we conclude that the condition (2.15) must imply the superordination given by (2.17). Furthermore, since the equation (2.16) has the univalent solution G, we see that from Lemma 3, it is the best subordinant of the given differential superordination. Therefore the proof of Theorem 3 is completed. □

The proof of Theorem 4 below is similar to that of Theorem 3, and so we omit the details.

Theorem 4. *Let $f,g \in \mathcal{A}_p$. Suppose that the condition (2.12) is satisfied, the function*

$$\beta \frac{\mathcal{L}_p(a+1,b)f(z)}{z^p} + (1-\beta)\frac{\mathcal{L}_p(a,b)f(z)}{z^p}$$

is univalent in $\triangle$ and $\mathcal{L}_p(a,b)f(z)/z^p \in \mathcal{H}[1,1]\cap Q$. Then the following subordination relation:

$$\beta \frac{\mathcal{L}_p(a+1,b)g(z)}{z^p} + (1-\beta)\frac{\mathcal{L}_p(a,b)g(z)}{z^p} \prec \beta \frac{\mathcal{L}_p(a+1,b)f(z)}{z^p} + (1-\beta)\frac{\mathcal{L}_p(a,b)f(z)}{z^p} \quad (0 \leq \beta \leq 1;\ z \in \triangle)$$

implies that

$$\frac{\mathcal{L}_p(a,b)g(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)f(z)}{z^p} \quad (z \in \triangle).$$

Moreover, the function $\frac{\mathcal{L}_p(a,b)g(z)}{z^p}$ is the best subordinant.

Combining Theorem 1 and Theorem 3, and Theorem 2 and Theorem 4, respectively, we get the following sandwich-type theorems.

Theorem 5. *Let $f,g_k \in \mathcal{A}_p$ $(k=1,2)$. Suppose that*

$$\frac{\mathcal{L}_p(a,b)g_k(z)}{z^p} \in \mathcal{M}_\beta^* \quad (\beta \geq 0;\ a > 0;\ z \in \triangle),$$

and the function

$$\left[\frac{\mathcal{L}_p(a,b)f(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a+1,b)f(z)}{z^p}\right]^{\beta}$$

is univalent and $\mathcal{L}_p(a,b)f(z)/z^p \in \mathcal{H}[1,1]\cap Q$. Then the following subordination relation:

$$\begin{aligned}\left[\frac{\mathcal{L}_p(a,b)g_1(z)}{z^p}\right]^{1-\beta}&\left[\frac{\mathcal{L}_p(a+1,b)g_1(z)}{z^p}\right]^{\beta}\\ &\prec \left[\frac{\mathcal{L}_p(a,b)f(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a+1,b)f(z)}{z^p}\right]^{\beta}\\ &\prec \left[\frac{\mathcal{L}_p(a,b)g_2(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a+1,b)g_2(z)}{z^p}\right]^{\beta} \quad (z \in \triangle)\end{aligned}$$

implies that

$$\frac{\mathcal{L}_p(a,b)g_1(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)f(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)g_2(z)}{z^p} \quad (z \in \triangle).$$

Moreover, the functions $\frac{\mathcal{L}_p(a,b)g_1(z)}{z^p}$ *and* $\frac{\mathcal{L}_p(a,b)g_2(z)}{z^p}$ *is the best subordinant and the best dominant.*

Theorem 6. *Let* $f, g_k \in \mathcal{A}_p$ $(k = 1,2)$*. Suppose that*

$$\frac{\mathcal{L}_p(a,b)g_k(z)}{z^p} \in \mathcal{M}_1^* \quad (a > 0;\ z \in \triangle),$$

and the function

$$\beta\frac{\mathcal{L}_p(a+1,b)f(z)}{z^p} + (1-\beta)\frac{\mathcal{L}_p(a,b)f(z)}{z^p}$$

is univalent in $\triangle$ *and* $\mathcal{L}_p(a,b)f(z)/z^p \in \mathcal{H}[1,1] \cap Q$*. Then the following subordination relation*:

$$\begin{aligned}
&\beta\frac{\mathcal{L}_p(a+1,b)g_1(z)}{z^p} + (1-\beta)\frac{\mathcal{L}_p(a,b)g_1(z)}{z^p} \prec \beta\frac{\mathcal{L}_p(a+1,b)f(z)}{z^p}\\
&+(1-\beta)\frac{\mathcal{L}_p(a,b)f(z)}{z^p} \prec \beta\frac{\mathcal{L}_p(a+1,b)g_2(z)}{z^p} + (1-\beta)\frac{\mathcal{L}_p(a,b)g_2(z)}{z^p}\\
&(0 \le \beta \le 1; \quad z \in \triangle)
\end{aligned}$$

implies that

$$\frac{\mathcal{L}_p(a,b)g_1(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)f(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)g_2(z)}{z^p} \quad (z \in \triangle).$$

Moreover, the functions $\frac{\mathcal{L}_p(a,b)g_1(z)}{z^p}$ *and* $\frac{\mathcal{L}_p(a,b)g_2(z)}{z^p}$ *is the best subordinant and the best dominant.*

If we take

$$a = p,\ c = p \text{ and } \beta = 1$$

in Theorem 5 or Theorem 6, then we have the following result.

Corollary 1. *Let* $f, g_k \in \mathcal{A}_p$ $(k = 1,2)$*. Suppose also that*

$$\frac{g_k(z)}{z^p} \in \mathcal{M}_1^* \quad (z \in \triangle;\ k = 1,2)$$

the function $f'(z)/pz^{p-1}$ *is univalent in* $\triangle$ *and* $f(z)/z^p \in \mathcal{H}[1,1] \cap Q$*. Then we have the following implication*:

$$\frac{g_1'(z)}{pz^{p-1}} \prec \frac{f'(z)}{pz^{p-1}} \prec \frac{g_2'(z)}{pz^{p-1}} \quad (z \in \triangle) \implies \frac{g_1(z)}{z^p} \prec \frac{f(z)}{z^p} \prec \frac{g_2(z)}{z^p} \quad (z \in \triangle).$$

Moreover, the functions $\frac{g_1(z)}{z^p}$ *and* $\frac{g_2(z)}{z^p}$ *is the best subordinant and the best dominant.*

Next, we study the extended Libera integral operator F_ν $(\nu > -p)$ defined by (cf.[3, 4, 11])

$$F_\nu(f)(z) := \frac{\nu+p}{z^\nu}\int_0^z t^{\nu-1}f(t)\,dt \quad (f \in \mathcal{A}_p;\ \Re\{\nu\} > -p) \tag{2.18}$$

Now, we get the sandwich-type result below involving the integral operator defined by (2.18).

Theorem 7. *Let $f, g_k \in \mathcal{A}$ $(k = 1,2)$. Suppose also that*

$$\frac{\mathcal{L}_p(a,b)F_\nu(g_k)(z)}{z^p} \in \mathcal{M}_\beta^* \quad (\nu > -p;\ \beta \geq 0;\ z \in \triangle;\ k = 1,2),$$

the function

$$\left[\frac{\mathcal{L}_p(a,b)F_\nu(f)(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a,b)f(z)}{z^p}\right]^{\beta}$$

is univalent in $\triangle$ and $\mathcal{L}_p(a,b)F_\nu(f)(z)/z^p \in \mathcal{H}[1,1] \cap Q$. Then the following subordination relation:

$$\begin{aligned}&\left[\frac{\mathcal{L}_p(a,b)F_\nu(g_1)(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a,b)g_1(z)}{z^p}\right]^{\beta}\\ &\prec \left[\frac{\mathcal{L}_p(a,b)F_\nu(f)(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a,b)f(z)}{z^p}\right]^{\beta}\\ &\prec \left[\frac{\mathcal{L}_p(a,b)F_\nu(g_2)(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a,b)g_2(z)}{z^p}\right]^{\beta} \quad (z \in \triangle)\end{aligned}$$

implies that

$$\frac{\mathcal{L}_p(a,b)F_\nu(g_1)(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)F_\nu(f)(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)F_\nu(g_2)(z)}{z^p} \quad (z \in \triangle)$$

Moreover, the functions $\frac{\mathcal{L}_p(a,b)F_\nu(g_1)(z)}{z^p}$ and $\frac{\mathcal{L}_p(a,b)F_\nu(g_1)(z)}{z^p}$ are the best subordinant and the best dominant.

Proof. Let us define the functions F and G_k $(k = 1,2)$ by

$$F(z) := \frac{\mathcal{L}_p(a,b)F_\nu(f)(z)}{z^p} \text{ and } G_k(z) := \frac{\mathcal{L}_p(a,b)F_\nu(g_k)(z)}{z^p} \quad (f, g \in \mathcal{A}_p;\ z \in \triangle). \tag{2.19}$$

By means of the definition of the integral operator F_ν defined by (2.19), we have

$$z(\mathcal{L}_p(a,b)F_\nu(f)(z))' = (\nu+p)\mathcal{L}_p(a+1,c)f(z) - \nu\mathcal{L}_p(a,b)F_\nu(f)(z) \tag{2.20}$$

Hence, by using (2.19), (2.20) and the same method as in the proof of Theorem 5, we can prove Theorem 7 and so we omit the details involved. □

Finally, we have the sandwich-type Theorem 8 below by using a similar method as in the proof of Theorem 6.

Theorem 8. *Let* $f, g_k \in \mathcal{A}$ $(k = 1,2)$. *Suppose that*

$$\frac{\mathcal{L}_p(a,b)F_\nu(g_k)(z)}{z^p} \in \mathcal{M}_1^* \quad (\nu > -p;\ \beta \geq 0;\ z \in \triangle;\ k = 1,2),$$

the function

$$\beta\frac{\mathcal{L}_p(a,b)f(z)}{z^p} + (1-\beta)\frac{\mathcal{L}_p(a,b)F_\nu(f)(z)}{z^p}$$

is univalent in $\triangle$ *and* $\mathcal{L}_p(a,b)f(z)/z^p \in \mathcal{H}[1,1] \cap Q$. *Then the following subordination relation*:

$$\begin{aligned}
&\beta\frac{\mathcal{L}_p(a,b)g_1(z)}{z^p} + (1-\beta)\frac{\mathcal{L}_p(a,b)F_\nu(g_1)(z)}{z^p}\\
&\prec \beta\frac{\mathcal{L}_p(a,b)f(z)}{z^p} + (1-\beta)\frac{\mathcal{L}_p(a,b)F_\nu(f)(z)}{z^p}\\
&\prec \beta\frac{\mathcal{L}_p(a,b)g_2(z)}{z^p} + (1-\beta)\frac{\mathcal{L}_p(a,b)F_\nu(g_2)(z)}{z^p} \quad (0 \leq \beta \leq 1;\ z \in \triangle)
\end{aligned}$$

implies that

$$\frac{\mathcal{L}_p(a,b)F_\nu(g_1)(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)F_\nu(f)(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)F_\nu(g_2)(z)}{z^p} \quad (z \in \triangle).$$

Moreover, the functions $\frac{\mathcal{L}_p(a,b)F_\nu(g_1)(z)}{z^p}$ *and* $\frac{\mathcal{L}_p(a,b)F_\nu(g_1)(z)}{z^p}$ *are the best subordinant and the best dominant.*

If we take

$$a = p,\ c = p \text{ and } \beta = 1$$

Theorem 7 or Theorem 8, then we have the following result.

Corollary 2. *Let* $f, g_k \in \mathcal{A}$ $(k = 1,2)$. *Suppose also that*

$$\frac{F_\nu(g_k)(z)}{z^p} \in \mathcal{M}_1^* \quad (\nu > -p;\ z \in \triangle;\ k = 1,2),$$

the function $f(z)/z^p$ *is univalent in* $\triangle$ *and* $F_\nu(f)(z)/z^p \in \mathcal{H}[1,1] \cap Q$. *Then we have the following implication*:

$$\frac{g_1(z)}{z^p} \prec \frac{f(z)}{z^p} \prec \frac{g_2(z)}{z^p} \quad (z \in \triangle) \Longrightarrow \frac{F_\nu(g_1)(z)}{z^p} \prec \frac{F_\nu(f)(z)}{z^p} \prec \frac{F_\nu(g_2)(z)}{z^p} \quad (z \in \triangle)$$

Moreover, the functions $\frac{F_\nu(g_1)(z)}{z^p}$ *and* $\frac{F_\nu(g_2)(z)}{z^p}$ *are the best subordinant and the best dominant.*

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