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M-SRIVASTAVA HYPERGEOMETRIC FUNCTIONS: INTEGRAL REPRESENTATIONS AND SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we introduce a new extensions of Srivastava's triple hypergeometric functions H_A, H_B and H_C, by using an modified beta function, which given with a generalized M-series in its kernel. We also introduce a new extension of Appell's hypergeometric function of the first kind by using the same modified beta function. Furthermore, we give some integral representations of the new extensions of Srivastava's triple hypergeometric functions. Finally, we obtain solutions of fractional differential equations involving new extensions of Srivastava's triple hypergeometric functions.

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1. INTRODUCTION AND PRELIMINARIES

Scientists have conducted a lot of research in recent years on various generalizations of special functions (see for example [2, 3, 5, 7–10, 16, 19, 21, 28] and reference therein). Particularly, the modified gamma function for $\Re(\alpha)>0$, $\Re(\rho)>0$, $\Re(\kappa)>0$ and $\xi_1,\ldots,\xi_p,\eta_1,\ldots,\eta_q \neq 0,-1,-2,\ldots$ was introduced by Ata in [4] as follows:

$$\begin{aligned}{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\kappa;\rho) &= {}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\kappa;\rho)\\ &= \int_0^\infty \Delta^{\kappa-1}\,{}_{p}^{\alpha}M_{q}^{\beta}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;-\Delta-\frac{\rho}{\Delta}\right)d\Delta.\end{aligned}$$

Also, the modified beta function for $\Re(\alpha)>0$, $\Re(\rho)>0$, $\Re(\kappa)>0$, $\Re(\omega)>0$ and $\xi_1,\ldots,\xi_p,\eta_1,\ldots,\eta_q \neq 0,-1,-2,\ldots$ was introduced by Ata in [4] as follows:

$${}^{M}B_{p,q}^{(\alpha,\beta)}(\kappa,\omega;\rho) = {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\kappa,\omega;\rho)$$

$$= \int_0^1 \Delta^{\kappa-1}(1-\Delta)^{\omega-1}\, {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\frac{-\rho}{\Delta(1-\Delta)}\right)d\Delta. \quad (1.1)$$

If we take $\Delta = (\sin\phi)^2$ in the equation (1.1), then

$$\begin{aligned} {}^{M}B_{p,q}^{(\alpha,\beta)}(\kappa,\omega;\rho) &= {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\kappa,\omega;\rho) \\ &= 2\int_0^{\frac{\pi}{2}}(\sin\phi)^{2\kappa-1}(\cos\phi)^{2\omega-1} \\ &\quad \times {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;-\rho(\sec\phi)^2(\csc\phi)^2\right)d\phi. \quad (1.2)\end{aligned}$$

If we take $\Delta = \frac{\Lambda}{1+\Lambda}$ in the equation (1.1), then

$$\begin{aligned} {}^{M}B_{p,q}^{(\alpha,\beta)}(\kappa,\omega;\rho) &= {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\kappa,\omega;\rho) \\ &= \int_0^{\infty}\frac{\Lambda^{\kappa-1}}{(1+\Lambda)^{\kappa+\omega}} \\ &\quad \times {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;-2\rho-\rho\left(\Lambda+\frac{1}{\Lambda}\right)\right)d\Lambda. \quad (1.3)\end{aligned}$$

If we take $\Delta = \frac{\Lambda-u}{v-u}$ in the equation (1.1), then

$$\begin{aligned} {}^{M}B_{p,q}^{(\alpha,\beta)}(\kappa,\omega;\rho) &= {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\kappa,\omega;\rho) \\ &= (v-u)^{1-\kappa-\omega}\int_u^{v}(\Lambda-u)^{\kappa-1}(v-\Lambda)^{\omega-1} \\ &\quad \times {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\frac{-\rho(v-u)^2}{(\Lambda-u)(v-\Lambda)}\right)d\Lambda. \quad (1.4)\end{aligned}$$

Moreover, the modified Gauss hypergeometric function for $\Re(\mu_3) > \Re(\mu_2) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$ and $\xi_1,\ldots,\xi_p,\eta_1,\ldots,\eta_q \neq 0,-1,-2,\ldots$ was introduced by Ata in [4] as follows:

$$\begin{aligned} {}^{M}F_{p,q}^{(\alpha,\beta)}(\mu_1,\mu_2;\mu_3;\tau;\rho) &= {}^{M}F_{p,q}^{(\alpha,\beta)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\mu_1,\mu_2;\mu_3;\tau;\rho) \\ &= \sum_{n=0}^{\infty}(\mu_1)_n\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\mu_2+n,\mu_3-\mu_2;\rho)}{B(\mu_2,\mu_3-\mu_2)}\frac{\tau^n}{n!}, \quad (|\tau|<1).\end{aligned}$$

The modified special functions given above were called M-gamma, M-beta and M-Gauss hypergeometric functions by Ata, respectively. If we put $\rho = 0$ and $p = q = \xi_1 = \eta_1 = \alpha = \beta = 1$ to the M-gamma, M-beta and M-Gauss hypergeometric functions, we get the classical special functions [1], respectively, as follows:

- The gamma function for $\Re(\kappa) > 0$:

$$\Gamma(\kappa) = \int_0^{\infty}\Delta^{\kappa-1}\exp(-\Delta)d\Delta.$$

- The beta function for $\Re(\kappa) > 0$ and $\Re(\omega) > 0$:

$$B(\kappa,\omega) = \int_0^1 \Delta^{\kappa-1}(1-\Delta)^{\omega-1} d\Delta.$$

- The Gauss hypergeometric function for $\Re(\mu_3) > \Re(\mu_2) > 0$:

$${}_2F_1(\mu_1,\mu_2;\mu_3;\tau) = \sum_{n=0}^{\infty} (\mu_1)_n \frac{B(\mu_2+n,\mu_3-\mu_2)}{B(\mu_2,\mu_3-\mu_2)} \frac{\tau^n}{n!}, \quad (|\tau| < 1).$$

The gamma and beta functions relation [1] is as follows:

$$B(\kappa,\omega) = \frac{\Gamma(\kappa)\Gamma(\omega)}{\Gamma(\kappa+\omega)}, \quad (\Re(\kappa) > 0, \Re(\omega) > 0). \tag{1.5}$$

The function ${}^{\alpha}_{p}M^{\beta}_{q}$ used above is known as the generalized M-series [23] for $\Re(\alpha) > 0$ and $\xi_1,\ldots,\xi_p,\eta_1,\ldots,\eta_q \neq 0,-1,-2,\ldots$ which defined as:

$${}^{\alpha}_{p}M^{\beta}_{q}(\tau) = {}^{\alpha}_{p}M^{\beta}_{q}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\tau) = \sum_{n=0}^{\infty} \frac{(\xi_1)_n \ldots (\xi_p)_n}{(\eta_1)_n \ldots (\eta_q)_n} \frac{\tau^n}{\Gamma(\alpha n+\beta)}.$$

The symbol $(\cdot)_n$ used above denotes the Pochhammer symbol [1] and is defined by

$$(\zeta)_n = \frac{\Gamma(\zeta+n)}{\Gamma(\zeta)} = \begin{cases} \zeta(\zeta+1)\ldots(\zeta+n-1), & n = 1,2,\ldots, \\ 1, & n = 0. \end{cases} \tag{1.6}$$

The equation (1.6) also yields [26]

$$(\zeta)_{n+m} = (\zeta)_n(\zeta+n)_m. \tag{1.7}$$

The binomial theorem [1] is as follows:

$$(1-\Delta)^{-\zeta} = \sum_{n=0}^{\infty} (\zeta)_n \frac{\Delta^n}{n!}, \quad (|\Delta| < 1). \tag{1.8}$$

The Caputo fractional derivative operator [18] for $\Re(\varepsilon) > 0$, $m-1 < \Re(\varepsilon) < m$, $(m \in \mathbb{N})$ is given by

$${}^{c}D^{\varepsilon}_{\rho}\{f(\rho)\} = \frac{1}{\Gamma(m-\varepsilon)} \int_0^{\rho} (\rho-\omega)^{m-\varepsilon-1} f^{(m)}(\omega) d\omega, \quad (\rho > 0).$$

The Laplace and inverse Laplace transforms for $\Re(s) > 0$ in [17], respectively, are defined as:

$$\mathfrak{L}\{f(\rho);s\} = F(s) = \int_0^{\infty} \exp(-s\rho) f(\rho) d\rho$$

and

$$\mathfrak{L}^{-1}\{F(s)\} = f(\rho) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp(s\rho) F(s) ds, \quad (c > 0).$$

The Laplace transform of the Caputo fractional derivative is as follows [20]:

$$\mathfrak{L}\left\{{}^{c}D_{\rho}^{\varepsilon}\{f(\rho)\};s\right\}=s^{\varepsilon}F(s)-\sum_{k=0}^{m-1}s^{\varepsilon-k-1}f^{(k)}(0),\quad (m-1<\Re(\varepsilon)\leq m).\tag{1.9}$$

We introduce the new extended Appell's hypergeometric function of the first kind for $\Re(\mu_4)>\Re(\mu_1)>0$, $\Re(\alpha)>0$, $\Re(\rho)>0$ and $\max\{|\kappa|,|\omega|\}<1$ as follows:

$$\begin{aligned}{}^{M}F_{1,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4;\kappa,\omega;\rho)&={}^{M}F_{1,p,q}^{(\alpha,\beta)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\mu_1,\mu_2,\mu_3;\mu_4;\kappa,\omega;\rho)\\&:=\sum_{m,n=0}^{\infty}(\mu_2)_m(\mu_3)_n\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\mu_1+m+n,\mu_4-\mu_1;\rho)}{B(\mu_1,\mu_4-\mu_1)}\frac{\kappa^m}{m!}\frac{\omega^n}{n!},\end{aligned}\tag{1.10}$$

which we called as M-Appell hypergeometric function F_1.

2. M-Srivastava hypergeometric functions

Srivastava defined triple hypergeometric functions H_A, H_B and H_C in [24, 25] and then scientists have studied on various extended of these functions [6, 11–15, 22, 27].

We introduce the new extended Srivastava's triple hypergeometric functions as follows:

$$\begin{aligned}&{}^{M}H_{A,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4,\mu_5;\kappa,\omega,\tau;\rho)\\&\quad={}^{M}H_{A,p,q}^{(\alpha,\beta)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\mu_1,\mu_2,\mu_3;\mu_4,\mu_5;\kappa,\omega,\tau;\rho)\\&\quad:=\sum_{m,n,k=0}^{\infty}\frac{(\mu_1)_{m+k}(\mu_2)_{m+n}}{(\mu_4)_m}\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\mu_3+n+k,\mu_5-\mu_3;\rho)}{B(\mu_3,\mu_5-\mu_3)}\frac{\kappa^m}{m!}\frac{\omega^n}{n!}\frac{\tau^k}{k!},\end{aligned}\tag{2.1}$$

$$\big(\Re(\alpha)>0,\Re(\rho)>0,\mathbf{r}<1,\mathbf{s}<1,\mathbf{t}<(1-\mathbf{r})(1-\mathbf{s})\big),$$

$$\begin{aligned}&{}^{M}H_{B,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4,\mu_5,\mu_6;\kappa,\omega,\tau;\rho)\\&\quad={}^{M}H_{B,p,q}^{(\alpha,\beta)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\mu_1,\mu_2,\mu_3;\mu_4,\mu_5,\mu_6;\kappa,\omega,\tau;\rho)\\&\quad:=\sum_{m,n,k=0}^{\infty}\frac{(\mu_1+\mu_2)_{2m+n+k}(\mu_3)_{n+k}}{(\mu_4)_m(\mu_5)_n(\mu_6)_k}\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\mu_1+m+k,\mu_2+m+n;\rho)}{B(\mu_1,\mu_2)}\frac{\kappa^m}{m!}\frac{\omega^n}{n!}\frac{\tau^k}{k!},\end{aligned}$$

$$\left(\Re(\alpha)>0,\Re(\rho)>0,\mathbf{r}+\mathbf{s}+\mathbf{t}+2\sqrt{\mathbf{rst}}<1\right),$$

and

$$\begin{aligned}&{}^{M}H_{C,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4;\kappa,\omega,\tau;\rho)\\&\quad={}^{M}H_{C,p,q}^{(\alpha,\beta)}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\mu_1,\mu_2,\mu_3;\mu_4;\kappa,\omega,\tau;\rho)\\&\quad:=\sum_{m,n,k=0}^{\infty}\frac{(\mu_2)_{m+n}(\mu_3)_{n+k}}{(\mu_4)_n}\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\mu_1+m+k,\mu_4+n-\mu_1;\rho)}{B(\mu_1,\mu_4+n-\mu_1)}\frac{\kappa^m}{m!}\frac{\omega^n}{n!}\frac{\tau^k}{k!},\end{aligned}\tag{2.2}$$

$$\left(\Re(\alpha) > 0, \Re(\rho) > 0, \mathbf{r} < 1, \mathbf{s} < 1, \mathbf{t} < 1, \mathbf{r}+\mathbf{s}+\mathbf{t}-2\sqrt{(1-\mathbf{r})(1-\mathbf{s})(1-\mathbf{t})} < 2\right),$$

where for brevity, the co-ordinates are written $(\mathbf{r},\mathbf{s},\mathbf{t})$ instead of $(|\kappa|,|\omega|,|\tau|)$.

Respectively, we called them as M-Srivastava hypergeometric function H_A, M-Srivastava hypergeometric function H_B, and M-Srivastava hypergeometric function H_C. Obviously for $\rho = 0$ and $p = q = \xi_1 = \eta_1 = \alpha = \beta = 1$, these functions are reduced to the Srivastava's triple hypergeometric functions H_A, H_B, and H_C.

The equations (2.1) and (2.2) can also be given with the following series representations:

$$\begin{aligned} {}^{M}H^{(\alpha,\beta)}_{A,p,q}&(\mu_1,\mu_2,\mu_3;\mu_4,\mu_5;\kappa,\omega,\tau;\rho) \\ &= \sum_{m=0}^{\infty} \frac{(\mu_1)_m(\mu_2)_m}{(\mu_4)_m}\, {}^{M}F^{(\alpha,\beta)}_{1,p,q}(\mu_3,\mu_2+m,\mu_1+m;\mu_5;\omega,\tau;\rho)\frac{\kappa^m}{m!}, \end{aligned}$$

$$\begin{aligned} {}^{M}H^{(\alpha,\beta)}_{C,p,q}&(\mu_1,\mu_2,\mu_3;\mu_4;\kappa,\omega,\tau;\rho) \\ &= \sum_{n=0}^{\infty} \frac{(\mu_2)_n(\mu_3)_n}{(\mu_4)_n}\, {}^{M}F^{(\alpha,\beta)}_{1,p,q}(\mu_1,\mu_2+n,\mu_3+n;\mu_4+n;\kappa,\tau;\rho)\frac{\omega^n}{n!}, \end{aligned}$$

where ${}^{M}F^{(\alpha,\beta)}_{1,p,q}$ is the extended Appell's hypergeometric function of the first kind given by (1.10).

3. Integral representations for M-Srivastava hypergeometric function H_A

Theorem 1. *Let $\Re(\mu_5) > \Re(\mu_3) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M-Srivastava hypergeometric function H_A holds true:*

$$\begin{aligned} {}^{M}H^{(\alpha,\beta)}_{A,p,q}&(\mu_1,\mu_2,\mu_3;\mu_4,\mu_5;\kappa,\omega,\tau;\rho) \\ &= \frac{\Gamma(\mu_5)}{\Gamma(\mu_3)\Gamma(\mu_5-\mu_3)} \int_0^1 \Delta^{\mu_3-1}(1-\Delta)^{\mu_5-\mu_3-1}(1-\omega\Delta)^{-\mu_2}(1-\tau\Delta)^{-\mu_1} \\ &\quad \times {}^{\alpha}_{p}M^{\beta}_{q}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) {}_2F_1\left(\mu_1,\mu_2;\mu_4;\frac{\kappa}{(1-\omega\Delta)(1-\tau\Delta)}\right) d\Delta. \end{aligned}$$

Proof. By using the formula (1.1) of the M-beta function in the definition of M-Srivastava hypergeometric function H_A, we have

$$\begin{aligned} {}^{M}H^{(\alpha,\beta)}_{A,p,q}&(\mu_1,\mu_2,\mu_3;\mu_4,\mu_5;\kappa,\omega,\tau;\rho) \\ &= \sum_{m,n,k=0}^{\infty} \frac{(\mu_1)_{m+k}(\mu_2)_{m+n}}{(\mu_4)_m} \frac{{}^{M}B^{(\alpha,\beta)}_{p,q}(\mu_3+n+k,\mu_5-\mu_3;\rho)}{B(\mu_3,\mu_5-\mu_3)} \frac{\kappa^m}{m!}\frac{\omega^n}{n!}\frac{\tau^k}{k!} \\ &= \sum_{m,n,k=0}^{\infty} \frac{(\mu_1)_{m+k}(\mu_2)_{m+n}}{(\mu_4)_m} \frac{1}{B(\mu_3,\mu_5-\mu_3)} \int_0^1 \Delta^{\mu_3+n+k-1}(1-\Delta)^{\mu_5-\mu_3-1} \end{aligned}$$

$$\times {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)\frac{\kappa^m}{m!}\frac{\omega^n}{n!}\frac{\tau^k}{k!}d\Delta.$$

By using equations (1.5) and (1.7), we get

$$\begin{aligned}
&{}^{M}H_{A,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4,\mu_5;\kappa,\omega,\tau;\rho)\\
&=\frac{\Gamma(\mu_5)}{\Gamma(\mu_3)\Gamma(\mu_5-\mu_3)}\sum_{m,n,k=0}^{\infty}\frac{(\mu_1)_{m+k}(\mu_2)_{m+n}}{(\mu_4)_m}\int_0^1\Delta^{\mu_3+n+k-1}(1-\Delta)^{\mu_5-\mu_3-1}\\
&\times {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)\frac{\kappa^m}{m!}\frac{\omega^n}{n!}\frac{\tau^k}{k!}d\Delta\\
&=\frac{\Gamma(\mu_5)}{\Gamma(\mu_3)\Gamma(\mu_5-\mu_3)}\int_0^1\Delta^{\mu_3-1}(1-\Delta)^{\mu_5-\mu_3-1}\,{}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)\\
&\times\sum_{m=0}^{\infty}\frac{(\mu_1)_m(\mu_2)_m}{(\mu_4)_m}\frac{\kappa^m}{m!}\sum_{n=0}^{\infty}(\mu_2+m)_n\frac{(\omega\Delta)^n}{n!}\sum_{k=0}^{\infty}(\mu_1+m)_k\frac{(\tau\Delta)^k}{k!}d\Delta.
\end{aligned}$$

By using equation (1.8), we obtain

$$\begin{aligned}
&{}^{M}H_{A,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4,\mu_5;\kappa,\omega,\tau;\rho)\\
&=\frac{\Gamma(\mu_5)}{\Gamma(\mu_3)\Gamma(\mu_5-\mu_3)}\int_0^1\Delta^{\mu_3-1}(1-\Delta)^{\mu_5-\mu_3-1}\,{}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)\\
&\times\sum_{m=0}^{\infty}\frac{(\mu_1)_m(\mu_2)_m}{(\mu_4)_m}\frac{\kappa^m}{m!}(1-\omega\Delta)^{-\mu_2-m}(1-\tau\Delta)^{-\mu_1-m}d\Delta\\
&=\frac{\Gamma(\mu_5)}{\Gamma(\mu_3)\Gamma(\mu_5-\mu_3)}\int_0^1\Delta^{\mu_3-1}(1-\Delta)^{\mu_5-\mu_3-1}(1-\omega\Delta)^{-\mu_2}(1-\tau\Delta)^{-\mu_1}\\
&\times {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right){}_2F_1\left(\mu_1,\mu_2;\mu_4;\frac{\kappa}{(1-\omega\Delta)(1-\tau\Delta)}\right)d\Delta,
\end{aligned}$$

which completes the proof. □

Theorem 2. *Let* $\Re(\mu_4)>\Re(\mu_2)>0$, $\Re(\mu_5)>\Re(\mu_3)>0$, $\Re(\alpha)>0$, $\Re(\rho)>0$. *The following integral representation for the M-Srivastava hypergeometric function* H_A *holds true:*

$$\begin{aligned}
&{}^{M}H_{A,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4,\mu_5;\kappa,\omega,\tau;\rho)\\
&=\frac{\Gamma(\mu_4)\Gamma(\mu_5)}{\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4-\mu_2)\Gamma(\mu_5-\mu_3)}\int_0^1\int_0^1\Lambda^{\mu_2-1}\Delta^{\mu_3-1}(1-\Lambda)^{\mu_4-\mu_2-1}\\
&\times(1-\Delta)^{\mu_5-\mu_3-1}(1-\omega\Delta)^{\mu_1-\mu_2}\left[(1-\omega\Delta)(1-\tau\Delta)-\kappa\Lambda\right]^{-\mu_1}\\
&\times {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Lambda(1-\Lambda)}\right){}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)d\Lambda d\Delta.
\end{aligned}$$

Proof. By using the formula (1.1) of the M-beta function in the definition of M-Srivastava hypergeometric function H_A and by making similar calculations in the proof of Theorem 1, the proof is completed. □

Theorem 3. *Let $\Re(\mu_4) > \Re(\mu_2) > 0$, $\Re(\mu_5) > \Re(\mu_3) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M-Srivastava hypergeometric function H_A holds true:*

$$
\begin{aligned}
&{}^{M}H_{A,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4,\mu_5;\kappa,\omega,\tau;\rho)\\
&= \frac{\Gamma(\mu_4)\Gamma(\mu_5)}{\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4-\mu_2)\Gamma(\mu_5-\mu_3)} \int_0^1\int_0^1 \Lambda^{\mu_2-1}\Delta^{\mu_3-1}(1-\Lambda)^{\mu_4-\mu_2-1}\\
&\quad\times(1-\Delta)^{\mu_5-\mu_3-1}(1-\omega\Delta)^{-\mu_2}(1-\kappa\Lambda-\tau\Delta)^{-\mu_1}\left(1-\frac{\kappa\omega\Lambda\Delta}{(1-\omega\Delta)(1-\kappa\Lambda-\tau\Delta)}\right)^{-\mu_1}\\
&\quad\times {}_{p}^{\alpha}M_{q}^{\beta}\left(\frac{-\rho}{\Lambda(1-\Lambda)}\right){}_{p}^{\alpha}M_{q}^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)d\Lambda d\Delta.
\end{aligned}
$$

Proof. By using the formula (1.1) of the M-beta function in the definition of M-Srivastava hypergeometric function H_A and by making similar calculations in the proof of Theorem 1, the proof is completed. □

Theorem 4. *Let $\Re(\mu_5) > \Re(\mu_3) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M-Srivastava hypergeometric function H_A holds true:*

$$
\begin{aligned}
&{}^{M}H_{A,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4,\mu_5;\kappa,\omega,\tau;\rho)\\
&\quad= \frac{2\Gamma(\mu_5)}{\Gamma(\mu_3)\Gamma(\mu_5-\mu_3)}\int_0^{\frac{\pi}{2}}(\sin\phi)^{2\mu_3-1}(\cos\phi)^{2\mu_5-2\mu_3-1}\left(1-\omega(\sin\phi)^2\right)^{-\mu_2}\\
&\quad\times\left(1-\tau(\sin\phi)^2\right)^{-\mu_1}\ {}_{p}^{\alpha}M_{q}^{\beta}\left(-\rho(\sec\phi)^2(\csc\phi)^2\right)\\
&\quad\times {}_2F_1\left(\mu_1,\mu_2;\mu_4;\frac{\kappa}{(1-\omega(\sin\phi)^2)(1-\tau(\sin\phi)^2)}\right)d\phi.
\end{aligned}
$$

Proof. By using the formula (1.2) of the M-beta function in the definition of M-Srivastava hypergeometric function H_A and by making similar calculations in the proof of Theorem 1, the proof is completed. □

Theorem 5. *Let $\Re(\mu_5) > \Re(\mu_3) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M-Srivastava hypergeometric function H_A holds true:*

$$
\begin{aligned}
&{}^{M}H_{A,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4,\mu_5;\kappa,\omega,\tau;\rho)\\
&\quad= \frac{\Gamma(\mu_5)}{\Gamma(\mu_3)\Gamma(\mu_5-\mu_3)}\int_0^{\infty}\Lambda^{\mu_3-1}(1+\Lambda)^{\mu_1+\mu_2-\mu_5}(1+\Lambda-\omega\Lambda)^{-\mu_2}\\
&\quad\times(1+\Lambda-\tau\Lambda)^{-\mu_1}\ {}_{p}^{\alpha}M_{q}^{\beta}\left(-2\rho-\rho\left(\Lambda+\frac{1}{\Lambda}\right)\right)
\end{aligned}
$$

$$\times {}_2F_1\left(\mu_1,\mu_2;\mu_4;\frac{\kappa(1+\Lambda)^2}{(1+\Lambda-\omega\Lambda)(1+\Lambda-\tau\Lambda)}\right)d\Lambda.$$

Proof. By using the formula (1.3) of the M-beta function in the definition of M-Srivastava hypergeometric function H_A and by making similar calculations in the proof of Theorem 1, the proof is completed. □

Theorem 6. *Let $\Re(\mu_5) > \Re(\mu_3) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M-Srivastava hypergeometric function H_A holds true:*

$$\begin{aligned}
&{}^{M}H^{(\alpha,\beta)}_{A,p,q}(\mu_1,\mu_2,\mu_3;\mu_4,\mu_5;\kappa,\omega,\tau;\rho)\\
&\quad=\frac{\Gamma(\mu_5)}{\Gamma(\mu_3)\Gamma(\mu_5-\mu_3)}(v-u)^{1+\mu_1+\mu_2-\mu_5}\int_u^v(\Lambda-u)^{\mu_3-1}(v-\Lambda)^{\mu_5-\mu_3-1}\\
&\quad\times\big(v-u-\omega(\Lambda-u)\big)^{-\mu_2}\big(v-u-\tau(\Lambda-u)\big)^{-\mu_1}\,{}^{\alpha}_{p}M^{\beta}_{q}\left(\frac{-\rho(v-u)^2}{(\Lambda-u)(v-\Lambda)}\right)\\
&\quad\times {}_2F_1\left(\mu_1,\mu_2;\mu_4;\frac{\kappa(v-u)^2}{\big(v-u-\omega(\Lambda-u)\big)\big(v-u-\tau(\Lambda-u)\big)}\right)d\Lambda.
\end{aligned}$$

Proof. By using the formula (1.4) of the M-beta function in the definition of M-Srivastava hypergeometric function H_A and by making similar calculations in the proof of Theorem 1, the proof is completed. □

4. Integral representations for M-Srivastava hypergeometric function H_B

Theorem 7. *Let $\Re(\mu_1) > 0$, $\Re(\mu_2) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M-Srivastava hypergeometric function H_B holds true:*

$$\begin{aligned}
&{}^{M}H^{(\alpha,\beta)}_{B,p,q}(\mu_1,\mu_2,\mu_3;\mu_4,\mu_5,\mu_6;\kappa,\omega,\tau;\rho)\\
&\quad=\frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_0^1\Delta^{\mu_1-1}(1-\Delta)^{\mu_2-1}\,{}^{\alpha}_{p}M^{\beta}_{q}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)\\
&\quad\times X_4\big(\mu_1+\mu_2,\mu_3;\mu_4,\mu_5,\mu_6;\kappa\Delta(1-\Delta),\omega(1-\Delta),\tau\Delta\big)d\Delta.
\end{aligned}$$

Proof. By using the formula (1.1) of the M-beta function in the definition of M-Srivastava hypergeometric function H_B, we have

$$\begin{aligned}
&{}^{M}H^{(\alpha,\beta)}_{B,p,q}(\mu_1,\mu_2,\mu_3;\mu_4,\mu_5,\mu_6;\kappa,\omega,\tau;\rho)\\
&\quad=\sum_{m,n,k=0}^{\infty}\frac{(\mu_1+\mu_2)_{2m+n+k}(\mu_3)_{n+k}}{(\mu_4)_m(\mu_5)_n(\mu_6)_k}\frac{{}^{M}B^{(\alpha,\beta)}_{p,q}(\mu_1+m+k,\mu_2+m+n;\rho)}{B(\mu_1,\mu_2)}\frac{\kappa^m}{m!}\frac{\omega^n}{n!}\frac{\tau^k}{k!}\\
&\quad=\frac{1}{B(\mu_1,\mu_2)}\sum_{m,n,k=0}^{\infty}\frac{(\mu_1+\mu_2)_{2m+n+k}(\mu_3)_{n+k}}{(\mu_4)_m(\mu_5)_n(\mu_6)_k}\int_0^1\Delta^{\mu_1+m+k-1}(1-\Delta)^{\mu_2+m+n-1}
\end{aligned}$$

$$\times {}_{p}^{\alpha}M_{q}^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)\frac{\kappa^m}{m!}\frac{\omega^n}{n!}\frac{\tau^k}{k!}d\Delta.$$

By using equation (1.5) and by making necessary arrangements, we get

$$\begin{aligned}
&{}^{M}H_{B,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4,\mu_5,\mu_6;\kappa,\omega,\tau;\rho)\\
&\quad=\frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_0^1\Delta^{\mu_1-1}(1-\Delta)^{\mu_2-1}\,{}_{p}^{\alpha}M_{q}^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)\\
&\quad\times\sum_{m,n,k=0}^{\infty}\frac{(\mu_1+\mu_2)_{2m+n+k}(\mu_3)_{n+k}}{(\mu_4)_m(\mu_5)_n(\mu_6)_k}\frac{(\kappa\Delta(1-\Delta))^m}{m!}\frac{(\omega(1-\Delta))^n}{n!}\frac{(\tau\Delta)^k}{k!}d\Delta\\
&\quad=\frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_0^1\Delta^{\mu_1-1}(1-\Delta)^{\mu_2-1}\,{}_{p}^{\alpha}M_{q}^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)\\
&\quad\times X_4\big(\mu_1+\mu_2,\mu_3;\mu_4,\mu_5,\mu_6;\kappa\Delta(1-\Delta),\omega(1-\Delta),\tau\Delta\big)d\Delta,
\end{aligned}$$

which completes the proof. □

Remark 1. The function X_4 used above is known as the Exton's function X_4 [26] and defined by

$$X_4\big(\mu_1,\mu_2;\mu_3,\mu_4,\mu_5;\kappa,\omega,\tau\big)=\sum_{m,n,k=0}^{\infty}\frac{(\mu_1)_{2m+n+k}(\mu_2)_{n+k}}{(\mu_3)_m(\mu_4)_n(\mu_5)_k}\frac{\kappa^m}{m!}\frac{\omega^n}{n!}\frac{\tau^k}{k!},$$

$$\left(2\sqrt{\mathbf{r}}+\left(\sqrt{\mathbf{s}}+\sqrt{\mathbf{t}}\right)^2<1\right),$$

where for brevity, the co-ordinates are written $(\mathbf{r},\mathbf{s},\mathbf{t})$ instead of $(|\kappa|,|\omega|,|\tau|)$.

Theorem 8. *Let $\Re(\mu_1)>0$, $\Re(\mu_2)>0$, $\Re(\alpha)>0$, $\Re(\rho)>0$. The following integral representation for the M-Srivastava hypergeometric function H_B holds true:*

$$\begin{aligned}
&{}^{M}H_{B,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4,\mu_5,\mu_6;\kappa,\omega,\tau;\rho)\\
&\quad=\frac{2\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_0^{\frac{\pi}{2}}(\sin\phi)^{2\mu_1-1}(\cos\phi)^{2\mu_2-1}\,{}_{p}^{\alpha}M_{q}^{\beta}\left(-\rho(\sec\phi)^2(\csc\phi)^2\right)\\
&\quad\times X_4\big(\mu_1+\mu_2,\mu_3;\mu_4,\mu_5,\mu_6;\kappa(\sin\phi)^2(\cos\phi)^2,\omega(\cos\phi)^2,\tau(\sin\phi)^2\big)d\phi.
\end{aligned}$$

Proof. By using the formula (1.2) of the M-beta function in the definition of M-Srivastava hypergeometric function H_B and by making similar calculations in the proof of Theorem 7, the proof is completed. □

Theorem 9. *Let $\Re(\mu_1)>0$, $\Re(\mu_2)>0$, $\Re(\alpha)>0$, $\Re(\rho)>0$. The following integral representation for M-Srivastava hypergeometric function H_B holds true:*

$$\begin{aligned}
&{}^{M}H_{B,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4,\mu_5,\mu_6;\kappa,\omega,\tau;\rho)\\
&\quad=\frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_0^{\infty}\frac{\Lambda^{\mu_1-1}}{(1+\Lambda)^{\mu_1+\mu_2}}\,{}_{p}^{\alpha}M_{q}^{\beta}\left(-2\rho-\rho\left(\Lambda+\frac{1}{\Lambda}\right)\right)
\end{aligned}$$

$$\times X_4\left(\mu_1+\mu_2,\mu_3;\mu_4,\mu_5,\mu_6;\frac{\kappa\Lambda}{(1+\Lambda)^2},\frac{\omega}{1+\Lambda},\frac{\tau\Lambda}{1+\Lambda}\right)d\Lambda.$$

Proof. By using the formula (1.3) of the M-beta function in the definition of M-Srivastava hypergeometric function H_B and by making similar calculations in the proof of Theorem 7, the proof is completed. □

Theorem 10. *Let $\Re(\mu_1)>0$, $\Re(\mu_2)>0$, $\Re(\alpha)>0$, $\Re(\rho)>0$. The following integral representation for the M-Srivastava hypergeometric function H_B holds true:*

$$\begin{aligned}&{}^{M}H_{B,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4,\mu_5,\mu_6;\kappa,\omega,\tau;\rho)\\&=\frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}(v-u)^{1-\mu_1-\mu_2}\int_u^v(\Lambda-u)^{\mu_1-1}(v-\Lambda)^{\mu_2-1}\,{}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho(v-u)^2}{(\Lambda-u)(v-\Lambda)}\right)\\&\quad\times X_4\left(\mu_1+\mu_2,\mu_3;\mu_4,\mu_5,\mu_6;\frac{\kappa(\Lambda-u)(v-\Lambda)}{(v-u)^2},\frac{\omega(v-\Lambda)}{v-u},\frac{\tau(\Lambda-u)}{v-u}\right)d\Lambda.\end{aligned}$$

Proof. By using the formula (1.4) of the M-beta function in the definition of M-Srivastava hypergeometric function H_B and by making similar calculations in the proof of Theorem 7, the proof is completed. □

5. Integral representations for M-Srivastava hypergeometric function H_C

Theorem 11. *Let $\Re(\mu_4)>\Re(\mu_1)>0$, $\Re(\alpha)>0$, $\Re(\rho)>0$. The following integral representation for the M-Srivastava hypergeometric function H_C holds true:*

$$\begin{aligned}&{}^{M}H_{C,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4;\kappa,\omega,\tau;\rho)\\&=\frac{\Gamma(\mu_4)}{\Gamma(\mu_1)\Gamma(\mu_4-\mu_1)}\int_0^1\Delta^{\mu_1-1}(1-\Delta)^{\mu_4-\mu_1-1}(1-\kappa\Delta)^{-\mu_2}(1-\tau\Delta)^{-\mu_3}\\&\quad\times{}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right){}_2F_1\left(\mu_2,\mu_3;\mu_4-\mu_1;\frac{\omega(1-\Delta)}{(1-\kappa\Delta)(1-\tau\Delta)}\right)d\Delta.\end{aligned}$$

Proof. By using the formula (1.1) of the M-beta function in the definition of M-Srivastava hypergeometric function H_C, we have

$$\begin{aligned}&{}^{M}H_{C,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4;\kappa,\omega,\tau;\rho)\\&=\sum_{m,n,k=0}^{\infty}\frac{(\mu_2)_{m+n}(\mu_3)_{n+k}}{(\mu_4)_n}\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\mu_1+m+k,\mu_4+n-\mu_1;\rho)}{B(\mu_1,\mu_4+n-\mu_1)}\frac{\kappa^m}{m!}\frac{\omega^n}{n!}\frac{\tau^k}{k!}\\&=\sum_{m,n,k=0}^{\infty}\frac{(\mu_2)_{m+n}(\mu_3)_{n+k}}{(\mu_4)_n}\frac{1}{B(\mu_1,\mu_4+n-\mu_1)}\int_0^1\Delta^{\mu_1+m+k-1}(1-\Delta)^{\mu_4+n-\mu_1-1}\\&\quad\times{}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)\frac{\kappa^m}{m!}\frac{\omega^n}{n!}\frac{\tau^k}{k!}d\Delta.\end{aligned}$$

By using equations (1.5), (1.6), (1.7) and multiplied by $\frac{\Gamma(\mu_4)\Gamma(\mu_4-\mu_1)}{\Gamma(\mu_4)\Gamma(\mu_4-\mu_1)}$, we get

$$
\begin{aligned}
&{}^{M}H_{C,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4;\kappa,\omega,\tau;\rho)\\
&= \frac{\Gamma(\mu_4)}{\Gamma(\mu_1)\Gamma(\mu_4-\mu_1)} \int_0^1 \Delta^{\mu_1-1}(1-\Delta)^{\mu_4-\mu_1-1}\, {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)\\
&\quad \times \sum_{m,n,k=0}^{\infty} \frac{(\mu_4)_n}{(\mu_4-\mu_1)_n} \frac{(\mu_2)_n(\mu_2+n)_m(\mu_3)_n(\mu_3+n)_k}{(\mu_4)_n} \frac{(\kappa\Delta)^m}{m!} \frac{(\omega(1-\Delta))^n}{n!} \frac{(\tau\Delta)^k}{k!} d\Delta\\
&= \frac{\Gamma(\mu_4)}{\Gamma(\mu_1)\Gamma(\mu_4-\mu_1)} \int_0^1 \Delta^{\mu_1-1}(1-\Delta)^{\mu_4-\mu_1-1}\, {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)\\
&\quad \times \sum_{n=0}^{\infty} \frac{(\mu_2)_n(\mu_3)_n}{(\mu_4-\mu_1)_n} \sum_{m=0}^{\infty} (\mu_2+n)_m \frac{(\kappa\Delta)^m}{m!} \sum_{k=0}^{\infty} (\mu_3+n)_k \frac{(\tau\Delta)^k}{k!} \frac{(\omega(1-\Delta))^n}{n!} d\Delta.
\end{aligned}
$$

By using equation (1.8), we obtain

$$
\begin{aligned}
&{}^{M}H_{C,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4;\kappa,\omega,\tau;\rho)\\
&= \frac{\Gamma(\mu_4)}{\Gamma(\mu_1)\Gamma(\mu_4-\mu_1)} \int_0^1 \Delta^{\mu_1-1}(1-\Delta)^{\mu_4-\mu_1-1}\, {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)\\
&\quad \times \sum_{n=0}^{\infty} \frac{(\mu_2)_n(\mu_3)_n}{(\mu_4-\mu_1)_n} (1-\kappa\Delta)^{-\mu_2-n}(1-\tau\Delta)^{-\mu_3-n} \frac{(\omega(1-\Delta))^n}{n!} d\Delta\\
&= \frac{\Gamma(\mu_4)}{\Gamma(\mu_1)\Gamma(\mu_4-\mu_1)} \int_0^1 \Delta^{\mu_1-1}(1-\Delta)^{\mu_4-\mu_1-1}(1-\kappa\Delta)^{-\mu_2}(1-\tau\Delta)^{-\mu_3}\\
&\quad \times {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) {}_2F_1\left(\mu_2,\mu_3;\mu_4-\mu_1;\frac{\omega(1-\Delta)}{(1-\kappa\Delta)(1-\tau\Delta)}\right) d\Delta,
\end{aligned}
$$

which completes the proof. □

Theorem 12. *Let* $\Re(\mu_1)>0$, $\Re(\mu_2)>0$, $\Re(\mu_4)>0$, $\Re(\mu_4-\mu_1-\mu_2)>0$, $\Re(\alpha)>0$, $\Re(\rho)>0$. *The following integral representation for the M-Srivastava hypergeometric function* H_C *holds true:*

$$
\begin{aligned}
&{}^{M}H_{C,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4;\kappa,\omega,\tau;\rho)\\
&= \frac{\Gamma(\mu_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_4-\mu_1-\mu_2)} \int_0^1\int_0^1 \Delta^{\mu_1-1}\Lambda^{\mu_2-1}(1-\Delta)^{\mu_4-\mu_1-1}\\
&\quad \times (1-\Lambda)^{\mu_4-\mu_1-\mu_2-1}(1-\kappa\Delta)^{\mu_3-\mu_2}\left(1-\kappa\Delta-\omega\Lambda-\tau\Delta+\omega\Delta\Lambda+\kappa\tau\Delta^2\right)^{-\mu_3}\\
&\quad \times {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Lambda(1-\Lambda)}\right) d\Delta d\Lambda.
\end{aligned}
$$

Proof. By using the formula (1.1) of the M-beta function in the definition of M-Srivastava hypergeometric function H_C and by making similar calculations in the proof of Theorem 11, the proof is completed. □

Theorem 13. *Let* $\Re(\mu_4) > \Re(\mu_1) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. *The following integral representation for the M-Srivastava hypergeometric function* H_C *holds true:*

$$
\begin{aligned}
&{}^{M}H_{C,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4;\kappa,\omega,\tau;\rho)\\
&\quad = \frac{2\Gamma(\mu_4)}{\Gamma(\mu_1)\Gamma(\mu_4-\mu_1)} \int_0^{\frac{\pi}{2}} (\sin\phi)^{2\mu_1-1}(\cos\phi)^{2\mu_4-2\mu_1-1}\left(1-\kappa(\sin\phi)^2\right)^{-\mu_2}\\
&\quad\quad \times \left(1-\tau(\sin\phi)^2\right)^{-\mu_3}\, {}_{p}^{\alpha}M_{q}^{\beta}\left(-\rho(\sec\phi)^2(\csc\phi)^2\right)\\
&\quad\quad \times {}_2F_1\left(\mu_2,\mu_3;\mu_4-\mu_1;\frac{\omega(\cos\phi)^2}{(1-\kappa(\sin\phi)^2)(1-\tau(\sin\phi)^2)}\right)d\phi.
\end{aligned}
$$

Proof. By using the formula (1.2) of the M-beta function in the definition of M-Srivastava hypergeometric function H_C and by making similar calculations in the proof of Theorem 11, the proof is completed. □

Theorem 14. *Let* $\Re(\mu_4) > \Re(\mu_1) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. *The following integral representation for the M-Srivastava hypergeometric function* H_C *holds true:*

$$
\begin{aligned}
&{}^{M}H_{C,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4;\kappa,\omega,\tau;\rho)\\
&\quad = \frac{\Gamma(\mu_4)}{\Gamma(\mu_1)\Gamma(\mu_4-\mu_1)} \int_0^{\infty} \Lambda^{\mu_1-1}(1+\Lambda)^{\mu_2+\mu_3-\mu_4}(1+\Lambda-\kappa\Lambda)^{-\mu_2}\\
&\quad\quad \times (1+\Lambda-\tau\Lambda)^{-\mu_3}\, {}_{p}^{\alpha}M_{q}^{\beta}\left(-2\rho-\rho\left(\Lambda+\frac{1}{\Lambda}\right)\right)\\
&\quad\quad \times {}_2F_1\left(\mu_2,\mu_3;\mu_4-\mu_1;\frac{\omega(1+\Lambda)}{(1+\Lambda-\kappa\Lambda)(1+\Lambda-\tau\Lambda)}\right)d\Lambda.
\end{aligned}
$$

Proof. By using the formula (1.3) of the M-beta function in the definition of M-Srivastava hypergeometric function H_C and by making similar calculations in the proof of Theorem 11, the proof is completed. □

Theorem 15. *Let* $\Re(\mu_4) > \Re(\mu_1) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. *The following integral representation for the M-Srivastava hypergeometric function* H_C *holds true:*

$$
\begin{aligned}
&{}^{M}H_{C,p,q}^{(\alpha,\beta)}(\mu_1,\mu_2,\mu_3;\mu_4;\kappa,\omega,\tau;\rho)\\
&\quad = \frac{\Gamma(\mu_4)}{\Gamma(\mu_1)\Gamma(\mu_4-\mu_1)}(v-u)^{1+\mu_2+\mu_3-\mu_4} \int_u^{v} (\Lambda-u)^{\mu_1-1}(v-\Lambda)^{\mu_4-\mu_1-1}\\
&\quad\quad \times \left(v-u-\kappa(\Lambda-u)\right)^{-\mu_2}\left(v-u-\tau(\Lambda-u)\right)^{-\mu_3}\, {}_{p}^{\alpha}M_{q}^{\beta}\left(\frac{-\rho(v-u)^2}{(\Lambda-u)(v-\Lambda)}\right)\\
&\quad\quad \times {}_2F_1\left(\mu_2,\mu_3;\mu_4-\mu_1;\frac{\omega(v-\Lambda)(v-u)}{\left(v-u-\kappa(\Lambda-u)\right)\left(v-u-\tau(\Lambda-u)\right)}\right)d\Lambda.
\end{aligned}
$$

Proof. By using the formula (1.4) of the M-beta function in the definition of M-Srivastava hypergeometric function H_C and by making similar calculations in the proof of Theorem 11, the proof is completed. □

6. Solution of fractional differential equations involving M-Srivastava hypergeometric functions

Example 1. Let $1 < \Re(\varepsilon) \leq 2$ and $\Re(\alpha) > 0$. Assume that, the fractional differential equation

$$^{c}D^{\varepsilon}_{\rho}\{f(\rho)\} = {}^{M}H^{(\alpha,\beta)}_{A,p,q}(\xi_1,\dots,\xi_p;\eta_1,\dots,\eta_q;\mu_1,\mu_2,\mu_3;\mu_4,\mu_5;\kappa,\omega,\tau;\varepsilon\rho)$$

with the initial conditions

$$f(0) = f'(0) = 0$$

are given. By considering equation (1.9) and by applying the Laplace transform to the fractional differential equation, we have

$$\mathfrak{L}\left\{^{c}D^{\varepsilon}_{\rho}\{f(\rho)\};s\right\} = \mathfrak{L}\left\{^{M}H^{(\alpha,\beta)}_{A,p,q}(\xi_1,\dots,\xi_p;\eta_1,\dots,\eta_q;\mu_1,\mu_2,\mu_3;\mu_4,\mu_5;\kappa,\omega,\tau;\varepsilon\rho);s\right\}$$

and then,

$$s^{\varepsilon}F(s) - s^{\varepsilon-1}f(0) - s^{\varepsilon-2}f'(0) = \frac{^{M}H^{(\alpha,\beta)}_{A,p+1,q}\left(\xi_1,\dots,\xi_p,1;\eta_1,\dots,\eta_q;\mu_1,\mu_2,\mu_3;\mu_4,\mu_5;\kappa,\omega,\tau;\frac{\varepsilon}{s}\right)}{s}.$$

By using the initial conditions, we get

$$F(s) = \frac{^{M}H^{(\alpha,\beta)}_{A,p+1,q}\left(\xi_1,\dots,\xi_p,1;\eta_1,\dots,\eta_q;\mu_1,\mu_2,\mu_3;\mu_4,\mu_5;\kappa,\omega,\tau;\frac{\varepsilon}{s}\right)}{s^{\varepsilon+1}}.$$

By applying the inverse Laplace transform to the last equation and by making the necessary calculations, we obtain the solution function as:

$$f(\rho) = \frac{^{M}H^{(\alpha,\beta)}_{A,p+1,q+1}(\xi_1,\dots,\xi_p,1;\eta_1,\dots,\eta_q,1+\varepsilon;\mu_1,\mu_2,\mu_3;\mu_4,\mu_5;\kappa,\omega,\tau;\varepsilon\rho)}{\Gamma(1+\varepsilon)\rho^{-\varepsilon}}.$$

Example 2. Let $1 < \Re(\varepsilon) \leq 2$ and $\Re(\alpha) > 0$. Assume that, the fractional differential equation

$$^{c}D^{\varepsilon}_{\rho}\{f(\rho)\} = {}^{M}H^{(\alpha,\beta)}_{B,p,q}(\xi_1,\dots,\xi_p;\eta_1,\dots,\eta_q;\mu_1,\mu_2,\mu_3;\mu_4,\mu_5,\mu_6;\kappa,\omega,\tau;\varepsilon\rho)$$

with the initial conditions

$$f(0) = f'(0) = 0$$

are given. By considering the equation (1.9) and by applying the Laplace transform to the fractional differential equation, we have

$$\mathfrak{L}\left\{^{c}D^{\varepsilon}_{\rho}\{f(\rho)\};s\right\} = \mathfrak{L}\left\{^{M}H^{(\alpha,\beta)}_{B,p,q}(\xi_1,\dots,\xi_p;\eta_1,\dots,\eta_q;\mu_1,\mu_2,\mu_3;\mu_4,\mu_5,\mu_6;\kappa,\omega,\tau;\varepsilon\rho);s\right\}$$

and then,

$$s^{\varepsilon}F(s)-s^{\varepsilon-1}f(0)-s^{\varepsilon-2}f'(0) = \frac{{}^{M}H^{(\alpha,\beta)}_{B,p+1,q}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q;\mu_1,\mu_2,\mu_3;\mu_4,\mu_5,\mu_6;\kappa,\omega,\tau;\frac{\varepsilon}{s}\right)}{s}.$$

By using the initial conditions, we get

$$F(s)=\frac{{}^{M}H^{(\alpha,\beta)}_{B,p+1,q}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q;\mu_1,\mu_2,\mu_3;\mu_4,\mu_5,\mu_6;\kappa,\omega,\tau;\frac{\varepsilon}{s}\right)}{s^{\varepsilon+1}}.$$

By applying the inverse Laplace transform to the last equation and by making the necessary calculations, we obtain the solution function as:

$$f(\rho)=\frac{{}^{M}H^{(\alpha,\beta)}_{B,p+1,q+1}(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1+\varepsilon;\mu_1,\mu_2,\mu_3;\mu_4,\mu_5,\mu_6;\kappa,\omega,\tau;\varepsilon\rho)}{\Gamma(1+\varepsilon)\rho^{-\varepsilon}}.$$

Example 3. Let $1<\Re(\varepsilon)\le 2$ and $\Re(\alpha)>0$. Assume that, the fractional differential equation

$${}^{c}D^{\varepsilon}_{\rho}\{f(\rho)\}={}^{M}H^{(\alpha,\beta)}_{C,p,q}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\mu_1,\mu_2,\mu_3;\mu_4;\kappa,\omega,\tau;\varepsilon\rho)$$

with the initial conditions

$$f(0)=f'(0)=0$$

are given. By considering the equation (1.9) and by applying the Laplace transform to the fractional differential equation, we have

$$\mathfrak{L}\left\{{}^{c}D^{\varepsilon}_{\rho}\{f(\rho)\};s\right\}=\mathfrak{L}\left\{{}^{M}H^{(\alpha,\beta)}_{C,p,q}(\xi_1,\ldots,\xi_p;\eta_1,\ldots,\eta_q;\mu_1,\mu_2,\mu_3;\mu_4;\kappa,\omega,\tau;\varepsilon\rho);s\right\}$$

and then,

$$s^{\varepsilon}F(s)-s^{\varepsilon-1}f(0)-s^{\varepsilon-2}f'(0) = \frac{{}^{M}H^{(\alpha,\beta)}_{C,p+1,q}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q;\mu_1,\mu_2,\mu_3;\mu_4;\kappa,\omega,\tau;\frac{\varepsilon}{s}\right)}{s}.$$

By using the initial conditions, we get

$$F(s)=\frac{{}^{M}H^{(\alpha,\beta)}_{C,p+1,q}\left(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q;\mu_1,\mu_2,\mu_3;\mu_4;\kappa,\omega,\tau;\frac{\varepsilon}{s}\right)}{s^{\varepsilon+1}}.$$

By applying the inverse Laplace transform to the last equation and by making the necessary calculations, we obtain the solution function as:

$$f(\rho)=\frac{{}^{M}H^{(\alpha,\beta)}_{C,p+1,q+1}(\xi_1,\ldots,\xi_p,1;\eta_1,\ldots,\eta_q,1+\varepsilon;\mu_1,\mu_2,\mu_3;\mu_4;\kappa,\omega,\tau;\varepsilon\rho)}{\Gamma(1+\varepsilon)\rho^{-\varepsilon}}.$$

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