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ON MODIFIED INTUITIONISTIC FUZZY SOFT METRIC SPACES AND APPLICATION

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Abstract. This paper introduces self maps satisfying (CLR_{α}) and $(CLR_{\alpha\beta})$ property in Modified Intuitionistic Fuzzy Soft Metric Space (MIFSMS). Moreover, we have also extended common fixed point theorems for weakly compatible maps satisfying common limit range property in the setting of MIFSMS. We have also given an application utilising our new results.

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1. INTRODUCTION

To deal with uncertainty Zadeh [15] presented Fuzzy Sets that includes the membership function only. Further K.T. Atanassov [1] gave Intuitionistic Fuzzy Set to overcome the limitations of Fuzzy sets. Moreover, in case of data consisting of parameters, Molodstov [13] gave Soft Sets to deal with the uncertainties. Das and Samanta [4, 5] applied the concept of soft sets to metric spaces and hence presented Soft Metric Spaces utilizing soft points of soft sets. Maji et al [12], in 2001, introduced Fuzzy Soft Sets. Beaula and Gunaseli [2] applied the metric space concept to Fuzzy Soft Sets and introduced Fuzzy Soft Metric Spaces using fuzzy soft point and defined some of its properties.

Saadiat et al [14] gave another important concept of Modified Intuitionistic Fuzzy Metric Spaces by using continuous $t-$representable norm. Saadiat et al [14] also gave fixed point results for compatible and weakly compatible maps in Modified Intuitionistic Fuzzy Metric Spaces. The soft metric space (in short, SMS) was a result of employing the soft set theory to metric space by Das and Samanta [4–6]. Later on, various FPT's were stated and proved by different researchers in SMS. Vishal and Aanchal [9] introduced Modified Intuitionistic Fuzzy Soft Metric Space (MIFSMS) and proved the fixed point theorems in its settings. S. Chauhan et al. [3]

uses the common limit range property to gave some fixed point theorems for weakly compatible maps in modified intuitionistic fuzzy metric spaces.

The main objective of our paper is to introduce common limit range property and fixed point theorems for two pairs of self maps satisfying the above said property in the setting of MIFSMS.

2. Preliminaries

The following section includes the basic definitions and results already proved in the literature that will be the foundation of our new results.

In this section, X is taken as universe, U represents the parameter set, $\bar{U}$ is taken as the absolute soft set and $SP(\bar{X})$ denotes the collection of all the soft points of $\bar{X}$.

Definition 1 ([13, Definition 2.1])**.** Soft set is a pair (S,U) on a universe X, where U represents the parameter set and S defines the map from U to power set of X i.e. $S : U \to P(X)$.

Definition 2 ([5, Definition 2.2.1])**.** A soft metric space is a 3-tuple $(\bar{X},\bar{\mu},U)$, where the soft metric $\bar{\mu}$ is defined from $SP(\bar{X}) \times SP(\bar{X})$ to $R(U)$. Here, $R(U)$ is the set containing non-negative soft real numbers and $\bar{\mu}$ satisfies the given conditions for all $\bar{u}_{e_1},\bar{v}_{e_2},\bar{w}_{e_3} \in SP(\bar{X})$:

(i) $\bar{\mu}(\bar{u}_{e_1},\bar{v}_{e_2}) > \bar{0}$,
(ii) $\bar{\mu}(\bar{u}_{e_1},\bar{v}_{e_2}) = \bar{0}$ iff $\bar{u}_{e_1} = \bar{v}_{e_2}$,
(iii) $\bar{\mu}(\bar{u}_{e_1},\bar{v}_{e_2}) = \bar{\mu}(\bar{v}_{e_2},\bar{u}_{e_1})$,
(iv) $\bar{\mu}(\bar{u}_{e_1},\bar{v}_{e_2}) \leq \bar{\mu}(\bar{u}_{e_1},\bar{w}_{e_3}) + \bar{\mu}(\bar{w}_{e_3},\bar{v}_{e_2})$.

Definition 3 ([12, Definition 2.1])**.** Fuzzy soft set is a pair (S,U) over a universe X, where U represents the parameter set and S defines the map from U to *F(X)*, where latter is the set containing fuzzy subsets in universe X i.e. $S : U \to$ *F(X)*.

Definition 4 ([8, Definition 3.5])**.** A soft fuzzy metric space is the 3-tuple $(\bar{X},S,*)$, where soft fuzzy metric on $\bar{X}$ is given by map $S : SP(\bar{X}) \times SP(\bar{X}) \times (0,\infty) \to [0,1]$ satisfying the given condition for all $\bar{u}_{e_1},\bar{v}_{e_2},\bar{w}_{e_3} \in SP(\bar{X}), s,t > 0$:

(i) $S(\bar{u}_{e_1},\bar{v}_{e_2},t) > 0$,
(ii) $S(\bar{u}_{e_1},\bar{v}_{e_2},t) = 1$ iff $\bar{u}_{e_1} = \bar{v}_{e_2}$,
(iii) $S(\bar{u}_{e_1},\bar{v}_{e_2},t) = S(\bar{v}_{e_2},\bar{u}_{e_1},t)$,
(iv) $S(\bar{u}_{e_1},\bar{v}_{e_2},t+s) \geq S(\bar{u}_{e_1},\bar{w}_{e_3},t) * S(\bar{w}_{e_3},\bar{v}_{e_2},s)$,
(v) $S(\bar{u}_{e_1},\bar{v}_{e_2},.) : (0,\infty) \to [0,1]$ is continuous.

Definition 5 ([14, Definition 1.6])**.** A *modified intuitionistic fuzzy metric space* is given by 3-tuple $(X,\varpi_{M,N},\Theta)$, where X is any arbitrary set, M,N are fuzzy sets from $X^2 \times (0,\infty)$ to $[0,1]$ satisfying $M(p,q,t) + N(p,q,t) \leq 1$ for all $p,q \in X$ and $t > 0$, continuous $t-$representable norm is given as Θ and $\varpi_{M,N}$ is a function from $X^2 \times (0,\infty)$ to L^* fulfilling the given conditions for all $p,q,r \in X$ and $t,s > 0$:

(i) $\varpi_{M,N}(p,q,t) >_{L^*} 0_{L^*}$;
(ii) $\varpi_{M,N}(p,q,t) = 1_{L^*}$ iff $p = q$;
(iii) $\varpi_{M,N}(p,q,t) = \varpi_{M,N}(q,p,t)$;
(iv) $\varpi_{M,N}(p,q,t+s) \geq_{L^*} \Theta(\varpi_{M,N}(p,r,t), \varpi_{M,N}(r,q,s))$;
(v) $\varpi_{M,N}(p,q,.) : (0,\infty) \to L^*$ is continuous.

Here, *modified intuitionistic fuzzy metric* $\varpi_{M,N}$ is given as

$$\varpi_{M,N}(p,q,t) = (M(p,q,t), N(p,q,t)).$$

Definition 6 ([7, Definition 2.1]). A map $S : X \to IF^U$, where X is an arbitrary set and IF^U is the set consisting of all the intuitionistic fuzzy subsets of U, then S is a function defined for every $a \in X$ as $S(a) = \{< u, \mu_{S(a)}, \nu_{S(a)} >: u \in U\}$, where degree of association and non association is given by $\mu_{S(a)}$ and $\nu_{S(a)}$ respectively.

Definition 7 ([7, Definition 4.1]). A map Θ defined from $(L^*)^2$ to L^* is a triangular norm $(t-\text{norm})$ if it satisfies the given conditions:
(i) $\Theta(p, 1_{L^*}) = p$, for all $p \in L^*$;
(ii) $\Theta(p,q) = \Theta(q,p)$, for all $(p,q) \in (L^*)^2$;
(iii) $\Theta(p, \Theta(q,r)) = \Theta(\Theta(p,q), r)$, for all $(p,q,r) \in (L^*)^3$;
(iv) $p \leq p'$ and $q \leq q' \Rightarrow \Theta(p,q) \leq_{L^*} \Theta(p',q')$, for all $(p,p',q,q') \in (L^*)^4$.

Definition 8 ([7, Definition 4.3]). A continuous $t-$representable norm is a continuous $t-\text{norm}$ Θ on L^* iff it implies the existence of a $t-\text{conorm}$ $\diamond$ on $[0,1]$, which is continuous, so that

$$\Theta(p,q) = (p_1 * q_1, p_2 \diamond q_2),$$

for all $p = (p_1, p_2)$ and $q = (q_1, q_2) \in L^*$.

Definition 9 ([9, Definition 3.3]). A MIFSMS is a 3-tuple $(\bar{X}, \varpi_{M,N}, \Theta)$, where $\bar{X}$ is any set, M and N are soft fuzzy sets, Θ is a continuous $t-$representable norm, $\varpi_{M,N}$ is a mapping from $SP(\bar{X}) \times SP(\bar{X}) \times (0,\infty)$ to L^* so that the given assertions are satisfied for all $\bar{p}_{e_1}, \bar{q}_{e_2}, \bar{r}_{e_3} \in SP(\bar{X})$ and $s,t > 0$:
(i) $\varpi_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, t) >_{L^*} 0_{L^*}$;
(ii) $\varpi_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, t) = 1_{L^*}$ iff $\bar{p}_{e_1} = \bar{q}_{e_2}$;
(iii) $\varpi_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, t) = \varpi_{M,N}(\bar{q}_{e_2}, \bar{p}_{e_1}, t)$;
(iv) $\varpi_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, t+s) \geq_{L^*} \Theta(\varpi_{M,N}(\bar{p}_{e_1}, \bar{r}_{e_3}, t), \varpi_{M,N}(\bar{r}_{e_3}, \bar{q}_{e_2}, s))$;
(v) $\varpi_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, .) : (0,\infty) \to L^*$ is continuous.

Then, $\varpi_{M,N}$ is called MIFSM on $\bar{X}$. Here the level of closeness, non closeness between $\bar{p}_{e_1}, \bar{q}_{e_2}$ w.r.t. t is given by the maps $M(\bar{p}_{e_1}, \bar{q}_{e_2}, t)$ and $N(\bar{p}_{e_1}, \bar{q}_{e_2}, t)$ respectively and metric $\varpi_{M,N}$ is given as

$$\varpi_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, t) = (M(\bar{p}_{e_1}, \bar{q}_{e_2}, t), N(\bar{p}_{e_1}, \bar{q}_{e_2}, t)).$$

Remark 1 ([9, Remark 3.1]). The function $M(\bar{p}_{e_1}, \bar{q}_{e_2}, t)$ is increasing and the function $N(\bar{p}_{e_1}, \bar{q}_{e_2}, t)$ is decreasing in a MIFSMS $\bar{X}$, for all $\bar{p}_{e_1}, \bar{q}_{e_2} \in SP(\bar{X})$.

Example 1 ([9, Example 3.1]). Consider $(\bar{X},\mu)$ be a soft metric space and M,N be soft fuzzy sets on $SP(\bar{X})\times SP(\bar{X})\times(0,\infty)$ as given below:

$$\begin{aligned}\varpi_{M,N}(\bar{p}_{e_1},\bar{q}_{e_2},t) &= (M(\bar{p}_{e_1},\bar{q}_{e_2},t),N(\bar{p}_{e_1},\bar{q}_{e_2},t))\\ &= \left(\frac{ht^n}{ht^n+m\mu(\bar{p}_{e_1},\bar{q}_{e_2})},\frac{m\mu(\bar{p}_{e_1},\bar{q}_{e_2})}{ht^n+m\mu(\bar{p}_{e_1},\bar{q}_{e_2})}\right)\end{aligned}$$

for all $h,t,m,n\in R^+;\Theta(\bar{r},\bar{s})=(\bar{r}_1\bar{s}_1,\min(\bar{r}_2+\bar{s}_2,1))$ for all $\bar{r}=(\bar{r}_1,\bar{r}_2)$ and $\bar{s}=(\bar{s}_1,\bar{s}_2)\in L^*$. The above $(\bar{X},\varpi_{M,N},\Theta)$ serves as an example of a MIFSMS.

Lemma 1 ([9, Lemma 3.3])**.** *Consider $(\bar{X},\varpi_{M,N},\Theta)$ be a MIFSMS. Then $\varpi_{M,N}(\bar{p}_{e_1},\bar{q}_{e_2},t)$ is increasing with respect to $t>0$ in $(L^*,\leq_{L^*})$ for all $\bar{p}_{e_1},\bar{q}_{e_2}\in SP(\bar{X})$.*

Definition 10 ([9, Definition 3.4])**.** Consider $(\bar{X},\varpi_{M,N},\Theta)$ be a MIFSMS. $\varpi_{M,N}$ is called continuous on $SP(\bar{X})\times SP(\bar{X})\times(0,\infty)$ if

$$\lim_{n\to\infty}\varpi_{M,N}(\bar{p}_{e_n},\bar{q}_{e_n},t_n)=\varpi_{M,N}(\bar{p}_{e_1},\bar{q}_{e_2},t),$$

where $\{(\bar{p}_{e_n},\bar{q}_{e_n},t_n)\}$ is a sequence converging to $(\bar{p}_{e_1},\bar{q}_{e_2},t)$.

Lemma 2 ([9, Lemma 3.4])**.** *For $(\bar{X},\varpi_{M,N},\Theta)$ be a MIFSMS, then $\varpi_{M,N}$ is continuous on $SP(\bar{X})\times SP(\bar{X})\times(0,\infty)$.*

Definition 11 ([9, Definition 4.1])**.** The open ball in a MIFSMS $(\bar{X},\varpi_{M,N},\Theta)$ is defined by $B(\bar{p}_{e_1},\alpha,t)$, having center $\bar{p}_{e_1}\in\bar{X}$ and radius $0<\alpha<1$ for any $t>0$,

$$B(\bar{p}_{e_1},\alpha,t)=\{\bar{q}_{e_2}\in\bar{X}:\varpi_{M,N}(\bar{p}_{e_1},\bar{q}_{e_2},t)>_{L^*}(N_s(\alpha),\alpha)\}.$$

Theorem 1 ([9, Theorem 4.1])**.** *Every open ball $B(\bar{p}_{e_1},\alpha,t)$ in MIFSMS is an open set.*

Remark 2 ([9, Remark 4.1]). The topology procured by MIFSM $\varpi_{M,N}$ on $\bar{X}$ in MIFSMS $(\bar{X},\varpi_{M,N},\Theta)$ is defined as

$$\begin{aligned}\tau_{M,N}=\{Y\subseteq\bar{X}:\ &\text{for every } \bar{p}_{e_1}\in Y,\ \text{there exist } t>0 \text{ and } \alpha\in(0,1)\|\\ &\text{so that } B(\bar{p}_{e_1},\alpha,t)\subseteq Y\}.\end{aligned}$$

Theorem 2 ([9, Theorem 4.2])**.** *A MIFSMS $(\bar{X},\varpi_{M,N},\Theta)$ is a Hausdorff space.*

Theorem 3 ([9, Theorem 4.4])**.** *If $(\bar{X},\varpi_{M,N},\Theta)$ be MIFSMS and $\tau_{M,N}$ is a topology on $\bar{X}$. Then $\bar{p}_{e_n}\to\bar{p}_{e_1}$ iff*

$$\lim_{n\to\infty}\varpi_{M,N}(\bar{p}_{e_n},\bar{p}_{e_1},t)=1_{L^*}$$

for sequence $\{\bar{p}_{e_n}\}$ in $\bar{X}$.

Definition 12 ([9, Definition 4.3])**.** Consider $(\bar{X},\varpi_{M,N},\Theta)$ be MIFSMS and $\{\bar{p}_{e_n}\}$ be any sequence in $\bar{X}$, then

1. the sequence is Cauchy iff for every $t > 0$, there exist $n_o \in N$ so that
$$\lim_{n_o \to \infty} \varpi_{M,N}(\bar{p}_{e_n}, \bar{p}_{e_{n+m}}, t) = 1_{L^*}$$
for each $n, m \geq n_o$.
2. the sequence converges to $\bar{p}$ iff for every $t > 0$,
$$\lim_{n \to \infty} \varpi_{M,N}(\bar{p}_{e_n}, \bar{p}, t) = 1_{L^*}.$$

Lemma 3 ([10, Lemma 3.1]). *Consider $(\bar{X}, \varpi_{M,N}, \Theta)$ be a MIFSMS and for all $\bar{p}_{e_1}, \bar{q}_{e_2} \in SP(\bar{X})$, $t > 0$, $0 < k < 1$, (2.1) holds, ie.*
$$\varpi_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, kt) \geq_{L^*} \varpi_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, t). \tag{2.1}$$
Then $\bar{p}_{e_1} = \bar{q}_{e_2}$.

Definition 13 ([10, Definition 3.2]). Consider two self maps A and S on MIFSMS $(\bar{X}, \varpi_{M,N}, \Theta)$. The maps are weakly compatible if
$$A\bar{p}_{e_1} = S\bar{p}_{e_1} \Rightarrow AS\bar{p}_{e_1} = SA\bar{p}_{e_1}$$
for some $\bar{p}_{e_1} \in SP(\bar{X})$.

3. Main Results

In this section we are defining common limit range property in the setting of MIFSMS as follows:

Definition 14. A pair of self maps (ϕ, α) in a MIFSMS $(\bar{X}, \varpi_{M,N}, \Theta)$ satisfies CLR_α property, known as common limit range property in relation to α, if there exists a sequence $\{\bar{p}_{e_n}\}$ in $\bar{X}$ so that (3.1) holds for some $\bar{p} \in \alpha(\bar{X})$ and $t > 0$,
$$\lim_{n \to \infty} \varpi_{M,N}(\phi\bar{p}_{e_n}, \bar{p}, t) = \lim_{n \to \infty} \varpi_{M,N}(\alpha\bar{p}_{e_n}, \bar{p}, t) = 1_{L^*}. \tag{3.1}$$

Definition 15. Two pairs of self maps (ϕ, α) and (η, β) in a MIFSMS $(\bar{X}, \varpi_{M,N}, \Theta)$ satisfies $(CLR_{\alpha\beta})$ property, known as common limit range property in relation to α and β, if there exists two sequences $\{\bar{p}_{e_n}\}$ and $\{\bar{q}_{e_n}\}$ in $\bar{X}$ so that (3.2) holds for some $\bar{h} \in \alpha(\bar{X}) \bigcap \beta(\bar{X})$ and $t > 0$,
$$\begin{aligned} \lim_{n \to \infty} \varpi_{M,N}(\phi\bar{p}_{e_n}, \bar{h}, t) &= \lim_{n \to \infty} \varpi_{M,N}(\alpha\bar{p}_{e_n}, \bar{h}, t) = \lim_{n \to \infty} \varpi_{M,N}(\eta\bar{q}_{e_n}, \bar{h}, t) \\ &= \lim_{n \to \infty} \varpi_{M,N}(\beta\bar{q}_{e_n}, \bar{h}, t) = 1_{L^*}. \end{aligned} \tag{3.2}$$

Example 2. Consider $(\bar{X}, \varpi_{M,N}, \Theta)$ be a MIFSMS, where $\bar{X} = [0, \infty)$ and
$$\varpi_{M,N}(\bar{\iota}, \bar{\kappa}, \rho) = (M(\bar{\iota}, \bar{\kappa}, \rho), N(\bar{\iota}, \bar{\kappa}, \rho)) = \left(\frac{\rho}{\rho + |\bar{\iota} - \bar{\kappa}|}, \frac{|\bar{\iota} - \bar{\kappa}|}{\rho + |\bar{\iota} - \bar{\kappa}|} \right),$$
for every $\bar{\iota}, \bar{\kappa} \in \bar{X}$ and $\rho > 0$.

The self maps ϕ and α on $\bar{X}$ are defined as $\phi(\bar{\iota}) = \bar{\iota} + 1$ and $\alpha(\bar{\iota}) = 3\bar{\iota}$, for every $\bar{\iota} \in \bar{X}$.

Take a sequence $\{\bar{\iota}_{e_n}\} = \{1 + \frac{1}{n}\}$ in $\bar{X}$, then
$\varpi_{M,N}(\phi\bar{\iota}_{e_n}, 3, \rho) = \varpi_{M,N}(\alpha\bar{\iota}_{e_n}, 3, \rho) = 1_{L^*}$, where $3 \in \alpha(\bar{X})$ and $\rho > 0$. Thus the maps ϕ and α satisfies (CLR_α) property.

Example 3. Consider $(\bar{X}, \varpi_{M,N}, \Theta)$ be a MIFSMS, where $\bar{X} = [3,27)$, $\Theta(\bar{p}, \bar{q}) = (\bar{p}_1\bar{q}_1, \min\{\bar{p}_2 + \bar{q}_2, 1\})$, for every $\bar{p} = (\bar{p}_1, \bar{p}_2)$ and $\bar{q} = (\bar{q}_1, \bar{q}_2) \in L^*$ with MIFSM ϖ defined as follows for every $\bar{\iota}, \bar{\kappa} \in \bar{X}$,

$$\varpi(\bar{\iota}, \bar{\kappa}, \rho) = \left(\frac{\rho}{\rho + |\bar{\iota} - \bar{\kappa}|}, \frac{|\bar{\iota} - \bar{\kappa}|}{\rho + |\bar{\iota} - \bar{\kappa}|} \right).$$

Consider $\Upsilon : L^* \to L^*$ defined as $\Upsilon(\rho_1, \rho_2) = (\sqrt{\rho_1}, 0)$ for every $\rho = (\rho_1, \rho_2) \in L^* \ \{0_{L^*},\ 1_{L^*})$. Let the self maps ϕ, η, α and β defined as

$$\phi(\bar{\iota}) = \begin{cases} 3 & \text{for } \bar{\iota} \in \{3\} \cup (11,27), \\ 21 & \text{for } 3 < \bar{\iota} \leq 11. \end{cases} \qquad \eta(\bar{\iota}) = \begin{cases} 3 & \text{for } \bar{\iota} \in \{3\} \cup (11,27), \\ 8 & \text{for } 3 < \bar{\iota} \leq 11. \end{cases}$$

$$\alpha(\bar{\iota}) = \begin{cases} 3 & \text{for } \bar{\iota} = 3, \\ 12 & \text{for } 3 < \bar{\iota} \leq 11, \\ \frac{\bar{\iota}+1}{4} & \text{for } 11 < \bar{\iota} < 27. \end{cases} \qquad \beta(\bar{\iota}) = \begin{cases} 3 & \text{for } \bar{\iota} = 3, \\ 20 & \text{for } 3 < \bar{\iota} \leq 11, \\ \bar{\iota} - 8 & \text{for } 11 < \bar{\iota} < 27. \end{cases}$$

Take sequences $\{\bar{\iota}_{e_n} = 11 + \frac{1}{n}\}$, $\{\bar{\kappa}_{e_n} = 3\}$, then we have

$$\lim_{n\to\infty} \phi(\bar{\iota}_{e_n}) = \lim_{n\to\infty} \alpha(\bar{\iota}_{e_n}) = \lim_{n\to\infty} \eta(\bar{\kappa}_{e_n}) = \lim_{n\to\infty} \beta(\bar{\kappa}_{e_n}) = 3 \in \alpha(\bar{X}) \cap \beta(\bar{X}).$$

Hence, (ϕ, α) and (η, β) satisfies $CLR_{\alpha\beta}$ property.

Definition 16. Two families of self mappings $\{\phi_i\}_{i\in\{1,2,3,\ldots,m\}}$ and $\{\alpha_k\}_{k\in\{1,2,3,\ldots,n\}}$ are said to be pairwise commuting if they satisfies the following conditions:

1. $\phi_i\phi_j = \phi_j\phi_i$ for every $i, j \in \{1,2,3,\ldots,m\}$,
2. $\alpha_k\alpha_l = \alpha_l\alpha_k$ for every $k, l \in \{1,2,3,\ldots,n\}$,
3. $\phi_i\alpha_k = \alpha_k\phi_i$ for every $i \in \{1,2,3,\ldots,m\}$ and $k \in \{1,2,3,\ldots,n\}$.

4. Fixed Point Theorems

Before giving the fixed point theorems, we are going to state and prove a lemma that will be used in our new theorems.

Lemma 4. *Consider* $\phi, \eta, \alpha, \beta$ *be self maps on a MIFSMS* $(\bar{X}, \varpi_{M,N}, \Theta)$*, satisfies the following conditions:*

1. *pair* (ϕ, α) *or* (η, β) *exhibits common limit range property with respect to* α *or* β *respectively,*
2. $\phi(\bar{X})$ *is strictly contained in* $\beta(\bar{X})$*, or* $\eta(\bar{X})$ *is strictly contained in* $\alpha(\bar{X})$*,*
3. $\beta(\bar{X})$ *or* $\alpha(\bar{X})$ *is a closed subset of* $\bar{X}$*,*
4. *for sequences* $\{\bar{p}_{e_n}\}$ *and* $\{\bar{q}_{e_n}\}$ *in* $\bar{X}$*, either* $\eta(\bar{q}_{e_n})$ *is convergent whenever* $\beta(\bar{q}_{e_n})$ *converges, or* $\phi(\bar{p}_{e_n})$ *is convergent whenever* $\alpha(\bar{p}_{e_n})$ *converges,*

5. *for a constant* $\varepsilon \in (0,1)$, (4.1) *holds for every* $\bar{p}, \bar{q} \in \bar{X}$ *and* $t > 0$,

$$\begin{aligned}\varpi_{M,N}(\phi\bar{p}, \eta\bar{q}, \varepsilon t) \geq_{L^*} \min\{&\varpi_{M,N}(\alpha\bar{p}, \beta\bar{q}, t), \varpi_{M,N}(\phi\bar{p}, \alpha\bar{p}, t), \varpi_{M,N}(\eta\bar{q}, \beta\bar{q}, t),\\ &\varpi_{M,N}(\phi\bar{p}, \beta\bar{q}, t), \varpi_{M,N}(\eta\bar{q}, \alpha\bar{p}, t)\}.\end{aligned} \tag{4.1}$$

Then pairs (ϕ, α) *and* (η, β) *exhibits common limit range property with respect to* α *and* β*, more precisely known as* $(CLR_{\alpha\beta})$ *property.*

Proof. Consider the pair (ϕ, α) exhibits (CLR_α) property, that implies the existence of a sequence $\{\bar{p}_{e_n}\}$ in $\bar{X}$ so that (4.2) holds for $\bar{h} \in \alpha(\bar{X})$,

$$\lim_{n\to\infty} \phi\bar{p}_{e_n} = \lim_{n\to\infty} \alpha\bar{p}_{e_n} = \bar{h}. \tag{4.2}$$

As $\phi(\bar{X}) \subset \beta(\bar{X})$ and $\beta(\bar{X})$ is a closed subset of $\bar{X}$, thus for every sequence $\{\bar{p}_{e_n}\}$ in $\bar{X}$ implies the existence of a sequence $\{\bar{q}_{e_n}\}$ in $\bar{X}$ so that $\phi\bar{p}_{e_n} = \beta\bar{q}_{e_n}$. Thus (4.3) holds for $\bar{h} \in \alpha(\bar{X}) \cap \beta(\bar{X})$,

$$\lim_{n\to\infty} \beta\bar{q}_{e_n} = \lim_{n\to\infty} \phi\bar{p}_{e_n} = \bar{h}. \tag{4.3}$$

Overall, we have $\phi\bar{p}_{e_n} \to \bar{h}$, $\alpha\bar{p}_{e_n} \to \bar{h}$ and $\beta\bar{q}_{e_n} \to \bar{h}$. Now to claim that $\eta\bar{q}_{e_n} \to \bar{h}$. Substituting $\bar{p} = \bar{p}_{e_n}$, $\bar{q} = \bar{q}_{e_n}$ in inequality (4.1), we have

$$\begin{aligned}\varpi_{M,N}(\phi\bar{p}_{e_n}, \eta\bar{q}_{e_n}, \varepsilon t) \geq_{L^*} \min\{&\varpi_{M,N}(\alpha\bar{p}_{e_n}, \beta\bar{q}_{e_n}, t), \varpi_{M,N}(\phi\bar{p}_{e_n}, \alpha\bar{p}_{e_n}, t),\\ &\varpi_{M,N}(\eta\bar{q}_{e_n}, \beta\bar{q}_{e_n}, t), \varpi_{M,N}(\phi\bar{p}_{e_n}, \beta\bar{q}_{e_n}, t),\\ &\varpi_{M,N}(\eta\bar{q}_{e_n}, \alpha\bar{p}_{e_n}, t)\}.\end{aligned}$$

Let $\lim_{n\to\infty} \varpi_{M,N}(\eta\bar{q}_{e_n}, \bar{l}, t) = 1_{L^*}$, where $\bar{l}(\neq \bar{h})$ for all $t > 0$. Then, taking $n \to \infty$, we get

$$\begin{aligned}\varpi_{M,N}(\bar{h}, \bar{l}, \varepsilon t) &\geq_{L^*} \min\{\varpi_{M,N}(\bar{h}, \bar{h}, t), \varpi_{M,N}(\bar{h}, \bar{h}, t), \varpi_{M,N}(\bar{l}, \bar{h}, t), \varpi_{M,N}(\bar{h}, \bar{h}, t),\\ &\qquad \varpi_{M,N}(\bar{l}, \bar{h}, t)\},\\ &\geq_{L^*} \min\{1_{L^*}, 1_{L^*}, \varpi_{M,N}(\bar{l}, \bar{h}, t), 1_{L^*}, \varpi_{M,N}(\bar{l}, \bar{h}, t)\},\\ &\geq_{L^*} \varpi_{M,N}(\bar{h}, \bar{l}, t).\end{aligned}$$

Thus, by Lemma 3 $\bar{h} = \bar{l}$. So, pairs (ϕ, α) and (η, β) satisfies $(CLR_{\alpha\beta})$ property. □

Now we are ready to prove our fixed point results, stated and proved as follows:

Theorem 4. *Consider* $\phi, \eta, \alpha, \beta$ *be self maps on a MIFSMS* $(\bar{X}, \varpi_{M,N}, \Theta)$ *satisfying inequality* (4.1) *of Lemma 4. The pair of self maps* (ϕ, α) *and* (η, β) *possess a coincidence point if they exhibit* $(CLR_{\alpha\beta})$ *property. Moreover, maps* ϕ, α, η *and* β *possess a unique common fixed point if pairs* (ϕ, α) *and* (η, β) *are weakly compatible.*

Proof. Since pair (ϕ, α) and (η, β) exhibit $(CLR_{\alpha\beta})$ property, this implies the existence of two sequences $\{\bar{p}_{e_n}\}$ and $\{\bar{q}_{e_n}\}$ in $\bar{X}$ so that (4.4) holds for $\bar{h} \in \alpha(\bar{X}) \cap \beta(\bar{X})$,

$$\lim_{n\to\infty} \phi\bar{p}_{e_n} = \lim_{n\to\infty} \alpha\bar{p}_{e_n} = \lim_{n\to\infty} \eta\bar{q}_{e_n} = \lim_{n\to\infty} \beta\bar{q}_{e_n} = \bar{h}. \tag{4.4}$$

Thus, $\alpha\bar{u} = \bar{h}$ and $\beta\bar{u} = \bar{h}$, where $\bar{u}, \bar{v} \in \bar{X}$. Now to claim that $\phi\bar{u} = \alpha\bar{u}$, substituting $\bar{p} = \bar{u}$ and $\bar{q} = \bar{q}_{e_n}$ in (4.1), we have

$$\varpi_{M,N}(\phi\bar{u}, \eta\bar{q}_{e_n}, \varepsilon t) \geq_{L^*} \min\{\varpi_{M,N}(\alpha\bar{u}, \beta\bar{q}_{e_n}, t), \varpi_{M,N}(\phi\bar{u}, \alpha\bar{u}, t), \varpi_{M,N}(\eta\bar{q}_{e_n}, \beta\bar{q}_{e_n}, t),$$
$$\varpi_{M,N}(\phi\bar{u}, \beta\bar{q}_{e_n}, t), \varpi_{M,N}(\eta\bar{q}_{e_n}, \alpha\bar{u}, t)\}.$$

Taking $n \to \infty$, we get

$$\varpi_{M,N}(\phi\bar{u}, \bar{h}, \varepsilon t) \geq_{L^*} \min\{\varpi_{M,N}(\bar{h}, \bar{h}, t), \varpi_{M,N}(\phi\bar{u}, \bar{h}, t), \varpi_{M,N}(\bar{h}, \bar{h}, t),$$
$$\varpi_{M,N}(\phi\bar{u}, \bar{h}, t), \varpi_{M,N}(\bar{h}, \bar{h}, t)\},$$

and so

$$\varpi_{M,N}(\phi\bar{u}, \bar{h}, \varepsilon t) \geq_{L^*} \min\{1_{L^*}, \varpi_{M,N}(\phi\bar{u}, \bar{h}, t), 1_{L^*}, \varpi_{M,N}(\phi\bar{u}, \bar{h}, t), 1_{L^*}\},$$
$$\varpi_{M,N}(\phi\bar{u}, \bar{h}, \varepsilon t) \geq_{L^*} \varpi_{M,N}(\phi\bar{u}, \bar{h}, t).$$

Thus, we have $\phi\bar{u} = \bar{h}$ and so $\phi\bar{u} = \alpha\bar{u} = \bar{h}$, this implies that pair (ϕ, α) has $\bar{u}$ as its coincidence point.

Claim that $\eta\bar{v} = \beta\bar{v}$, substituting $\bar{p} = \bar{u}$ and $\bar{q} = \bar{v}$ in (4.1), we have

$$\varpi_{M,N}(\phi\bar{u}, \eta\bar{v}, \varepsilon t) \geq_{L^*} \min\{\varpi_{M,N}(\alpha\bar{u}, \beta\bar{v}, t), \varpi_{M,N}(\phi\bar{u}, \alpha\bar{u}, t), \varpi_{M,N}(\eta\bar{v}, \beta\bar{v}, t),$$
$$\varpi_{M,N}(\phi\bar{u}, \beta\bar{v}, t), \varpi_{M,N}(\eta\bar{v}, \alpha\bar{u}, t)\},$$

so we get

$$\varpi_{M,N}(\bar{h}, \eta\bar{v}, \varepsilon t) \geq_{L^*} \min\{\varpi_{M,N}(\bar{h}, \bar{h}, t), \varpi_{M,N}(\bar{h}, \bar{h}, t), \varpi_{M,N}(\eta\bar{v}, \bar{h}, t),$$
$$\varpi_{M,N}(\bar{h}, \bar{h}, t), \varpi_{M,N}(\eta\bar{v}, \bar{h}, t)\},$$

and

$$\varpi_{M,N}(\bar{h}, \eta\bar{v}, \varepsilon t) \geq_{L^*} \min\{1_{L^*}, 1_{L^*}, \varpi_{M,N}(\eta\bar{v}, \bar{h}, t), 1_{L^*}, \varpi_{M,N}(\eta\bar{v}, \bar{h}, t)\},$$
$$\geq_{L^*} \varpi_{M,N}(\bar{h}, \eta\bar{v}, t).$$

Thus, we have $\bar{h} = \eta\bar{v}$ and hence $\eta\bar{v} = \beta\bar{v} = \bar{h}$, this implies that pair (η, β) has $\bar{v}$ as its coincidence point.

Since (ϕ, α) is weakly compatible and $\phi\bar{u} = \alpha\bar{u}$, hence $\phi\bar{h} = \phi\alpha\bar{u} = \alpha\phi\bar{u} = \alpha\bar{h}$. Now to claim that $\bar{h}$ is a common fixed point of (ϕ, α), substituting $\bar{p} = \bar{h}$ and $\bar{q} = \bar{v}$ in (4.1), we have

$$\varpi_{M,N}(\phi\bar{h}, \eta\bar{v}, \varepsilon t) \geq_{L^*} \min\{\varpi_{M,N}(\alpha\bar{h}, \beta\bar{v}, t), \varpi_{M,N}(\phi\bar{h}, \alpha\bar{h}, t), \varpi_{M,N}(\eta\bar{v}, \beta\bar{v}, t),$$
$$\varpi_{M,N}(\phi\bar{h}, \beta\bar{v}, t), \varpi_{M,N}(\eta\bar{v}, \alpha\bar{h}, t)\},$$

so we get

$$\varpi_{M,N}(\phi\bar{h}, \bar{h}, \varepsilon t) \geq_{L^*} \min\{\varpi_{M,N}(\phi\bar{h}, \bar{h}, t), \varpi_{M,N}(\phi\bar{h}, \phi\bar{h}, t), \varpi_{M,N}(\bar{h}, \bar{h}, t),$$
$$\varpi_{M,N}(\phi\bar{h}, \bar{h}, t), \varpi_{M,N}(\bar{h}, \phi\bar{h}, t)\},$$

and

$$\begin{aligned}\varpi_{M,N}(\phi\bar{h},\bar{h},\varepsilon t) &\geq_{L^*} \min\{\varpi_{M,N}(\phi\bar{h},\bar{h},t),1_{L^*},1_{L^*},\varpi_{M,N}(\phi\bar{h},\bar{h},t),\varpi_{M,N}(\phi\bar{h},\bar{h},t)\},\\ &\geq_{L^*} \varpi_{M,N}(\phi\bar{h},\bar{h},t).\end{aligned}$$

Thus, we have $\phi\bar{h} = \bar{h} = \alpha\bar{h}$, this implies, the pair (ϕ,α) has $\bar{h}$ as its common fixed point.

As pair (η,β) is weakly compatible, hence $\eta\bar{h} = \eta\beta\bar{v} = \beta\eta\bar{v} = \beta\bar{h}$. Substituting $\bar{p} = \bar{u}$ and $\bar{q} = \bar{h}$ in (4.1), we have

$$\begin{aligned}\varpi_{M,N}(\phi\bar{u},\eta\bar{h},\varepsilon t) \geq_{L^*} \min\{&\varpi_{M,N}(\alpha\bar{u},\beta\bar{h},t),\varpi_{M,N}(\phi\bar{u},\alpha\bar{u},t),\varpi_{M,N}(\eta\bar{h},\beta\bar{h},t),\\ &\varpi_{M,N}(\phi\bar{u},\beta\bar{h},t),\varpi_{M,N}(\eta\bar{h},\alpha\bar{u},t)\},\end{aligned}$$

so, we get

$$\begin{aligned}\varpi_{M,N}(\bar{h},\eta\bar{h},\varepsilon t) \geq_{L^*} \min\{&\varpi_{M,N}(\bar{h},\eta\bar{h},t),\varpi_{M,N}(\bar{h},\bar{h},t),\varpi_{M,N}(\eta\bar{h},\eta\bar{h},t),\\ &\varpi_{M,N}(\bar{h},\eta\bar{h},t),\varpi_{M,N}(\eta\bar{h},\bar{h},t)\},\end{aligned}$$

and

$$\begin{aligned}\varpi_{M,N}(\bar{h},\eta\bar{h},\varepsilon t) &\geq_{L^*} \min\{\varpi_{M,N}(\bar{h},\eta\bar{h},t),1_{L^*},1_{L^*},\varpi_{M,N}(\bar{h},\eta\bar{h},t),\varpi_{M,N}(\eta\bar{h},\bar{h},t)\},\\ &\geq_{L^*} \varpi_{M,N}(\bar{h},\eta\bar{h},t).\end{aligned}$$

Thus, we have $\eta\bar{h} = \bar{h} = \beta\bar{h}$, this implies that pair (η,β) has $\bar{h}$ as their common fixed point. Hence, maps ϕ,η,α and β have $\bar{h}$ as their common fixed point.

Uniqueness: Consider maps ϕ,η,α and β have another common fixed point say $\bar{w}(\neq \bar{h})$. Substitute $\bar{p} = \bar{w}$ and $\bar{q} = \bar{h}$ in (4.1), we have

$$\begin{aligned}\varpi_{M,N}(\phi\bar{w},\eta\bar{h},\varepsilon t) \geq_{L^*} \min\{&\varpi_{M,N}(\alpha\bar{w},\beta\bar{h},t),\varpi_{M,N}(\phi\bar{w},\alpha\bar{w},t),\varpi_{M,N}(\eta\bar{h},\beta\bar{h},t),\\ &\varpi_{M,N}(\phi\bar{w},\beta\bar{h},t),\varpi_{M,N}(\eta\bar{h},\alpha\bar{w},t)\},\end{aligned}$$

so, we get

$$\begin{aligned}\varpi_{M,N}(\bar{w},\bar{h},\varepsilon t) \geq_{L^*} \min\{&\varpi_{M,N}(\bar{w},\bar{h},t),\varpi_{M,N}(\bar{w},\bar{w},t),\varpi_{M,N}(\bar{h},\bar{h},t),\\ &\varpi_{M,N}(\bar{w},\bar{h},t),\varpi_{M,N}(\bar{h},\bar{w},t)\},\end{aligned}$$

and

$$\begin{aligned}\varpi_{M,N}(\bar{w},\bar{h},\varepsilon t) &\geq_{L^*} \min\{\varpi_{M,N}(\bar{w},\bar{h},t),1_{L^*},1_{L^*},\varpi_{M,N}(\bar{w},\bar{h},t),\varpi_{M,N}(\bar{w},\bar{h},t)\},\\ &\geq_{L^*} \varpi_{M,N}(\bar{w},\bar{h},t).\end{aligned}$$

Thus, we have $\bar{w} = \bar{h}$. Therefore ϕ,η,α and β possess a unique common fixed point. □

Theorem 5. *Consider ϕ,η,α,β be self maps on a MIFSMS $(\bar{X},\varpi_{M,N},\Theta)$ satisfying conditions 1-5 of Lemma 4. The maps $\phi,\alpha,\eta,$ and β possess a unique common fixed point if the pairs (ϕ,α) and (η,β) are weakly compatible.*

Proof. The pairs (ϕ,α) and (η,β) exhibit $(CLR_{\alpha\beta})$ property by Lemma 4, implies the existence of two sequences $\{\bar{p}_{e_n}\}$ and $\{\bar{q}_{e_n}\}$ in $\bar{X}$ so that (4.5) holds for $\bar{h} \in \alpha(\bar{X}) \cap \beta(\bar{X})$,

$$\lim_{n\to\infty} \phi\bar{p}_{e_n} = \lim_{n\to\infty} \alpha\bar{p}_{e_n} = lt_{n\to\infty}\eta\bar{q}_{e_n} = \lim_{n\to\infty} \beta\bar{q}_{e_n} = \bar{h}. \tag{4.5}$$

The rest can be proved on the similar lines to Theorem 4. □

Following are the two illustration making use of recently proved theorems:

Example 4. Consider $\bar{X} = [1,15)$ in MIFSMS $(\bar{X}, \varpi_{M,N}, \Theta)$, $t-$representable norm is given by $\Theta(\bar{a},\bar{b}) = (\bar{a}_1\bar{b}_1, \min\{\bar{a}_2+\bar{b}_2, 1\})$ for all $\bar{a} = (\bar{a}_1,\bar{a}_2)$ and $\bar{b} = (\bar{b}_1,\bar{b}_2) \in L^*$ so that for all $\bar{p},\bar{q} \in \bar{X}$ and $t > 0$, MIFSM is defined as

$$\varpi_{M,N}(\bar{p},\bar{q},t) = \left(\frac{t}{t+|\bar{p}-\bar{q}|}, \frac{|\bar{p}-\bar{q}|}{t+|\bar{p}-\bar{q}|}\right).$$

Define self maps ϕ,η,α,β by

$$\phi(\bar{p}) = \begin{cases} 1 & \text{if } \bar{p} \in \{1\}\cup(3,15); \\ 2 & \text{if } \bar{p} \in (1,3]. \end{cases} \qquad \eta(\bar{p}) = \begin{cases} 1 & \text{if } \bar{p} \in \{1\}\cup(3,15); \\ 3 & \text{if } \bar{p} \in (1,3]. \end{cases}$$

$$\alpha(\bar{p}) = \begin{cases} 1 & \text{if } \bar{p} = 1; \\ 5 & \text{if } \bar{p} \in (1,3]; \\ \dfrac{\bar{p}+1}{4} & \text{if } \bar{p} \in (3,15). \end{cases} \qquad \beta(\bar{p}) = \begin{cases} 1 & \text{if } \bar{p} = 1; \\ 9+\bar{p} & \text{if } \bar{p} \in (1,3]; \\ \bar{p}-2 & \text{if } \bar{p} \in (3,15). \end{cases}$$

Taking $\{\bar{p}_{e_n} = 3+\frac{1}{n}\}$, $\{\bar{q}_{e_n} = 1\}$ or $\{\bar{p}_{e_n} = 1\}$, $\{\bar{q}_{e_n} = 3+\frac{1}{n}\}$, it is clear that pairs (ϕ,α) and (η,β) exhibits $(CLR_{\alpha\beta})$ property.

$$\lim_{n\to\infty} \phi\bar{p}_{e_n} = \lim_{n\to\infty} \alpha\bar{p}_{e_n} = \lim_{n\to\infty} \eta\bar{q}_{e_n} = \lim_{n\to\infty} \beta\bar{q}_{e_n} = 1 \in \alpha(\bar{X}) \cap \beta(\bar{X}).$$

It is also clear that $\phi(\bar{X}) \subset \beta(\bar{X})$ and $\eta(\bar{X}) \subset \alpha(\bar{X})$. Thus, Theorem 4 is applicable and hence (ϕ,α) and (η,β) have 1 as their unique common fixed point, which is also a point of coincidence.

Theorem 5 cannot be applied in the above example, as $\alpha(\bar{X}) = [1,4) \cup \{5\}$ and $\beta(\bar{X}) = [1,13)$ are not closed subsets of $\bar{X}$.

Example 5. If maps α,β in Example 4 are replaced by the following maps, keeping rest the same:

$$\alpha(\bar{p}) = \begin{cases} 1 & \text{if } \bar{p} = 1; \\ 4 & \text{if } \bar{p} \in (1,3]; \\ \dfrac{\bar{p}+1}{4} & \text{if } \bar{p} \in (3,15). \end{cases} \qquad \beta(\bar{p}) = \begin{cases} 1 & \text{if } \bar{p} = 1; \\ 13 & \text{if } \bar{p} \in (1,3); \\ \bar{p}-2 & \text{if } \bar{p} \in [3,15). \end{cases}$$

Now, $\phi(\bar{X}) \subset \beta(\bar{X})$ and $\eta(\bar{X}) \subset \alpha(\bar{X})$ also $\alpha(\bar{X}) = [1,4]$ and $\beta(\bar{X}) = [1,13]$ are closed subsets of $\bar{X}$. Thus Theorem 5 is applicable here, hence (ϕ,α) and (η,β) have 1 as their unique common fixed point.

Next, we are giving a corollary:

Corollary 1. *Consider ϕ and α be self maps on a MIFSMS $(\bar{X},\varpi_{M,N},\Theta)$ and suppose that*

1. *(ϕ,α) satisfies (CLR_α) property.*
2. *there exist $\varepsilon \in (0,1)$ so that (4.6) holds for all $\bar{p},\bar{q} \in \bar{X}$ and $t > 0$,*

$$\begin{aligned} \varpi_{M,N}(\phi\bar{p},\phi\bar{q},\varepsilon t) \geq_{L^*} \min\{&\varpi_{M,N}(\alpha\bar{p},\alpha\bar{q},t),\varpi_{M,N}(\phi\bar{p},\alpha\bar{p},t),\varpi_{M,N}(\phi\bar{q},\alpha\bar{q},t),\\ &\varpi_{M,N}(\phi\bar{p},\alpha\bar{q},t),\varpi_{M,N}(\phi\bar{q},\alpha\bar{p},t)\}. \end{aligned} \tag{4.6}$$

Then the pair (ϕ,α) possess a coincidence point. Moreover, (ϕ,α) possess a unique common fixed point in $\bar{X}$ if they are weakly compatible.

Proof. Take $\phi = \eta$ and $\alpha = \beta$ in Theorem 4, the result will be proved on the similar lines. □

5. Application

In this section we are going to give an application utilising our new results.

Theorem 6. *Consider $(\bar{X},\varpi_{M,N},\Theta)$ be MIFSMS. Consider ϕ,η,α,β are self maps satisfies the following conditions:*

(i) *pairs (ϕ,α) and (η,β) exhibit $(CLR_{\alpha\beta})$ property.*

(ii) *for $\bar{p},\bar{q} \in \bar{X}$, $\rho > 0$,*

$$\int_0^{s_1} \Upsilon(\rho)d\rho \geq_{L^*} \int_0^{s_2} \Upsilon(\rho)d\rho,$$

where $\Upsilon(\rho)$ is Lebesgue integrable function and

$$\begin{aligned} s_1 &= \varpi_{M,N}(\phi\bar{p},\eta\bar{q},\varepsilon\rho),\\ s_2 &= \min\{\varpi_{M,N}(\phi\bar{p},\beta\bar{q},\rho),\varpi_{M,N}(\eta\bar{q},\alpha\bar{p},\rho),\\ &\qquad \varpi_{M,N}(\phi\bar{p},\alpha\bar{p},\rho),\varpi_{M,N}(\eta\bar{q},\beta\bar{q},\rho),\\ &\qquad \varpi_{M,N}(\alpha\bar{p},\beta\bar{q},\rho)\}. \end{aligned}$$

Then, pairs (ϕ,α) and (η,β) possess a coincidence point. Moreover, maps ϕ,α,η and β possess a unique common fixed point if (ϕ,α) and (η,β) are weakly compatible.

Proof. Since, (ϕ,α) and (η,β) satisfies $(CLR_{\alpha\beta})$ property, thus equation (4.4) of Theorem 4 is satisfied. From condition (ii) we have

$$\int_0^{\varpi_{M,N}(\phi\bar{p}_{en},\eta\bar{h},\rho)} \Upsilon(\rho)d\rho \geq_{L^*} \int_0^{s_2} \Upsilon(\rho)d\rho,$$

where

$$\begin{aligned}s_2 = \min\{&\varpi_{M,N}(\phi\bar{p}_{e_n}, \beta\bar{h}, \rho), \varpi_{M,N}(\eta\bar{h}, \alpha\bar{p}_{e_n}, \rho),\\ &\varpi_{M,N}(\phi\bar{p}_{e_n}, \alpha\bar{p}_{e_n}, \rho), \varpi_{M,N}(\eta\bar{h}, \beta\bar{h}, \rho),\\ &\varpi_{M,N}(\alpha\bar{p}_{e_n}, \beta\bar{h}, \rho)\}.\end{aligned}$$

Taking $n \to \infty$, we have

$$\eta\bar{h} = \beta\bar{h}. \tag{5.1}$$

From condition (ii) and taking $n \to \infty$, we have

$$\int_0^{\varpi_{M,N}(\phi\bar{h}, \eta\bar{q}_{e_n}, \rho)} \Upsilon(\rho)d\rho \geq_{L^*} \int_0^{s_2} \Upsilon(\rho)d\rho,$$

where

$$\begin{aligned}s_2 = \min\{&\varpi_{M,N}(\phi\bar{h}, \beta\bar{q}_{e_n}, \rho), \varpi_{M,N}(\eta\bar{q}_{e_n}, \alpha\bar{h}, \rho),\\ &\varpi_{M,N}(\phi\bar{h}, \alpha\bar{h}, \rho), \varpi_{M,N}(\eta\bar{q}_{e_n}, \beta\bar{q}_{e_n}, \rho),\\ &\varpi_{M,N}(\alpha\bar{h}, \beta\bar{q}_{e_n}, \rho)\}.\end{aligned}$$

Thus, we have

$$\phi\bar{h} = \alpha\bar{h}. \tag{5.2}$$

From equations (5.1) and (5.2), we have

$$\phi\bar{h} = \alpha\bar{h} = \eta\bar{h} = \beta\bar{h}. \tag{5.3}$$

Thus pairs (ϕ, α) and (η, β) possess a common coincidence point.

Now, let us consider

$$\phi\bar{h} = \alpha\bar{h} = \eta\bar{h} = \beta\bar{h} = \bar{u}. \tag{5.4}$$

The weak compatibility of pairs (ϕ, α) and (η, β) implies

$$\phi\bar{u} = \alpha\bar{u}, \eta\bar{u} = \beta\bar{u}. \tag{5.5}$$

By condition (ii) and equations (5.3), (5.4) and (5.5), we get

$$\int_0^{\varpi_{M,N}(\phi\bar{u}, \eta\bar{h}, \rho)} \Upsilon(\rho)d\rho \geq_{L^*} \int_0^{s_2} \Upsilon(\rho)d\rho,$$

where

$$\begin{aligned}s_2 = \min\{&\varpi_{M,N}(\phi\bar{u}, \beta\bar{h}, \rho), \varpi_{M,N}(\eta\bar{h}, \alpha\bar{u}, \rho),\\ &\varpi_{M,N}(\phi\bar{u}, \alpha\bar{u}, \rho), \varpi_{M,N}(\eta\bar{h}, \beta\bar{h}, \rho),\\ &\varpi_{M,N}(\alpha\bar{u}, \beta\bar{h}, \rho)\}.\end{aligned}$$

So, we have

$$\varpi_{M,N}(\phi\bar{u}, \bar{u}, \rho) >_{L^*} \varpi_{M,N}(\phi\bar{u}, \bar{u}, \rho).$$

Hence, we obtain

$$\bar{u} = \phi\bar{u} = \alpha\bar{u}.$$

Using condition (ii), we have

$$\int_0^{\varpi_{M,N}(\phi\bar{h},\eta\bar{u},\rho)} \Upsilon(\rho)d\rho \geq_{L^*} \int_0^{s_2} \Upsilon(\rho)d\rho,$$

where

$$\begin{aligned} s_2 = \min\{&\varpi_{M,N}(\phi\bar{h},\beta\bar{u},\rho),\varpi_{M,N}(\eta\bar{u},\alpha\bar{h},\rho),\\ &\varpi_{M,N}(\phi\bar{h},\alpha\bar{h},\rho),\varpi_{M,N}(\eta\bar{u},\beta\bar{u},\rho),\\ &\varpi_{M,N}(\alpha\bar{h},\beta\bar{u},\rho)\}. \end{aligned}$$

Thus, we get

$$\bar{u} = \phi\bar{u} = \alpha\bar{u} = \eta\bar{u} = \beta\bar{u}.$$

Hence, ϕ,α,η and β possess $\bar{u}$ as their common fixed point. The uniqueness of fixed point is easy to prove by using condition (ii). □

Following example validates the Theorem 6.

Example 6. Replacing maps α and β in example (3) by

$$\alpha(\imath) = \begin{cases} 3 & for\ \imath = 3,\\ 3+\imath & for\ 3<\imath\leq 11,\\ \frac{\imath+1}{4} & for\ 11<\imath<27. \end{cases} \qquad \beta(\imath) = \begin{cases} 3 & for\ \imath = 3,\\ 11+\imath & for\ 3<\imath\leq 11,\\ \imath-8 & for\ 11<\imath<27. \end{cases}$$

Then, $\phi(\bar{X}) = \{3,21\} \subset [3,22] = \beta(\bar{X})$ and $\eta(\bar{X}) = \{3,8\} \subset [3,14] = \alpha(\bar{X})$.

Therefore, every assertion of Theorem 6 is satisfied with $\Upsilon(\rho) = 1$. Thus, maps ϕ,α,η and β possess 3 as their common invariant point.

6. Conclusion

We have defined self maps satisfying $(CLR_{\alpha\beta})$ property in Modified Intuitionistic Fuzzy Soft Metric Space in this paper. Results of Saadati et al. [14] and Imdad et al. [11] have been generalized to MIFSMS. Examples have been given to prove the applicability of our new results. An application to integral type contraction have also been proved.

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REFERENCES

[1] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, pp. 87–96, 1986, doi: 10.1016/S0165-0114(86)80034-3.
[2] T. Beaula and C. Gunaseeli, "On fuzzy soft metric spaces," *Malaya Journal of Mathematics*, vol. 2, no. 3, pp. 197–202, 2014, doi: 10.26637/mjm203/003.
[3] S. Chauhan, M. Imdad, and B. Samet, "Coincidence and common fixed point theorems in modified intuitionistic fuzzy metric spaces," *Mathematical and Computer Modelling*, vol. 58, pp. 898–906, 2013, doi: 10.1016/j.mcm.2013.03.010.
[4] S. Das and S. K. Samanta, "Soft real sets, soft real numbers and their properties," *J. Fuzzy Math.*, vol. 20, no. 3, pp. 551–576, 2012.
[5] S. Das and S. K. Samanta, "On soft metric spaces," *J. Fuzzy Math.*, vol. 21, no. 3, pp. 707–734, 2013.
[6] S. Das and S. K. Samanta, "Soft metric," *Ann. Fuzzy Math. Inform*, vol. 6, no. 1, pp. 77–94, 2013.
[7] G. Deschrijver, C. Cornelis, and E. E. Kerre, "On the representation of intuitionistic fuzzy t-norms and t-conorms," *IEEE Trans Fuzzy System*, vol. 12, pp. 45–61, 2004, doi: 10.1109/TFUZZ.2003.822678.
[8] F. S. Erduran, E. Yigit, R. Alar, and A. Gezici, "Soft fuzzy metric spaces," *General Letters in Mathematics*, vol. 3, no. 2, pp. 91–101, 2017.
[9] V. Gupta and Aanchal, "Existence of fixed points in modified intuitionistic fuzzy soft metric space with application," *Communicated*, 2023.
[10] V. Gupta and A. Gondhi, "Fixed points of weakly compatible maps on modified intuitionistic fuzzy soft metric spaces," *International Journal of System Assurance Engineering and Management*, vol. 13, no. 16, pp. 1232–1238, 2021, doi: 10.1007/s13198-021-01423-1.
[11] M. Imdad, J. Ali, and M. Hasan, "Common fixed point theorems in modified intuitionistic fuzzy metric spaces," *Iran. J. Fuzzy Syst.*, vol. 9, no. 5, pp. 77–92, 2012, doi: 10.22111/IJFS.2012.105.
[12] P. K. Maji, R. Biswas, and A. R. Roy, "Fuzzy soft set," *J.Fuzzy Math.*, vol. 9, no. 3, pp. 589–602, 2001.
[13] D. Molodtsov, "Soft set theory–first results," *Comput. Math. Appl.*, vol. 37, no. 4-5, pp. 19–31, 1999, doi: 10.1016/S0898-1221(99)00056-5.
[14] R. Saadati, S. Sedgi, and N. Shobe, "Modified intuitionistic fuzzy metric spaces and fixed point theorems," *Chaos Solitons and Fractals*, vol. 38, pp. 36–47, 2008, doi: 10.1016/j.chaos.2006.11.008.
[15] L. A. Zadeh, "Fuzzy sets," *Fuzzy Sets Information and Control*, vol. 8, pp. 338–353, 1965, doi: 10.1016/s0019-9958(65)90241-x.

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