Miskolc Mathematical Notes
Vol. 25 (2024), No. 1, pp. 523–536
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4321

EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A $p(x)$-KIRCHHOFF TYPE EQUATION WITH STEKLOV BOUNDARY

ZEHRA YÜCEDAG

Received 06 July, 2022

Abstract. In this paper we study a class of $p(x)$-Kirchhoff type problem with Steklov boundary value in variable exponent Sobolev spaces. Precisely, we show the existence of at least three solutions and a nontrivial weak solution.

2010 *Mathematics Subject Classification:* 35A15; 35J60; 47J30

Keywords: $p(x)$-Kirchhoff-type equation, variable exponent Sobolev spaces, Steklov boundary value, variational principle

1. INTRODUCTION

The purpose of this article is to establish the existence of at least three weak solutions for the following equation Steklov boundary value involving $p(x)$-Laplacian

$$\begin{cases} M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\right)\Delta_{p(x)}u = |u|^{p(x)-2}u & \text{in } x\in\Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \upsilon} = \lambda f(x,u) & \text{on } x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 2)$ is a smooth bounded domain, λ is a positive parameter, υ is the outward normal vector on $\partial\Omega$, $p\in C\left(\overline{\Omega}\right)$, $\Delta_{p(x)}u = div\left(|\nabla u|^{p(x)-2}\nabla u\right)$ is a $p(x)$-Laplacian operator, $f:\partial\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function and $M:(0,\infty)\to(0,\infty)$ is a continuous Kirchhoff function.

Problem (1.1) is related to the stationary version of a model, the so-called Kirchhoff equation, introduced by Kirchhoff [21]. To be more precise, Kirchhoff established a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where ρ, P_0, h, E, L are constants. This equation is an extension of the classical D'Alambert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. For $p(x)$-Kirchhoff type problems see, for example, [10, 11, 28].

Differential and partial differential problems with variable exponent growth condition have been received considerable attention in recent years. Moreover, these type of problems are used interesting topic like electrorheological fluids [23], image processing and stationary thermo-rheological viscous flows of non-Newtonian fluids [9], elastic mechanics [31], the mathematical description of the processes filtration of an idea barotropic gas through a porous medium [6]. Many results have been obtained on this kind of problems, for example, we refer to [13, 14, 16, 17, 20, 22, 27].

The Steklov problems involving $p(x)$-Laplacian have been intensively studied using variational methods by many authors [3, 4, 12, 19, 24, 25, 30]. Especially, some of the authors have studied the problems of type (1.1) in the case of $M(t) = 1$. For example, In [7], the author investigated the existence and multiplicity of solutions for Steklov problem standard growth condition without Ambrosetti-Rabinowitz type condition by means of critical point theorems with Cerami condition under weaker conditions. In [5], the authors considered the existence and multiplicity of solutions for the nonlinear Steklov boundary-value problem, by using Mountain Pass, Fountain and Ricceri three-critical-points theorems. In [4], the authors obtained the existence and multiplicity of solutions of problem (1.2) when $f(x,u) = |u|^{q(x)-2}u$, by applying two versions of the mountain pass theorem and Ekeland's variational principle.

Moreover, in [1], under suitable assumptions on the nonlinearity, they studied the existence and multiplicity of solutions for a Steklov problem involving the $p(x)$-Laplacian using variational methods. In [29], the authors proved the existence and multiplicity of a-harmonic solutions for the elliptic Steklov problem with variable exponents. Their approach is based on variational methods. In [18], the authors dealt with nontrivial weak solution a weighted Steklov problem involving the $p(x)$-Laplacian operator, by variational method and Ekeland's variational principle. In [2], the author proved some results on the existence of weak solutions for non-linear Steklov boudary value problem in Sobolev spaces with variable exponents, by using min-max method and Ekeland's variational principle.

Inspired by the papers above mentioned, we studied problem (1.1). The present article is composed of two sections. In Section 2, we recall the definition of variable exponent Lebesgue-Sobolev spaces. In Section 3, we give the main results and their proofs are presented. We approach is based on the Ricceri Theorem and the Mountain Pass Theorem.

2. Preliminaries

In order to discuss problem (1.1), we need some definitions and basic properties of variable exponent Lebesgue-Sobolev spaces which will be used later [16, 22, 28].

Set

$$C_+\left(\overline{\Omega}\right) = \left\{p : \ p \in C\left(\overline{\Omega}\right), \ p(x) > 1 \text{ for all } x \in \overline{\Omega}\right\}.$$

For any $p(x) \in C_+(\overline{\Omega})$, we denote

$$1 < p^- := \inf_{x\in\overline{\Omega}} p(x) \le p \le p^+ := \sup_{x\in\overline{\Omega}} p(x) < \infty.$$

We define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u \,|\, u : \Omega \to \mathbb{R} \text{ is a measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

We recall the following so-called Luxemburg norm on this space defined by the formula

$$|u|_{L^{p(x)}(\Omega)} := |u|_{p(x)} = \inf\left\{ \rho > 0 : \int_\Omega \left| \frac{u(x)}{\rho} \right|^{p(x)} dx \le 1 \right\}.$$

Similarly, define the space $L^{p(x)}(\partial\Omega)$ for any $p(x) \in C_+(\partial\Omega)$

$$L^{p(x)}(\partial\Omega) = \left\{ u \,|\, u : \partial\Omega \to \mathbb{R} \text{ is a measurable and } \int_{\partial\Omega} |u(x)|^{p(x)}\, d\sigma < \infty \right\},$$

with the norm

$$|u|_{L^{p(x)}(\partial\Omega)} = |u|_{p(x),\partial\Omega} := \inf\left\{ \zeta > 0 : \int_{\partial\Omega} \left| \frac{u(x)}{\zeta} \right|^{p(x)} d\sigma \le 1 \right\},$$

where $d\sigma$ is the measure on the boundary. Then, $L^{p(x)}(\partial\Omega)$ is a Banach space.

Proposition 1 ([22, Theorem 2.1]). *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ and $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\left| \int_\Omega uv dx \right| \le \left(\frac{1}{p^-} + \frac{1}{(p^-)'} \right) |u|_{p(x)} |v|_{p'(x)}.$$

The modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\psi : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \int_\Omega |u(x)|^{p(x)}\, dx, \ \forall u \in L^{p(x)}(\Omega).$$

Proposition 2 ([17, Theorem 1.3]). *If $u, u_n \in L^{p(x)}(\Omega)$, $n = 1, 2, \ldots$, then we have*

(i) $|u|_{p(x)} > 1\ (= 1, < 1) \Leftrightarrow \psi(u) > 1\ (= 1, < 1)$,

(ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \le \psi(u) \le |u|_{p(x)}^{p^+}$,

(iii) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \le \psi(u) \le |u|_{p(x)}^{p^-}$,

(iv) $|u_n - u|_{p(x)} \to 0\ (\to \infty) \Leftrightarrow \psi(u_n - u) \to 0\ (\to \infty)$.

Proposition 3 ([12, Proposition 2.4]; [25, Proposition 2.5]). *Set $\phi(u) = \int_{\partial\Omega} |u(x)|^{p(x)}\, d\sigma$. For $u, u_n \in L^{p(x)}(\partial\Omega)$, $n = 1, 2, \ldots$, we have*

(i) $|u|_{L^{p(x)}(\partial\Omega)} \ge 1 \Rightarrow |u|_{L^{p(x)}(\partial\Omega)}^{p^-} \le \phi(u) \le |u|_{L^{p(x)}(\partial\Omega)}^{p^+}$,

(ii) $|u|_{L^{p(x)}(\partial\Omega)} \leq 1 \Rightarrow |u|^{p^+}_{L^{p(x)}(\partial\Omega)} \leq \phi(u) \leq |u|^{p^-}_{L^{p(x)}(\partial\Omega)}$,
(iii) $|u_n - u|_{p(x)} \to 0 \ (\to \infty) \Leftrightarrow \phi(u_n - u) \to 0 \ (\to \infty)$.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : \ |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

and equipped with the norm

$$\|u\|_{1,p(x)} := \inf \left\{ \kappa > 0 : \int_\Omega \left(\left| \frac{\nabla u(x)}{\kappa} \right|^{p(x)} + \left| \frac{u(x)}{\kappa} \right|^{p(x)} \right) dx \leq 1 \right\},$$

or

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \forall u \in W^{1,p(x)}(\Omega).$$

The space $W_0^{1,p(x)}(\Omega)$ is denoted by the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. We can define an equivalent norm

$$\|u\| = |\nabla u|_{p(x)} \text{ for all } u \in W_0^{1,p(x)}(\Omega).$$

Proposition 4 ([14, Lemma 2.1]). *Let $p(x)$ and $q(x)$ be measurable functions such that $1 \leq p(x)q(x) \leq \infty$ and $p(x) \in L^\infty(\Omega)$ for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then*

$$\min\left(|u|^{p^-}_{p(x)q(x)}, |u|^{p^+}_{p(x)q(x)} \right) \leq \left| |u|^{p(x)} \right|_{q(x)} \leq \max\left(|u|^{p^-}_{p(x)q(x)}, |u|^{p^+}_{p(x)q(x)} \right).$$

In particular, if $p(x) = p$ is constant, then

$$||u|^p|_{q(x)} = |u|^p_{pq(x)}.$$

Proposition 5 ([12, Theorem 2.1]; [14, Lemma 3.1]; [16, Theorem 1.1]). *; ([22, Theorem 3.1]; [25, Proposition 2.2])*

(i) *If $1 < p^- \leq p^+ < \infty$ then, the spaces $L^{p(x)}(\Omega)$, $L^{p(x)}(\partial\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces,*
(ii) *if $q(x) \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous, where*

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)}, & N > p(x), \\ \infty, & N \leq p(x), \end{cases}$$

(iii) *if $q(x) \in C_+(\partial\Omega m)$ and $q(x) < p^\partial(x)$ for all $x \in \partial\Omega$, then the trace embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$ is compact and continuous, where*

$$p^\partial(x) := \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & N > p(x), \\ \infty, & N \leq p(x), \end{cases}$$

(iv) *(Poincaré inequality) There is a positive constant* $C > 0$ *such that*

$$|u|_{p(x)} \leq C\|u\| \text{ for all } u \in W_0^{1,p(x)}(\Omega).$$

Remark 1. If $N < p^- \leq p(x)$ for any $x \in \overline{\Omega}$, by Theorem 2.2 in [17], we deduce that $W_0^{1,p^-}(\Omega)$ is compact and continuous embedded in $C\left(\overline{\Omega}\right)$. Thus, we deduce that $W_0^{1,p(x)}(\Omega)$ is compactly embedded in $C\left(\overline{\Omega}\right)$. Defining $\|u\|_{C(\overline{\Omega})} = \sup_{x\in\overline{\Omega}} |u(x)|$, we find that there exists a positive constant $c > 0$ such that

$$|u|_{C(\overline{\Omega})} \leq c\|u\| \text{ for all } u \in W_0^{1,p(x)}(\Omega).$$

Throughout this paper, we assume that M and f satisfy the following conditions:
($\mathbf{M}_0$) $M : (0,\infty) \to (0,\infty)$ is a continuous function such that

$$m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\alpha-1} \text{ for all } t > 0$$

where m_1, m_2 and α are real numbers such that $0 < m_1 \leq m_2$ and $\alpha > 1$.

(**F**1) $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition and

$$|f(x,t)| \leq a(x) + b|t|^{s(x)-1} \text{ for all } (x,t) \in \partial\Omega \times \mathbb{R},$$

where $a(x) \in L^{\frac{s(x)}{s(x)-1}}(\partial\Omega)$, $b \geq 0$ is a constant and $s(x) \in C_+(\partial\Omega)$ such that

$$1 < s^- := \inf_{x\in\overline{\Omega}} s(x) \leq s(x) \leq s^+ := \sup_{x\in\overline{\Omega}} p(x) < p^- \text{and } N < p^-.$$

(**F**2) If $|t| \in (0,1)$, then $F(x,t) < 0$ and $t \in (t_0,\infty)$ for $t_0 > 1$, then $F(x,t) > \vartheta > 0$.

(**F**3) $F : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition and

$$F(x,tu) = |t|^{q(x)} F(x,u) \text{ for all } t > 0 \text{ and } (x,u) \in \partial\Omega \times \mathbb{R}$$

where $q(x) \in C(\partial\Omega)$ and $F(x,t) = \int_0^t f(x,k)\,dk$ for all $t > 0$ and $(x,k) \in \partial\Omega \times \mathbb{R}$.

(**F**4) There exists $\Omega_0 \subset\subset \partial\Omega$ with $\text{meas}_\sigma(\Omega_0) > 0$ such that $F(x,t) > 0$ for all $(x,t) \in \Omega_0 \times \mathbb{R}$.

(**F**5) $1 < q^+ < p^-$ and $p^- < p^\partial(x)$ for all $x \in \partial\Omega$.

Theorem 1 ([8, Theorem 1])**.** *Let X be a separable and reflexive real Banach space; $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that,*

(i) $\lim_{\|u\|\to\infty} J(u) = \lim_{\|u\|\to\infty} (\Phi(u) + \lambda\Psi(u)) = \infty$ *for all* $\lambda > 0$,

(ii) *There are* $r \in \mathbb{R}$ *and* u_0, $u_1 \in X$ *such that* $\Phi(u_0) < r < \Phi(u_1)$,

(iii) $\inf_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) > \frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)}$.

Then there exist an open interval $\Lambda \subset (0,\infty)$ *and a positive real number* ρ *such that for each* $\lambda \in \Lambda$ *the equation*

$$\Phi'(u) + \lambda \Psi'(u) = 0$$

has at least three solutions in X *whose norms are lees than* ρ.

3. Main Results and Proofs

Let X denote the variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$. Our main results in this paper are the proofs of the following theorems, which are based on results showed in the papers by Bonanno and Candito [8, Theorem 1] and by Willem [26] .

Theorem 2. *Assume that* $(\mathbf{M}_0)$, $(\mathbf{F}1)$ *and* $(\mathbf{F}2)$ *hold. Then there exist an open interval* $\Lambda \subset (0,\infty)$ *and a constant* $\rho > 0$ *such that for any* $\lambda \in \Lambda$, *problem* (1.1) *has at least three weak solutions in* X *whose norms are less than* ρ.

Theorem 3. *Assume that* $(\mathbf{M}_0)$, $(\mathbf{F}3)$, $(\mathbf{F}4)$ *and* $(\mathbf{F}5)$ *hold. Then there exists* $\lambda^* > 0$ *such that* $\lambda \in (0,\lambda^*)$, *problem* (1.1) *has a nontrivial weak solution in* X.

Proposition 6 ([5, Theorem 2.9])**.** *Let* $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function satisfying* $(\mathbf{F}1)$. *For each* $u \in X$, *set* $\varkappa(u) = \int_{\partial\Omega} F(x,u)\,d\sigma$. *Then* $\varkappa(u) \in C^1(X,\mathbb{R})$ *and*

$$\left\langle \varkappa'(u), \upsilon \right\rangle = \int_{\partial\Omega} f(x,u)\,\upsilon\, d\sigma,$$

for all $\upsilon \in X$. *Moreover, the operator* $\varkappa : X \to X^*$ *compact.*

Proposition 7 ([30, Theorem 3.1])**.** *If one denotes*

$$\varphi(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx$$

then, $\varphi \in C^1(X,\mathbb{R})$ *and the derivative operator of* φ, *denoted by* φ', *is*

$$\langle \varphi'(u), v \rangle = \int_\Omega |\nabla u|^{p(x)-2} uv dx,$$

for all $u,v \in X$ *and one has*

(i) $\varphi' : X \to X^*$*is a continuous, bounded, strictly monotone operator and homeomorphism,*

(ii) φ' *is a mapping of of* (S_+) *type, that is,*

if $u_n \rightharpoonup u$ *in* X *and* $\limsup_{n\to\infty} \langle \varphi'(u_n), u_n - u \rangle \le 0$ *implies, then* $u_n \to u$ *in* X.

In order to apply Theorem 1, we define the functionals ; $\Phi, \Psi : X \to \mathbb{R}$

$$\Phi(u) = \widehat{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx \right) + \int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx$$

$$\Psi(u) = -\int_{\partial\Omega} F(x,u)\, d\sigma,$$

where $\widehat{M}(t)$ and $F(x,t)$ are denoted by

$$\widehat{M}(t) = \int_0^t M(k)\, dk, \;\; F(x,t) = \int_0^t f(x,k)\, dk \text{ for all } t > 0 \text{ and } (x,k) \in \partial\Omega \times \mathbb{R}.$$

Then energy functional associated to the problem (1.1) is $J_\lambda(u) = \Phi(u) + \lambda\Psi(u)$. Moreover, from $(\mathbf{F}1)$, $(\mathbf{M}_0)$, Proposition 6 and Proposition 7, it is easy to see that the functional $\Phi, \Psi \in C^1(X,\mathbb{R})$ and we can infer that critical points of functional J are the weak solutions for problem (1.1). Then, we have

$$\langle \Phi'(u), \upsilon\rangle = M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx\right) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \upsilon dx + \int_\Omega |u|^{p(x)-2} u\upsilon dx$$

$$\langle \Psi'(u), \upsilon\rangle = -\int_{\partial\Omega} f(x,u)\, v d\sigma,$$

for any $u, \upsilon \in X$.

We say that $u \in X$ is a weak solution of (1.1) if

$$M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx\right) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_\Omega |u|^{p(x)-2} uv dx$$
$$= \lambda \int_{\partial\Omega} f(x,u) v d\sigma$$

where $v \in X$.

Proof of Theorem 2. To prove this theorem, we first show the condition (i) of Theorem 1. For any $u \in X$ with $\|u\| > 1$, using $(\mathbf{M}_0)$ and Proposition 2, we have

$$\begin{aligned}\Phi(u) &= \widehat{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx\right) + \int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx \\ &\geq \frac{m_1}{\alpha (p^+)^\alpha} \|u\|^{\alpha p^-} + \frac{1}{p^+} \|u\|^{p^-} \\ &\geq \frac{m_1}{\alpha (p^+)^\alpha} \|u\|^{\alpha p^-}. \end{aligned} \tag{3.1}$$

On the other hand, by $(\mathbf{F}_1)$ and Proposition 1, we get

$$\begin{aligned}\lambda\Psi(u) &= -\lambda \int_{\partial\Omega} F(x,u)\, d\sigma = -\lambda \int_{\partial\Omega} \left(\int_0^{u(x)} f(x,t)\, dt\right) d\sigma \\ &\geq -\lambda c_0 \int_{\partial\Omega} \left(a(x)\,|u(x)| + \frac{b}{s(x)} |u(x)|^{s(x)}\right) d\sigma\end{aligned}$$

$$\geq -\lambda c_0 \|a(x)\|_{\frac{s(x)}{s(x)-1},\partial\Omega} \|u(x)\|_{s(x),\partial\Omega} - \frac{\lambda b}{s^-}\int_{\partial\Omega} |u(x)|^{s(x)}\,d\sigma. \tag{3.2}$$

Since we have that X is continuous embedding $L^{s^+}(\partial\Omega)$ from Proposition 4, there exist positive constant c_1 such that

$$|u|_{L^{s^+}(\partial\Omega)} \leq c_1 \|u\|. \tag{3.3}$$

If we use (3.2) and (3.3), we get

$$\lambda\Psi(u) \geq -\lambda c_0 \|a(x)\|_{\frac{s(x)}{s(x)-1},\partial\Omega} \|u(x)\|_{s(x),\partial\Omega} - \frac{\lambda b c_1}{s^-}\|u\|^{s^+}.$$

Combining (3.1) and (3.3) for any $\lambda > 0$, we obtain

$$J_\lambda(u) \geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{p^-} - \lambda c_0 \|a(x)\|_{\frac{s(x)}{s(x)-1},\partial\Omega} \|u(x)\|_{s(x),\partial\Omega} - \frac{\lambda b c_1}{s^-}\|u\|^{s^+}.$$

Since $p^- > s^+$, it follows that

$$\lim_{\|u\|\to\infty} J_\lambda(u) = \lim_{\|u\|\to\infty} (\Phi(u) + \lambda\Psi(u)) = \infty.$$

Then, condition (i) of Theorem 1 is satisfied.

Now, we show the condition (ii) of Theorem 1. By $f(x,t) = \frac{\partial F(x,t)}{\partial t}$ and the condition (**F**2), it is obtained that $F(x,t)$ is increasing for $t \in (t_0,\infty)$ and decreasing for $t \in (0,1)$ uniformly for $x \in \partial\Omega$, and $F(x,0) = 0$ is obvious, $F(x,t) \to \infty$ when $t \to \infty$, since $F(x,t) \geq \vartheta t$ uniformly on x. Then, there exists a real number $\delta > t_0$ such that

$$F(x,t) \geq 0 = F(x,0) \geq F(x,\eta), \quad \forall u \in X,\ t > \delta,\ \eta \in (0,1). \tag{3.4}$$

Let β,γ be two real numbers such that $0 < \beta < \min\{1,c\}$, with c given in Remark 1. Also, we choose $\gamma > \delta$ satisfying $\gamma^{p^-}|\Omega| > 1$. If we use (3.4), we have

$$F(x,t) \leq F(x,0) = 0 \text{ for } t \in [0,\beta].$$

Then

$$\int_{\partial\Omega} \sup_{0<t<\beta} F(x,t)d\sigma \leq \int_{\partial\Omega} F(x,0)d\sigma = 0. \tag{3.5}$$

Moreover, since $\gamma > \delta$, we write

$$\int_{\partial\Omega} F(x,\gamma)d\sigma > 0$$

and

$$\frac{1}{c^{p^+}}\cdot\frac{\beta^{p^+}}{\gamma^{p^-}}\int_{\partial\Omega} F(x,\gamma)d\sigma > 0. \tag{3.6}$$

Combining (3.5) and (3.6), we obtain

$$\int_{\partial\Omega} \sup_{0<t<\beta} F(x,t) \leq 0 < \frac{1}{c_2^{p^+}}\cdot\frac{\beta^{p^+}}{\gamma^{p^-}}\int_{\partial\Omega} F(x,\gamma)d\sigma.$$

Let $u_0, u_1 \in X,\ u_0(x) = 0$ and $u_1(x) = \gamma$ for any $x \in \overline{\Omega}$. If we define $r = \frac{1}{p^+}.\left(\frac{\beta}{c}\right)^{p^+}$, then $r \in (0,1)$, $\Phi(u_0) = \Psi(u_0) = 0$,

$$\Phi(u_1) \geq \int_\Omega \frac{1}{p(x)}\gamma^{p(x)}dx \geq \frac{1}{p^+}.\gamma^{p^-}|\Omega| > \frac{1}{p^+}.1 > \frac{1}{p^+}.\left(\frac{\beta}{c}\right)^{p^+} = r$$

and

$$\Psi(u_1) = -\int_{\partial\Omega} F(x, u_1(x))d\sigma = -\int_{\partial\Omega} F(x,\gamma)d\sigma < 0.$$

So, we deduce that $\Phi(u_0) < r < \Phi(u_1)$, that is, condition (ii) of Theorem 1 is satisfied.

Finally, we show the condition (iii) of Theorem 1, we have

$$-\frac{(\Phi(u_1) - r)\Psi(u_0) + (r - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)} = -\frac{r\,\Psi(u_1)}{\Phi(u_1)}$$
$$= r\frac{\int_{\partial\Omega} F(x,\gamma)d\sigma}{\int_\Omega \frac{1}{p(x)}\gamma^{p(x)}dx} > 0.$$

Let $u \in X$ such that $\Phi(u) \leq r < 1$. Since $\frac{1}{p^+}\psi_{p(x)}(u) \leq \Phi(u) \leq r$ for $u \in X$, we obtain

$$\psi_{p(x)}(u) \leq p^+ r = \left(\frac{\beta}{c}\right)^{p^+} < 1.$$

Then by Proposition 2 for $\|u\| < 1$, we have

$$\frac{1}{p^+}\|u\|^{p^+} \leq \frac{1}{p^+}\psi_{p(x)}(u) \leq \Phi(u) \leq r.$$

On the other hand, from Remark 1 and (3.5), we get

$$\|u\|_{C(\overline{\Omega})} \leq c\|u\| \leq c\left(p^+ r\right)^{\frac{1}{p^+}} = \beta,$$

for all $u \in X$ and $x \in \overline{\Omega}$ with $\Phi(u) \leq r$.

The above inequality shows that

$$-\inf_{u \in \Phi^{-1}((-\infty,r])} \Psi(u) = \sup_{u \in \Phi^{-1}((-\infty,r])} -\Psi(u) = \int_{\partial\Omega} \sup_{0<t<\beta} F(x,t) \leq 0.$$

Thus we have

$$-\inf_{u \in \Phi^{-1}((-\infty,r])} \Psi(u) < r\frac{\int_{\partial\Omega} F(x,\gamma)d\sigma}{\int_\Omega \frac{1}{p(x)}\gamma^{p(x)}dx}$$

or

$$\inf_{u \in \Phi^{-1}((-\infty,r])} \Psi(u) > \frac{(\Phi(u_1) - r)\Psi(u_0) + (r - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)}.$$

So, condition (iii) of Theorem 1 is obtained. Since all conditions of Theorem 1 are verified, there exists an open interval $\Lambda \subset (0,\infty)$ and a positive real number ρ such that for each $\lambda \in \Lambda$ the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0$$

has at least three solutions in X whose norms are *less than* ρ. The proof of Theorem 2 is complete. ☐

Lemma 1. *Suppose* $(\mathbf{M}_0), (\mathbf{F}3)$ *and* $(\mathbf{F}5)$ *hold. Then, there exist two positive real numbers* τ, r *and* $\lambda^* > 0$ *such that for any* $\lambda \in (0,\lambda^*)$, *we have* $J_\lambda(u) \geq r > 0$, $u \in X$ *with* $\|u\| = \tau$ *for all* $\tau \in (0,1)$.

Proof. Since we have the continuous embedding $X \hookrightarrow L^{q(x)}(\partial\Omega)$ from Proposition 4, there exist positive constants c_2 such that

$$|u|_{L^{q(x)}(\partial\Omega)} \leq c_2 \|u\| \text{ for all } u \in X. \tag{3.7}$$

On the other hand, using assumption $(\mathbf{F}3)$, there exist c_3 and c_4 are positive constant such that

$$|F(x,u)| \leq c_3 |u|^{q(x)} \text{ and } f(x,u) \leq c_4 |u|^{q(x)-1} \text{ for all } x \in \partial\Omega. \tag{3.8}$$

By taking into account $(\mathbf{M}_0), (\mathbf{F}3)$, (3.7), (3.8) and Proposition 4, we get

$$\begin{aligned}
J_\lambda(u) &\geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^+} + \frac{1}{p^+}\int_\Omega |u|^{p(x)}dx - \lambda c_3 \int_{\partial\Omega} |u|^{q(x)} d\sigma \\
&\geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^+} - \lambda c_3 c_2^{q^-}\|u\|^{q^-} \\
&\geq \left(\frac{m_1}{\alpha(p^+)^\alpha}\tau^{\alpha p^+ - q^-} - \lambda c_3 c_2^{q^-}\right)\tau^{q^-}.
\end{aligned}$$

By the above inequality, we remark that $q^- < \alpha p^+$ from $(\mathbf{F}5)$ and if we choose

$$\lambda^* = \frac{m_1}{2c_3 c_2^{q^-}\alpha(p^+)^\alpha}\tau^{\alpha p^+ - q^-},$$

then, there exist two positive real numbers τ and r such that $\lambda \in (0,\lambda^*)$, we obtain

$$J_\lambda(u) \geq r > 0, \forall u \in X \text{ with } \|u\| = \tau \in (0,1).$$

The proof of Lemma 1 is complete. ☐

Lemma 2. *Assume that* $(\mathbf{M}_0), (\mathbf{F}3), (\mathbf{F}4)$ *and* $(\mathbf{F}5)$. *Then, there exist* $\omega \in X$ *such that* $\omega \geq 0$, $\omega \neq 0$ *and* $J_\lambda(t\omega) < 0$ *for* $t > 0$ *small enough.*

Proof. From assumption $(\mathbf{F}5)$, we know that $q^- < p^-$. Let $\varepsilon_0 > 0$ be such that $q^- + \varepsilon_0 < p^-$. On the other hand, since $q \in C(\overline{\Omega})$ it follows that there exists an open set $\Omega_0 \subset \Omega \subset\subset \partial\Omega$ such that $|q(x) - q^-| < \varepsilon_0$ for all $x \in \overline{\Omega}_0$. Hence, we conclude that

$$q(x) \leq q^- + \varepsilon_0 < p^- \text{ for all } x \in \overline{\Omega}_0.$$

Let $\omega \in C_0^\infty(\Omega)$ be such that $supp(\omega) \supset \overline{\Omega}_0$, $\omega(x) = 1$ for all $x \in \overline{\Omega}_0$ and $0 \leq \omega(x) \leq 1$ in Ω. Then, by the above information for any $t \in (0,1)$, we have

$$\begin{aligned} J_\lambda(t\omega) &= \widehat{M}\left(\int_\Omega \frac{|\nabla t\omega|^{p(x)}}{p(x)} dx\right) + \int_\Omega \frac{1}{p(x)} |t\omega|^{p(x)} dx - \lambda \int_{\partial\Omega} F(x, t\omega)\, d\sigma \\ &\leq \frac{m_2 t^{\alpha p^-}}{\alpha (p^-)^\alpha} \int_\Omega |\nabla\omega|^{p(x)} dx + \frac{t^{p^-}}{p^-} \int_\Omega |\omega|^{p(x)} dx - \lambda \frac{t^{q^- + \varepsilon_0}}{q^+} \int_{\Omega_0} F(x,\omega)\, d\sigma \\ &\leq \frac{m_2 t^{p^-}}{p^-} \int_\Omega \left(|\nabla\omega|^{p(x)} + |\omega|^{p(x)}\right) dx - \lambda t^{q^- + \varepsilon_0} \int_{\Omega_0} F(x,\omega)\, d\sigma. \end{aligned}$$

Thus

$$J_\lambda(t\omega) < 0,$$

for $t < \xi^{1/(p^- - q^- - \varepsilon_0)}$ with

$$0 < \xi < \min\left\{1, \frac{\lambda p^-}{m_2} . \frac{\int_{\Omega_0} F(x,\omega)\, d\sigma}{\int_\Omega \left(|\nabla\omega|^{p(x)} + |\omega|^{p(x)}\right) dx}\right\}.$$

On the other hand, we point out that $\int_\Omega \left(|\nabla\omega|^{p(x)} + |\omega|^{p(x)}\right) dx > 0$ from Proposition 2. If $\int_\Omega \left(|\nabla\omega|^{p(x)} + |\omega|^{p(x)}\right) dx$, then $\|\omega\| = 0$ i.e. $\omega = 0$ in Ω which is a contradiction. Moreover, we know that $\int_{\Omega_0} F(x,\omega)\, d\sigma > 0$ from $(\mathbf{F}4)$. The proof of Lemma 2 is complete. □

Proposition 8.

$$\lim_{n\to\infty} \int_{\partial\Omega} f(x, u_n)\, u_n (u_n - u)\, d\sigma = 0.$$

Proof. By Proposition 1, Proposition 4, Proposition 5 and inequality (3.8), we obtain

$$\begin{aligned} \left|\int_{\partial\Omega} f(x,u_n)(u_n - u)\, d\sigma\right| &\leq c_4 \left|\int_{\partial\Omega} \left(|u_n|^{q(x)-1}\right)(u_n - u)\, d\sigma\right| \\ &\leq c_5 \left|\,|u_n|^{q(x)-1}\right|_{L^{q'(x)}} |u_n - u|_{L^{q(x)}(\partial\Omega)}. \end{aligned}$$

where $c_5 > 0$ is a constant. If we consider the compact embedding $X \hookrightarrow L^{q(x)}(\partial\Omega)$, that is, $|u_n - u|_{L^{q(x)}(\partial\Omega)} \to 0$ as $n \to \infty$, we have

$$\int_{\partial\Omega} f(x,u_n)(u_n - u)\, d\sigma \to 0.$$

□

Proof of Theorem 3. By Lemma 1, we infer that there exists on the boundary of the ball centered at the origin and of radius ρ in X such that $\inf_{\partial B_\rho(0)} J_\lambda > 0$. Moreover, from (3.6), there exists $\omega \in X$ such that $J_\lambda(t\omega) < 0$, for all $t > 0$ small enough. Thus, take into account inequality (3.7), we obtain the following

$$-\infty < c_6 := \inf_{B_\rho(0)} J_\lambda < 0.$$

Let choose $\varepsilon > 0$. Then, it follows

$$0 < \varepsilon < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{B_\rho(0)} J_\lambda.$$

Applying Ekeland's variational principle [15] to the functional $J_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, we can find $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$J_\lambda(u_\varepsilon) < \inf_{B_\rho(0)} J_\lambda + \varepsilon,$$

$$J_\lambda(u_\varepsilon) < J_\lambda(u) + \varepsilon \|u - u_\varepsilon\|, \; u \neq u_\varepsilon.$$

By the fact that

$$J_\lambda(u_\varepsilon) < \inf_{B_\rho(0)} J_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} J_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} J_\lambda,$$

we can infer that $u_\varepsilon \in B_\rho(0)$. Now, we define $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $I_\lambda(u) = J_\lambda(u) + \varepsilon \|u - u_\varepsilon\|$. It is clear that u_ε is a minimum point of I_λ, and thus

$$\frac{I_\lambda(u_\varepsilon + tv) - I_\lambda(u_\varepsilon)}{t} \geq 0,$$

for $t > 0$ small enough and any $v \in B_1(0)$. By the above relation, we have

$$\frac{J_\lambda(u_\varepsilon + tv) - J_\lambda(u_\varepsilon)}{t} + \varepsilon \|v\| \geq 0.$$

Letting $t \to 0$, we have that $\langle J_\lambda'(u_\varepsilon), \upsilon \rangle + \varepsilon \|v\| > 0$ and we infer that $\|J_\lambda'(u_\varepsilon)\| \leq \varepsilon$. We show that there exists a sequence $\{u_n\} \subset B_\rho(0)$ such that

$$J_\lambda(u_n) \to c = \inf_{B_\rho(0)} J_\lambda < 0 \text{ and } J_\lambda'(u_n) \to 0.$$

Hence, we have that the sequence $\{u_n\}$ is bounded in X. Thus, there exists $u \in X$ such that, up to a subsequence, $\{u_n\}$ converges weakly to $u \in X$. So $\langle J_\lambda'(u_n), u_n - u \rangle \to 0$ as $n \to \infty$. On the other hand, we have

$$\begin{aligned}\langle J_\lambda'(u_n), u_n - u \rangle = {} & M\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)} dx\right) \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) \, dx \\ & + \int_\Omega |u_n|^{p(x)-2} u_n (u_n - u) \, dx - \lambda \int_{\partial\Omega} f(x, u_n) u_n (u_n - u) \, d\sigma \to 0.\end{aligned} \tag{3.9}$$

By Proposition 1, Proposition 4, Proposition 5, we have

$$\int_{\Omega} |u_n|^{p(x)-2} u_n (u_n - u)\, dx \to 0. \tag{3.10}$$

From (3.9), (3.10) and Proposition 8, we obtain

$$M\left(\int_{\Omega} \frac{|\nabla u_n|^{p(x)}}{p(x)} dx\right) \int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) \to 0.$$

Moreover, from $(\mathbf{M}_0)$, we conclude that

$$\int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) \to 0.$$

Eventually, from Proposition (7), we get $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, so we conclude that u is a nontrivial weak sollution for problem (1.1). The proof of Theorem 3 is complete. □

ACKNOWLEDGEMENTS

I would like to express my thanks to an anonymous referee for useful comments.

REFERENCES

[1] G. A. Afrouzi, A. Hadjian, and S. Heidarkhani, “Steklov problem involving the p(x)-Laplacian.” vol. 134, pp. 1–11, 2014, doi: 10.1080/17476933.2015.1031122.
[2] K. B. Ali, “Existence results for Steklov problem involving the p(x)-Laplace operator.” vol. 63, no. 12, pp. 1675–1686, 2018, doi: 10.7494/OpMath.2017.37.6.779.
[3] K. B. Ali, A. Ghanmi, and K. Kefi, “On the Steklov problem involving the p (x)-Laplacian with indefinite weight.” vol. 37, no. 6, 2017, doi: 10.7494/OpMath.2017.37.6.779.
[4] M. Allaoui, “Continuous spectrum of Steklov nonhomogeneous elliptic problem.” vol. 35, no. 6, pp. 853–866, 2015, doi: 10.7494/OpMath.2015.35.6.853.
[5] M. Allaoui, A. R. El Amrouss, and A. Ourraoui, “Existence and multiplicity of solutions for a Steklov problem involving the p(x)-Laplace operator.” vol. 132, pp. 1–12, 2012.
[6] S. N. Antontsev and S. I. Shmarev, “A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions.” vol. 60, no. 3, pp. 515–545, 2005, doi: 10.1016/j.na.2004.09.026.
[7] A. Ayoujil, “On the superlinear Steklov problem involving the p(x)-Laplacian.” vol. 2014, no. 38, pp. 1–13, 2014, doi: 10.14232/ejqtde.2014.1.38.
[8] G. Bonanno and P. Candito, “Three solutions to a Neumann problem for elliptic equations involving the p-Laplacian.” vol. 80, no. 4, pp. 424–429, 2003, doi: 10.1007/s00013-003-0479-8.
[9] Y. Chen, S. Levine, and M. Rao, “Variable exponent, linear growth functionals in image restoration.” vol. 66, no. 4, pp. 1383–1406, 2006, doi: 10.1137/050624522.
[10] N. T. Chung, “Multiple solutions for an anisotropic elliptic equation of Kirchhoff type in bounded domain.” vol. 1, no. 3, pp. 116–127, 2018.
[11] G. Dai and R. Hao, “Existence of solutions for a p(x)-Kirchhoff-type equation.” vol. 359, no. 1, pp. 275–284, 2009, doi: 10.1016/j.jmaa.2009.05.031.
[12] S.-G. Deng, “Eigenvalues of the p(x)-Laplacian Steklov problem.” vol. 339, no. 2, pp. 925–937, 2008, doi: 10.1016/j.jmaa.2007.07.028.

[13] N. C. Eddine and M. A. Ragusa, "Generalized critical Kirchhoff-type potential systems with Neumann Boundary conditions." *Applicable Analysis*, pp. 1–31, 2022, doi: 10.1080/00036811.2022.2057305.

[14] D. Edmunds and J. Rákosník, "Sobolev embeddings with variable exponent." vol. 3, no. 143, pp. 267–293, 2000.

[15] I. Ekeland, "On the variational principle." vol. 47, no. 2, pp. 324–353, 1974.

[16] X. L. Fan, J. Shen, and D. Zhao, "Sobolev embedding theorems for spaces Wk,p(x)." vol. 262, pp. 749–760, 2001, doi: 10.1006/jmaa.2001.7618.

[17] X. Fan and D. Zhao, "On the spaces Lp(x) and Wm,p(x)." vol. 263, pp. 424–446, 2001, doi: 10.1006/jmaa.2000.7617.

[18] M. Hsini, N. Irzi, and K. Kefi, "Nonhomogeneous p(x)-Laplacian Steklov problem with weights." 2019" doi: 10.1080/17476933.2019.1597070.

[19] B. Karim, A. Zerouali, and O. Chakrone, "Steklov eigenvalue problem with a-harmonic solutions and variable exponents." vol. 28, no. 3, pp. 363–373, 2021, doi: 10.1515/gmj-2019-2079.

[20] M. Khodabakhshi, S. M. Vaezpour, and M. Tavani, "Existence results for a Kirchhoff-type problem with singularity." *Miskolc Mathematical Notes*, vol. 22, no. 1, pp. 351–362, 2021, doi: 10.18514/MMN.2021.3429.

[21] G. Kirchhoff, *Mechanik.* Teubner, Leipzig, Germany, 1883, vol. 1.

[22] O. Kováćik and J. Rákosnik, "On the spaces Lp(x) and Wk,p(x)." vol. 41, pp. 1592–618, 1991.

[23] M. Mihăilescu and V. Rădulescu, "A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids." vol. 462, no. 2073, pp. 2625–2641, 2006, doi: 10.1016/j.jmaa.2007.09.015.

[24] A. Ourraoui, "Existence and uniqueness of solutions for Steklov problem with variable exponent." vol. 5, no. 1, pp. 158–166, 2021, doi: 10.31197/atnaa.688047.

[25] Z. Wei and Z. Chen, "Existence results for the p(x)-Laplacian with nonlinear boundary condition." vol. 2012, 2012, doi: 10.5402/2012/727398.

[26] M. Willem, *Minimax theorems.* Springer Science & Business Media, 1997, vol. 24.

[27] Y. Wu, Z. Qiao, M. K. Hamdani, B. Kou, and L. Yang, "A class of variable-order fractional-Kirchhoff-type systems." *Journal of Function Spaces*, vol. 2021, 2021, doi: 10.1155/2021/5558074.

[28] Z. Yucedag, M. Avci, and R. Mashiyev, "Existence of weak solutions for p (x)-Kirchhoff-type equation." vol. 92, pp. 61–71, 2014, doi: 10.12732/ijpam.v92i1.5.

[29] A. A. Zerouali, B. Karim, and O. Chakrone, "Existence and multiplicity of a-harmonic solutions for a Steklov problem with variable exponents." vol. 36, no. 2, pp. 125–136, 2018, doi: 10.5269/bspm.v36i2.31071.

[30] A. A. Zerouali, B. Karim, O. Chakrone, and A. Anane, "Existence and multiplicity results for elliptic problems with nonlinear boundary conditions and variable exponents." vol. 33, no. 2, pp. 123–133, 2015, doi: 10.5269/bspm.v33i2.23355.

[31] V. V. Zhikov, "Averaging of functionals of the calculus of variations and elasticity theory." vol. 29, no. 1, p. 33, 1987, doi: 10.1070/IM1987v029n01ABEH000958.

Author's address

Zehra Yücedag
Dicle University, Department of Mathematics, 21280 Diyarbakir, Turkey
E-mail address: zyucedag@dicle.edu.tr