Miskolc Mathematical Notes
Vol. 23 (2022), No. 2, pp. 691–701
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2022.4320

DECOMPOSITION OF FUNCTIONAL EQUATIONS WITH APPLICATIONS

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Received 07 July, 2022

Abstract. In this paper the Decomposition Theorem for functional equations is shown. As an application of this Theorem the two times continuously differentiable solution of the functional equation

$$G_1(x(x+y)) + F_1(y) = G_2(y(x+y)) + F_2(y)$$

can be given with unknown functions G_i, $F_i : \mathbb{R}_+ \to \mathbb{R}$ $(i = 1,2)$ where the Equation is fulfilled for all $x,y \in \mathbb{R}_+$ (where $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$).

2010 *Mathematics Subject Classification:* Primary 39B22

Keywords: functional equations, additive and logarithmic functions, linear differential equations, measurable solutions

1. Introduction

The main purpose of this paper is to show the decomposition of Pexider functional equations.

The Pexider equations are functional equations contained more than one unknown functions. For example, consider the well-known Lobachevsky functional equation

$$f(x+y)f(x-y) = f^2(x)$$

where the unknown function $f : \mathbb{R} \to \mathbb{R}$ satisfies the Equation for all $x,y \in \mathbb{R}$ [1]. The Pexider version [17] of this Equation is

$$F(x+y)G(x-y) = H^2(x),$$

where the unknown functions F, G, $H : \mathbb{R} \to \mathbb{R}$ satisfy the Equation for all $x,y \in \mathbb{R}$.

The decomposition of functional equations is a method for Pexider equations with certain symmetric structure. Such an equation results in two other, simpler equations. The general (twice continuously differentiable) solution of the original function equation can be expressed by the general (twice continuously differentiable) solutions of the obtained equations.

In section 2 the Decomposition Theorem is given. In the rest of the paper any applications of the Decomposition Theorem can be found.

2. The Decomposition Theorems

Theorem 1. *Let X be a set; $\circ$ and $*$ be binary operations on the set X; $(Y,+)$ be a uniquely two divisible Abelian group.*

(1) *If the functions G_i, $F_i : X \to Y$ $(i = 1,2)$ are solutions of the functional equation*

$$G_1(x\circ y)+F_1(x*y) = G_2(y\circ x)+F_2(y*x) \qquad (x,y\in X), \tag{2.1}$$

and the functions g, γ, f, $\varphi : X \to Y$ are defined by

$$\begin{aligned} g &:= \frac{1}{2}(G_1+G_2), \quad f := \frac{1}{2}(F_1+F_2), \\ \gamma &:= \frac{1}{2}(G_1-G_2), \quad \varphi := \frac{1}{2}(F_1-F_2), \end{aligned} \tag{2.2}$$

then this functions satisfy the Equation

$$\begin{aligned} g(x\circ y)+f(x*y) &= g(y\circ x)+f(y*x), \\ \gamma(x\circ y)+\gamma(y\circ x) &= -\varphi(x*y)-\varphi(y*x) \end{aligned} \qquad (x,y\in X).$$

(2) *If the functions g, γ, f, $\varphi : X \to Y$ are solutions of Equation (1), and the functions G_i, $F_i : X \to Y$ are defined by*

$$\begin{aligned} G_1 &:= g+\gamma, \quad F_1 := f+\varphi, \\ G_2 &:= g-\gamma, \quad F_2 := f-\varphi, \end{aligned} \tag{2.3}$$

then this functions satisfy Equation (2.1).

Proof. (1) Assume that the functions G_i, $F_i : X \to Y$ $(i = 1,2)$ are solutions of the functional equation (2.1). Consider the Equation (2.1) and the Equation which can be obtained from Equation (2.1) by changing x and y.

$$\begin{aligned} G_1(x\circ y)+F_1(x*y) &= G_2(y\circ x)+F_2(y*x) \\ G_2(x\circ y)+F_2(x*y) &= G_1(y\circ x)+F_1(y*x) \end{aligned} \tag{2.4}$$

for all $x,y\in X$. Add and subtract the Equations of (2.4). Thus we have

$$\begin{aligned} (G_1+G_2)(x\circ y)+(F_1+F_2)(y) &= (G_1+G_2)(y\circ x)+(F_1+F_2)(x) \\ (G_1-G_2)(x\circ y)+(F_1-F_2)(y) &= (G_2-G_1)(y\circ x)+(F_2-F_1)(x) \end{aligned} \tag{2.5}$$

for all $x,y\in X$. Define the functions g, γ, f, $\varphi : X \to Y$ by (2.2) thus by Equation (2.5) we obtain the Equations of (1).
(2) Assume that the functions g, γ, f, $\varphi : X \to Y$ are solutions of the Equations of (2.3) and define the functions G_i, $F_i : X \to Y$ $(i = 1,2)$ by (1). Then

$$\begin{aligned} &G_1(x\circ y)+F_1(x*y) \\ &\qquad = (g(x\circ y)+\gamma(x\circ y))+(f(x*y)+\varphi(x*y)) \\ &\qquad = (g(x\circ y)+f(x*y))+(\gamma(x\circ y)+\varphi(x*y)) \end{aligned}$$

$$\begin{aligned}&= (g(y\circ x)+f(y*x))+(-\gamma(y\circ x)-\varphi(y*x))\\ &= (g(y\circ x)-\gamma(y\circ x))+(f(y*x)-\varphi(y*x))\\ &= G_2(y\circ x)+F_2(y*x)\end{aligned}$$

for all $x,y\in X$ which completes the proof. □

The following Theorem is a special case of Theorem 1 with $x*y:=y$ for all $x,y\in X$.

Theorem 2. *Let X be a set $\circ$ be a binary operation on the set X, $(Y,+)$ be a uniquely two divisible Abelian group. The functions $G_i,F_i : X\to Y$ $(i=1,2)$ are solutions of the functional equation*

$$G_1(x\circ y)+F_1(y) = G_2(y\circ x)+F_2(x) \qquad (x,y\in X) \tag{2.6}$$

if and only if they are of the form

$$\begin{aligned}G_1(x) &= g(x)+\gamma(x), \qquad & F_1(x) &= f(x)-\gamma(x\circ x),\\ G_2(x) &= g(x)-\gamma(x), \qquad & F_2(x) &= f(x)+\gamma(x\circ x)\end{aligned}$$

for all $x\in X$ where the functions f, g, $\gamma : X\to Y$ are solutions of the Equations

$$\begin{aligned}g(x\circ y)+f(y) &= g(y\circ x)+f(x) \qquad && (x,y\in X)\\ \gamma(x\circ x)+\gamma(y\circ y) &= \gamma(x\circ y)+\gamma(y\circ x) \qquad && (x,y\in X).\end{aligned}$$

Proof. This Theorem can be similarly proven as Theorem 1. □

We investigate the case $X:=\mathbb{R}_+:=\{x\in\mathbb{R}\mid x>0\}$, $x\circ y:=x(x+y)$ for all $x,y\in\mathbb{R}_+$, $Y(+):=\mathbb{R}(+)$, and we want to find the twice continuously differentiable solutions. By Theorem 2 instead of Equation

$$G_1(x(x+y))+F_1(y) = G_2(y(x+y))+F_2(x) \qquad (x,y\in\mathbb{R}_+) \tag{2.7}$$

with unknown functions G_i, $F_i : \mathbb{R}_+\to\mathbb{R}$ $(i=1,2)$ where the Equation is fulfilled for all $x,y\in\mathbb{R}_+$ it is enough to give the twice continuously differentiable solutions of the Equations

$$g(x(x+y))+f(y) = g(y(x+y))+f(y) \qquad (x,y\in\mathbb{R}_+) \tag{2.8}$$

and

$$\gamma(x(x+y))+\gamma(y(x+y)) = \gamma(2x^2)+\gamma(2y^2) \qquad (x,y\in\mathbb{R}_+). \tag{2.9}$$

In Equation (2.9) we will use the function $\delta:\mathbb{R}_+\to\mathbb{R}$ defined by $\delta(x):=\gamma(2x^2)$ for all $x\in\mathbb{R}_+$.

Equation (2.9) (and with it its Pexider version, Equation 2.7) is related to equations containing means and equations containing the Gauss composition of these means (see [3], [4], [5], [7], and [8]), so these equations are very important.

The measurable solutions of Equation of (2.7) can be easily obtained by using the method of A. Járai [15], [16], but we do not deal with finding the measurable solutions, because we want to know the general solutions in more general settings see our Conjecture in the last section of this paper.

3. The twice continuously differentiable solutions of Equation (2.8)

In his paper [18] S. Narumi was the first to use differential calculus to solve functional equations. Nowadays, there are many different methods for constructing differential operators that can be used to reduce a functional equation to a differential equation [10], [14], [19]. Instead of them we will use only two similar and natural differential operators.

Proposition 1. *If the functions F_i, $G_i : \mathbb{R}_+ \to \mathbb{R}$ are twice differentiable solutions of Equation(2.7) then*

$$0 = \frac{-2x+y}{x+2y}F_1'(y) + \frac{-2x-y}{x+2y}F_1''(y) + \frac{-x+2y}{x+2y}F_2'(x) + \frac{x}{y}F_2''(x) \tag{3.1}$$

for all $x,y \in \mathbb{R}_+$.

Proof. Apply the differential operator

$$D_1 := \frac{1}{2x+y}\frac{\partial}{\partial x} - \frac{1}{x}\frac{\partial}{\partial y}$$

for Equation (2.7), whence we have that

$$-\frac{1}{x}F_1'(y) = -\frac{2(x+y)^2}{x(2x+y)}G_2'(xy+y^2) + \frac{1}{2x+y}F_2'(x) \qquad (x,y \in \mathbb{R}_+),$$

whence by ordering we have that

$$G_2'(xy+y^2) = \frac{2x+y}{2(x+y)^2}F_1'(y) + \frac{x}{2(x+y)^2}F_2'(x) \qquad (x,y \in \mathbb{R}_+). \tag{3.2}$$

Apply the differential operator

$$D_{2:=}\frac{1}{y}\frac{\partial}{\partial x} - \frac{1}{x+2y}\frac{\partial}{\partial y}$$

for Equation (3.2) thus we obtain Equation (3.1). □

Proposition 2. *If the functions f, $g : \mathbb{R}_+ \to \mathbb{R}$ are twice differentiable solutions of the Equation (2.8), and the constants c_1, c_2; and functions a_0, a_1, $\Phi : \mathbb{R}_+ \to \mathbb{R}$ are defined by*

$$\begin{aligned} c_1 &:= f'(1), \quad c_2 := f''(1), \\ a_0(x) &:= c_1\frac{2x-1}{x(x+2)} + c_2\frac{2x+1}{x(x+2)} \qquad (x \in \mathbb{R}_+) \\ a_1(x) &:= \frac{x-2}{x(x+2)} \qquad (x \in \mathbb{R}_+), \\ \Phi(x) &:= f'(x) \qquad (x \in \mathbb{R}_+), \end{aligned} \tag{3.3}$$

then

$$\Phi'(x) = a_0(x) + a_1(x)\Phi(x) \qquad (x \in \mathbb{R}_+) \tag{3.4}$$

Proof. Write $f(y)$ instead of $F_1(y)$ and $f(x)$ instead of $F_2(x)$ in Equation(3.1) thus we have that

$$0 = \frac{-2x+y}{x+2y} f_1'(y) + \frac{-2x-y}{x+2y} f''(y) + \frac{-x+2y}{x+2y} f'(x) + \frac{x}{y} f''(x) \tag{3.5}$$

for all $x, y \in \mathbb{R}_+$. Write 1 instead of y in Equation (3.5) after order the obtained Equation. Thus we have that

$$f''(x) = f'(1)\frac{2x-1}{x(x+2)} + f''(1)\frac{2x+1}{x(x+2)} + \frac{x-2}{x(x+2)} f'(x) \qquad (x \in \mathbb{R}_+),$$

whence by the notations of (3.3) we obtain the differential Equation (3.4). □

For to solve the above differential equation we can apply the well-known formula

$$\Phi(x) = \exp\left(\int_1^x a_1(u)du\right)\left(c_1 + \int_1^x \frac{a_0(v)}{\exp(\int_1^v a_1(u)du)} dv\right) \qquad (x \in \mathbb{R}_+).$$

Proposition 3. *Preserve the notation of Proposition 2. The solution of differential equation (3.4) (with boundary values $\Phi(1) = c_1$, $\Phi'(1) = c_2$) is*

$$\Phi(x) = c_1 \frac{x^2+1}{2x} + c_2 \frac{x^2-1}{2x} \qquad (x \in \mathbb{R}_+). \tag{3.6}$$

Proof. It is easy to see that

$$\exp\left(\int_1^x a_1(u)du\right) = \frac{(x+2)^2}{9x},$$

$$\int_1^x \frac{a_0(v)}{\exp(\int_1^v a_1(u)du)} dv = c_1 \int_1^x \frac{18v-9}{(v+2)^3} dv + c_2 \int_1^x \frac{18v+9}{(v+2)^3} dv$$
$$= c_1\left(-\frac{-7x^2+8x-1}{2(x+2)^2}\right) + c_2\left(-\frac{9(x^2-1)}{2(x+2)^2}\right),$$

whence we obtain Equation (3.6). □

Theorem 3. *The twice continuously differentiable solution of Equation (2.8) is*

$$\begin{aligned} g(x) &= C_1 x + C_2 \ln(x) + C_4 \qquad (x \in \mathbb{R}_+), \\ f(x) &= C_1 x^2 + C_2 \ln(x) + C_3 \qquad (x \in \mathbb{R}_+) \end{aligned} \tag{3.7}$$

where C_i are arbitrary constants for $i = 1, 2, 3$.

Proof. Since $\Phi(x) = f'(x)$ for all $x \in \mathbb{R}_+$ thus by Equation (3.6) we obtain that

$$f(x) = c_1 \int_1^x \frac{u^2+1}{2u} du + c_2 \int_1^x \frac{u^2-1}{2u} du + f(1)$$
$$= \frac{1}{4}(c_1+c_2)x^2 + \frac{1}{2}(c_1-c_2)\ln(x) - \frac{1}{4}(c_1+c_2) + f(1)$$

for all $x \in \mathbb{R}_+$ whence we obtain the second Equation of (3.7) with constants $C_1 := \frac{1}{4}(c_1+c_2)$, $C_2 := \frac{1}{2}(c_1-c_2)$ and $C_3 := -\frac{1}{4}(c_1+c_2)+f(1)$.

Take the substitution $x \leftarrow \frac{x}{\sqrt{x+1}}$, $y \leftarrow \frac{1}{\sqrt{x+1}}$,that is, take $x \leftarrow \frac{u}{\sqrt{u+1}}$ and $y \leftarrow \frac{1}{\sqrt{u+1}}$ after $u \leftarrow x$ thus we obtain the Equation

$$g(x) = -f\left(\frac{1}{\sqrt{x+1}}\right) + f\left(\frac{x}{\sqrt{x+1}}\right) + g(1) \qquad (x \in \mathbb{R}_+). \tag{3.8}$$

From the second Equation of (3.7) and Equation (3.8) we obtain that

$$\begin{aligned} g(x) &= -C_1\frac{1}{x+1} - C_2 \ln\left(\frac{1}{\sqrt{x+1}}\right) \\ &\quad + C_1\frac{x^2}{x+1} + C_2 \ln\left(\frac{x}{\sqrt{x+1}}\right) + C_3 + g(1) - C_1 \\ &= C_1 x + C_2 \ln(x) + C_3 + g(1) - C_1, \end{aligned}$$

which is the first Equation of (3.7) with constants $C_4 := C_3 + g(1) - C_1$.

The converse statement can be obtained by trivial calculation. □

4. The twice continuously differentiable solutions of Equation (2.9) and (2.7)

Proposition 4. *If the function $\gamma : \mathbb{R}_+ \to \mathbb{R}$ is twice differentiable solutions of Equation (2.9), and the functions b_0, b_1, $\Phi : \mathbb{R}_+ \to \mathbb{R}$ and the constants d_1, d_2 are defined by*

$$\begin{aligned} \delta(x) &:= \gamma(2x^2), \\ d_1 &:= \delta'(1), \quad d_2 := \delta''(1), \\ b_0(x) &:= d_1\frac{2x-1}{x(x+2)} + d_2\frac{2x+1}{x(x+2)} \qquad (x \in \mathbb{R}_+) \\ b_1(x) &:= \frac{x-2}{x(x+2)} \qquad (x \in \mathbb{R}_+), \\ \Psi(x) &:= \delta'(x) \qquad (x \in \mathbb{R}_+), \end{aligned} \tag{4.1}$$

then

$$\Psi'(x) = b_0(x) + b_1(x)\Phi(x) \qquad (x \in \mathbb{R}_+) \tag{4.2}$$

Proof. Write $-\delta(y)$ instead of $F_1(y)$ and $\delta(x)$ instead of $F_2(x)$ in Equation(3.1) thus we have that

$$0 = -\frac{-2x+y}{x+2y}\delta'(y) + \frac{2x+y}{x+2y}\delta''(y) + \frac{-x+2y}{x+2y}\delta'(x) + \frac{x}{y}\delta''(x)$$

for all $x, y \in \mathbb{R}_+$. Write 1 instead of y in Equation (3.5) after order the obtained Equation. Thus we have that

$$\delta''(x) = \delta'(1)\frac{2x+1}{x(x+2)} + \delta''(1)\frac{-2x-1}{x(x+2)} + \frac{x-2}{x(x+2)}f'(x) \qquad (x \in \mathbb{R}_+),$$

whence by the notations of (4.1) we obtain the differential equation (4.2). □

Proposition 5. *Preserve the notation of Proposition 4. The solution of differential equation (3.4) (with boundary values $\Psi(1) = d_1$, $\Psi'(1) = d_2$) is*

$$\Psi(x) = d_1 \frac{-5x^2+16x+7}{18x} + d_2 \frac{-(x-2)^2}{2x} \qquad (x \in \mathbb{R}_+). \tag{4.3}$$

Proof. It is easy to see that

$$\exp\left(\int_1^x b_1(u)du\right) = \frac{(x+2)^2}{9x},$$

$$\int_1^x \frac{b_0(v)}{\exp(\int_1^v b_1(u)du)}dv = d_1 \int_1^x \frac{-18v+9}{(v+2)^3}dv + d_2 \int_1^x \frac{-18v-9}{(v+2)^3}dv$$
$$= d_1\left(-\frac{-7x^2+8x-1}{2(x+2)^2}\right) + d_2\left(-\frac{9(x^2-1)}{2(x+2)^2}\right),$$

whence we obtain Equation (4.3). □

In the proof of following Theorem we shall use the difference operator Δ_λ defined by

$$\Delta_\lambda F(x) := F(\lambda x) - F(x) \qquad (x \in \mathbb{R}_+),$$

where $\lambda > 0$ and $F : \mathbb{R}_+ \to \mathbb{R}$ is an arbitrary function. This difference operator was used first in [6] where was applied to give the general solution of Vajzović equation, see also [9] and have been also used in [11].

Theorem 4. *If the function $\gamma : \mathbb{R}_+ \to \mathbb{R}$ is a twice continuously differentiable solution of Equation (2.9), then it is a constant function.*

Proof. Preserve the notations of Proposition 4 and Proposition 5. Since $\Phi(x) = \delta'(x)$ for all $x \in \mathbb{R}_+$ thus by Equation (4.3) we obtain that

$$\delta(x) = d_1 \int_1^x \frac{-5u^2+16u+7}{18u}du + d_2 \int_1^x \frac{-(u-2)^2}{2u}du + \delta(1)$$
$$= \frac{d_1}{36}(-5x^2+32x+14\ln(x)-27) + \frac{d_2}{4}(-x^2+2\ln(x)+1) + \delta(1)$$

for all $x \in \mathbb{R}_+$. Since $\gamma(x) = \delta(\sqrt{\frac{x}{2}})$ for all $x \in \mathbb{R}_+$ thus we have that

$$\gamma(x) = \left(-\frac{5}{72}d_1 - \frac{1}{8}d_2\right) + \frac{4\sqrt{2}}{9}d_1\sqrt{x}$$
$$+ \left(\frac{7}{36}d_1 + \frac{1}{4}d_2\right)(\ln(x) - \ln(2)) + \left(-\frac{3}{4}d_1 + \frac{1}{2}d_2 + \delta(1)\right) \tag{4.4}$$

for all $x \in \mathbb{R}_+$. Define the function $\gamma_2 : \mathbb{R}_+ \to \mathbb{R}$ by $\gamma_2 := \Delta_2\gamma$. Thus by Equation (4.4) we have that

$$\begin{aligned}\gamma_2(x) &= \left(-\frac{5}{72}d_1 - \frac{1}{8}d_2\right)x + \frac{4\sqrt{2}}{9}\left(\sqrt{2}-1\right)d_1\sqrt{x} \\ &+ \left(\frac{7}{36}d_1 + \frac{1}{4}d_2\right)\ln(2) \\ &= D_1x + D_2\sqrt{x} + D_3 \qquad (x \in \mathbb{R}_+)\end{aligned} \tag{4.5}$$

with constants

$$\begin{aligned}D_1 := -\frac{5}{72}d_1 - \frac{1}{8}d_2, \qquad D_2 := \frac{4\sqrt{2}}{9}\left(\sqrt{2}-1\right)d_1, \\ D_3 := \left(\frac{7}{36}d_1 + \frac{1}{4}d_2\right)\ln(2).\end{aligned} \tag{4.6}$$

□

It is easy to see that the function γ_2 is satisfies the Equation (2.9) thus we can write the function γ_2 instead of the function γ in this Equation. Thus by simple calculation we obtain that

$$D_1 = D_2\frac{\left(\sqrt{x}+\sqrt{y}\right)\sqrt{x+y} - \sqrt{2}(x+y)}{(x-y)_2} \qquad (x, y \in \mathbb{R}_+, x \neq y)$$

which shows that $D_1 = D_2 = 0$ whence by (4.6) we have that $d_1 = d_2 = 0$ whence by Equation (4.4) we obtain that $\gamma(x) = \delta(1) = \gamma(2)$ for all $x \in \mathbb{R}_+$ which was to be proven.

Now, we can give the twice continuously differentiable solution of Equation (2.7).

Theorem 5. *The twice continuously differentiable solution of Equation (2.7) is*

$$\begin{aligned}G_1(x) &= C_1x + C_2\ln(x) + D_1, \qquad & F_1(x) &= C_1x^2 + C_2\ln(x) + D_2, \\ G_2(x) &= C_1x + C_2\ln(x) + D_3, \qquad & F_2(x) &= C_1x^2 + C_2\ln(x) + D_4\end{aligned}$$

for all $x \in \mathbb{R}_+$ where C_i $i = 1,2$, and $D_j \in \mathbb{R}$ $j = 1,2,3,4$ are arbitrary constants such that $D_1 + D_2 = D_3 + D_4$.

Proof. The assertion can be easily obtained by Theorem 2, Theorem 3, and Theorem 4. □

5. Additional applications and conjectures

In this chapter we shell apply the notations of Theorem 2.

In paper [13] was investigated the case when $X = \mathbb{F}_+$ where $\mathbb{F} = \mathbb{F}(+,\cdot)$ is an ordered field (the operation $\cdot$ is commutative) the $\mathbb{T}_+$ is the set of positive elements of the $\mathbb{F}$, $x \circ y := \frac{x}{y+1}$ for all $x, y \in \mathbb{F}_+$, $Y = Y(+)$ is a uniquely two-divisible Abelian group. In this settings the general solution of Equation (2.6) is

$$\begin{aligned}
G_1(x) &= l_1 x + l_2(x+1) + l_3(x) + d_1,\\
G_2(x) &= l_1 x + l_2(x+1) - l_3(x) + d_3,\\
F_1(x) &= l_1(x(x+1)) + l_2(x) - l_3\left(\frac{x+1}{x}\right) + d_2,\\
F_2(x) &= l_1(x(x+1)) + l_2(x) + l_3\left(\frac{x+1}{x}\right) + d_4,
\end{aligned}$$

for all $x \in \mathbb{F}_+$ where $l_i : \mathbb{F}_+ \to Y$ $(i = 1,2,3)$ are arbitrary logarithmic functions, $d_i \in Y$ $(i = 1,2,3,4)$ are arbitrary constants such that $d_1 + d_2 = d_3 + d_4$. (Concerning the logarithmic functions see [1], and [17]). This result can not be improved.

In paper [11] was investigated the case when $X = \mathbb{F}_+$ where $\mathbb{F} = \mathbb{F}(+,\cdot)$ is an Archimedean ordered field (the operation $\cdot$ is commutative), $x \circ y := x(y+1)$ for all $x, y \in \mathbb{F}_+$, $Y = Y(+)$ is a uniquely two-divisible Abelian group. In this settings the general solution of Equation (2.6) is

$$\begin{aligned}
G_1(x) &= a(x) + l_2(x) + l_3(x) + d_1,\\
G_2(x) &= a(x) + l_2(x) - l_3(x) + d_3,\\
F_1(x) &= -a(x) + l_2\left(\frac{x}{x+1}\right) - l_3(x(x+1)) + d_2,\\
F_2(x) &= -a(x) + l_2\left(\frac{x}{x+1}\right) + l_3(x(x+1)) + d_4,
\end{aligned} \tag{5.1}$$

for all $x \in \mathbb{F}_+$ where $a : \mathbb{F}_+ \to Y$ is an arbitrary additive function, $l_i : \mathbb{F}_+ \to Y$ $(i = 2,3)$ are arbitrary logarithmic functions, $d_i \in Y$ $(i = 1,2,3,4)$ are arbitrary constants such that $d_1 + d_2 = d_3 + d_4$. (Concerning the additive and logarithmic functions see [1], and [17]). This result could perhaps be improved writing simply ordered field instead of Archimedean ordered field, that is,

Conjecture 1. *If $X = \mathbb{F}_+$ where $\mathbb{F} = \mathbb{F}(+,\cdot)$ is an ordered field, $x \circ y := x(y+1)$ for all $x, y \in \mathbb{F}_+$, $Y = Y(+)$ is a uniquely two-divisible Abelian group, then the general solution of Equation (2.6) is of the form (5.1) for all $x \in \mathbb{F}_+$ where $a : \mathbb{F}_+ \to Y$ is an arbitrary additive function, $l_i : \mathbb{F}_+ \to Y$ $(i = 2,3)$ are arbitrary logarithmic functions, $d_i \in Y$ $(i = 1,2,3,4)$ are arbitrary constants such that $d_1 + d_2 = d_3 + d_4$.*

The above two problems have probability background see [18], [2], [13], and [11]. Two pairs of new equations can be derived from the above two equations, the Pexider versions of the second of these equations (which contain the unknown gamma function) characterize the logarithmic functions see [12].

Finally, we can formulate our Conjecture concerning the main equation of this paper.

Conjecture 2. *If $X = \mathbb{F}_+$ where $\mathbb{F} = \mathbb{F}(+,\cdot)$ is an ordered field, $x \circ y := x(x+y)$ for all $x,y \in \mathbb{F}_+$, $Y = Y(+)$ is a uniquely two-divisible Abelian group, then the general solution of Equation (2.6) is*

$$G_1(x) = a(x) + \mathrm{l}(x) + d_1, \qquad F_1(x) = a(x^2) + \mathrm{l}(x) + d_2,$$
$$G_2(x) = a(x) + \mathrm{l}(x) + d_3, \qquad F_2(x) = a(x^2) + \mathrm{l}(x) + d_4$$

for all $x \in \mathbb{F}_+$ where $a : \mathbb{F}_+ \to Y$ is an arbitrary additive function, $\mathrm{l} : \mathbb{F}_+ \to Y$ is an arbitrary logarithmic function, $d_i \in Y$ $i = 1,2,3,4$ are arbitrary constants such that $d_1 + d_2 = d_3 + d_4$.

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