Miskolc Mathematical Notes
Vol. 24 (2023), No. 2, pp. 779–788
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.4315

CONNECTION OF BALANCING NUMBERS WITH SOLUTION OF A SYSTEM OF TWO HIGHER-ORDER DIFFERENCE EQUATIONS

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Received 04 July, 2022

Abstract. We provide some theoretical justifications pertaining to the representation for the solution of the system of the higher-order rational difference equations

$$x_{n+1} = \frac{1}{6 - y_{n-k}}, \quad y_{n+1} = \frac{1}{6 - x_{n-k}}, \qquad n,k \in \mathbb{N}_0.$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and the initial conditions x_{-k}, $x_{-k+1}, \ldots,$ x_0, y_{-k}, $y_{-k+1}, \ldots,$ y_0 are non zero real numbers such that their solution is related to Balancing numbers. We also study the stability character and asymptotic behavior of this system.

2010 *Mathematics Subject Classification:* 39A10; 40A05

Keywords: general solution, Balancing numbers, stability, system of difference equations

1. INTRODUCTION

The notion of Balancing numbers was first suggested by Behera and Panda [4] in relation to a Diophantine equation in 1999. A positive integer n is known as a Balancing number if

$$1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \ldots + (n+r)$$

for some $r \in \mathbb{N}$. Here r is known as the balancer corresponding to the Balancing number n. Behera and Panda [4] demonstrated in a joint study that the Balancing numbers satisfy the following recurrence relation

$$B_{n+2} = 6B_{n+1} - B_n, \quad n \in \mathbb{N},$$

where $B_1 = 1$ and $B_2 = 6$. The Binet's formula for Balancing numbers is given by

$$B_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \in \mathbb{N}_0,$$

where

$$\alpha = 3 + 2\sqrt{2}, \quad \beta = 3 - 2\sqrt{2}.$$

The first author was supported by DGRSDT (Algeria).

We always have the quotient of two consecutive terms of Balancing numbers tends to α, that is

$$\lim_{n\to\infty} \frac{B_{n+1}}{B_n} = \alpha.$$

Many papers have been published previously focusing on the forme of the solution of system of difference equations, see for example [1,2,5–8,10,12,15,19,20]. Curiously, some of the solution forms of these systems can even be expressed in terms of well-known sequences such as Fibonacci numbers, Horadam numbers, Padovan numbers, Lucas numbers and Pell numbers, see for example [3, 9, 11, 13, 14, 16–18] .

In this paper we provide some theoretical justifications pertaining to the representation for the solution of the system of the higher-order rational difference equations

$$x_{n+1} = \frac{1}{6 - y_{n-k}}, \quad y_{n+1} = \frac{1}{6 - x_{n-k}}, \qquad n,k \in \mathbb{N}_0. \tag{1.1}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0, y_{-k}, y_{-k+1}, \ldots, y_0$ are non zero real numbers such that their solution is related to Balancing numbers.

2. Auxiliary Results

To prove our main results in Sections 3 and 4, we will need the following two lemmas.

Lemma 1. *Consider the two homogenous second order linear autonomous difference equations*

$$y_{n+2} - 6y_{n+1} + y_n = 0, \quad n \in \mathbb{N}_0, \tag{2.1}$$

$$S_{n+2} + 6S_{n+1} + S_n = 0, \quad n \in \mathbb{N}_0. \tag{2.2}$$

Then, we have for all $n \in \mathbb{N}_0$

$$y_n = -y_0 B_{n-1} + y_1 B_n, \qquad S_n = (-1)^n (S_0 B_{n-1} + S_1 B_n).$$

Proof. As is well-known, the equation

$$y_{n+2} - 6y_{n+1} + y_n = 0, \quad n \in \mathbb{N}_0,$$

where y_0 and $y_1 \in \mathbb{R}$, is usually solved by using the characteristic roots α and β of the characteristic polynomial $\varphi^2 - 6\varphi + 1$, so

$$\alpha = 3 + 2\sqrt{2}, \qquad \beta = 3 - 2\sqrt{2},$$

and the formulas of general solution is

$$y_n = c_1\alpha^n + c_2\beta^n.$$

Using the initial conditions y_0 and y_1 with some calculations we get

$$c_1 = \frac{y_0\beta - y_1}{\beta - \alpha}, \qquad c_2 = \frac{y_1 - y_0\alpha}{\beta - \alpha}.$$

So
$$y_n = y_0\frac{\alpha^{n-1}-\beta^{n-1}}{-(\alpha-\beta)} + y_1\frac{\alpha^n-\beta^n}{\alpha-\beta}.$$
Hence
$$y_n = -y_0B_{n-1} + y_1B_n.$$
By the same argument, we get
$$S_n = (-1)^n(S_0B_{n-1} + S_1B_n).$$
□

Lemma 2. *Consider the linear system of second order linear autonomous difference equations*
$$t_{n+2} = 6p_{n+1} - t_n, \quad p_{n+2} = 6t_{n+1} - p_n, \quad n \in \mathbb{N}_0. \tag{2.3}$$
Then, we have for all $n \in \mathbb{N}_0$
$$\begin{cases} t_{2n} = -p_0B_{2n-1} + t_1B_{2n}, & t_{2n+1} = -t_0B_{2n} + p_1B_{2n+1}, \\ p_{2n} = -t_0B_{2n-1} + p_1B_{2n}, & p_{2n+1} = -p_0B_{2n} + t_1B_{2n+1}. \end{cases}$$

Proof. From (2.3) we get the following system
$$\begin{cases} t_{n+2} + p_{n+2} = 6p_{n+1} - t_n + 6t_{n+1} - p_n, \\ t_{n+2} - p_{n+2} = 6p_{n+1} - t_n - 6t_{n+1} + p_n. \end{cases}$$
So
$$\begin{cases} t_{n+2} + p_{n+2} = 6(p_{n+1} + t_{n+1}) - (t_n + p_n), \\ t_{n+2} - p_{n+2} = -6(t_{n+1} - p_{n+1}) - (t_{n+} - p_n). \end{cases} \tag{2.4}$$
By posing the following changes of the variables
$$R_n = t_n + p_n, \quad S_n = t_n - p_n. \tag{2.5}$$
System (2.4) becomes
$$R_{n+2} = 6R_{n+1} - R_n, \qquad S_{n+2} = -6S_{n+1} - S_n.$$
which are in the form of equations (2.1) and (2.2). Then it follows from Lemma 1 that
$$\begin{cases} R_{2n} = -R_0B_{n-1} + R_1B_n, & R_{2n+1} = -R_0B_{n-1} + R_1B_n, \\ S_{2n} = S_0B_{n-1} + S_1B_n, & S_{2n+1} = -S_0B_{n-1} - S_1B_n. \end{cases}$$
From (2.5), we have
$$t_n = \frac{1}{2}(R_n - S_n), \quad p_n = \frac{1}{2}(R_n + S_n).$$
So
$$t_{2n} = \frac{1}{2}(R_{2n} - S_{2n}), \quad t_{2n+1} = \frac{1}{2}(R_{2n+1} - S_{2n+1}),$$

$$p_{2n}=\frac{1}{2}(R_{2n}+S_{2n}),\quad p_{2n+1}=\frac{1}{2}(R_{2n+1}+S_{2n+1}).$$

Hence

$$\begin{aligned}
t_{2n}&=\frac{1}{2}\left(B_{2n-1}(-R_0-S_0)+B_{2n}(R_1-S_1)\right),\\
t_{2n+1}&=\frac{1}{2}\left(B_{2n}(-R_0+S_0)+B_{2n+1}(R_1+S_1)\right),\\
p_{2n}&=\frac{1}{2}\left(B_{2n-1}(-R_0+S_0)+B_{2n}(R_1+S_1)\right),\\
p_{2n+1}&=\frac{1}{2}\left(B_{2n}(-R_0-S_0)+B_{2n+1}(R_1-S_1)\right).
\end{aligned}$$

So

$$\begin{cases}
t_{2n}=-p_0B_{2n-1}+t_1B_{2n},\\
t_{2n+1}=-t_0B_{2n}+p_1B_{2n+1},\\
p_{2n}=-t_0B_{2n-1}+p_1B_{2n},\\
p_{2n+1}=-p_0B_{2n}+t_1B_{2n+1}.
\end{cases}$$

□

3. Main result

In this section we give the explicit formula of solution to system (1.1) in terms of Balancing numbers.

From (1.1), we can write

$$\begin{cases}
x_{(k+1)(n+1)-j}=\dfrac{1}{6-y_{(k+1)n-j}},\\
y_{(k+1)(n+1)-j}=\dfrac{1}{6-x_{(k+1)n-j}}.
\end{cases}$$

Let

$$x_n^{(j)}=x_{(k+1)n-j},\quad y_n^{(j)}=y_{(k+1)n-j},\quad j=0,1,\ldots,k-1.$$

So, the system (1.1) becomes

$$x_{n+1}^{(j)}=\frac{1}{6-y_n^{(j)}},\quad y_{n+1}^{(j)}=\frac{1}{6-x_n^{(j)}},\quad n\in\mathbb{N}_0. \tag{3.1}$$

To find the form of the solution of the system (3.1) we consider the following changes of variables

$$x_n^{(j)}=\frac{W_n}{U_{n+1}},\quad y_n^{(j)}=\frac{U_n}{W_{n+1}}.$$

So

$$x_{n+1}^{(j)}=\frac{W_{n+1}}{U_{n+2}},\quad y_{n+1}^{(j)}=\frac{U_{n+1}}{W_{n+2}}.$$

Hence

$$\frac{W_{n+1}}{U_{n+2}}=\frac{1}{6-\dfrac{U_n}{W_{n+1}}},\quad \frac{W_{n+1}}{U_{n+2}}=\frac{W_{n+1}}{6W_{n+1}-U_n},$$
$$\frac{U_{n+1}}{W_{n+2}}=\frac{1}{6-\dfrac{W_n}{U_{n+1}}},\quad \frac{U_{n+1}}{W_{n+2}}=\frac{U_{n+1}}{6U_{n+1}-W_n}.$$

So the system (3.1) becomes

$$U_{n+2}=6W_{n+1}-U_n,\quad W_{n+2}=6U_{n+1}-W_n.$$

Then it follows from Lemma 2 that

$$\begin{cases} U_{2n}=-W_0B_{2n-1}+U_1B_{2n}, \\ U_{2n+1}=-U_0B_{2n}+W_1B_{2n+1}, \\ W_{2n}=-U_0B_{2n-1}+W_1B_{2n}, \\ W_{2n+1}=-W_0B_{2n}+U_1B_{2n+1}. \end{cases}$$

So,

$$x_{2n+1}^{(j)}=\frac{W_{2n+1}}{U_{2n+2}}=\frac{-W_0B_{2n}+U_1B_{2n+1}}{-W_0B_{2n+1}+U_1B_{2n+2}}=\frac{\dfrac{-W_0B_{2n}+U_1B_{2n+1}}{U_1}}{\dfrac{-W_0B_{2n+1}+U_1B_{2n+2}}{U_1}}.$$

Hence

$$x_{2n+1}^{(j)}=\frac{-x_0^{(j)}B_{2n}+B_{2n+1}}{-x_0^{(j)}B_{2n+1}+B_{2n+2}},$$

and

$$x_{2n}^{(j)}=\frac{W_{2n}}{U_{2n+1}}=\frac{-U_0B_{2n-1}+W_1B_{2n}}{-U_0B_{2n}+W_1B_{2n+1}}=\frac{\dfrac{-U_0B_{2n-1}+W_1B_{2n}}{W_1}}{\dfrac{-U_0B_{2n}+W_1B_{2n+1}}{W_1}}.$$

So

$$x_{2n}^{(j)}=\frac{-y_0^{(j)}B_{2n-1}+B_{2n}}{-y_0^{(j)}B_{2n}+B_{2n+1}}.$$

By a similar calculation, we find $y_n^{(j)}$. We have

$$y_{2n+1}^{(j)}=\frac{U_{2n+1}}{W_{2n+2}}=\frac{-U_0B_{2n}+W_1B_{2n+1}}{-U_0B_{2n+1}+W_1B_{2n+2}}=\frac{\dfrac{-U_0B_{2n}+W_1B_{2n+1}}{W_1}}{\dfrac{-U_0B_{2n+1}+W_1B_{2n+2}}{W_1}}.$$

So

$$y_{2n+1}^{(j)} = \frac{-y_0^{(j)} B_{2n} + B_{2n+1}}{-y_0^{(j)} B_{2n+1} + B_{2n+2}},$$

and

$$y_{2n}^{(j)} = \frac{U_{2n}}{W_{2n+1}} = \frac{-W_0 B_{2n-1} + U_1 B_{2n}}{-W_0 B_{2n} + U_1 B_{2n+1}} = \frac{\dfrac{-W_0 B_{2n-1} + U_1 B_{2n}}{U_1}}{\dfrac{-W_0 B_{2n} + U_1 B_{2n+1}}{U_1}}.$$

Hence

$$y_{2n}^{(j)} = \frac{-x_0^{(j)} B_{2n-1} + B_{2n}}{-x_0^{(j)} B_{2n} + B_{2n+1}}.$$

According to all the above, we have the following theorem.

Theorem 1. *Let $\left\{x_n^{(j)}, y_n^{(j)}\right\}_{n\geq 0}$ be the solution to system (3.1). Then for $n \in \mathbb{N}$ and $j = 0, 1 \ldots, k$*

$$\begin{cases} x_{2n+1}^{(j)} = \dfrac{-x_0^{(j)} B_{2n} + B_{2n+1}}{-x_0^{(j)} B_{2n+1} + B_{2n+2}}, & x_{2n}^{(j)} = \dfrac{-y_0^{(j)} B_{2n-1} + B_{2n}}{-y_0^{(j)} B_{2n} + B_{2n+1}}, \\ y_{2n+1}^{(j)} = \dfrac{-y_0^{(j)} B_{2n} + B_{2n+1}}{-y_0^{(j)} B_{2n+1} + B_{2n+2}}, & y_{2n}^{(j)} = \dfrac{-x_0^{(j)} B_{2n-1} + B_{2n}}{-x_0^{(j)} B_{2n} + B_{2n+1}}. \end{cases}$$

The following corollary is our main result which give the explicit formula of solution to system (1.1).

Corollary 1. *Let $\{x_n, y_n\}_{n\geq 0}$ be the solution to system (1.1). Then for $n \in \mathbb{N}$ and $j = 0, 1 \ldots, k$*

$$\begin{cases} x_{(k+1)(2n+1)-j} = \dfrac{-x_{-j} B_{2n} + B_{2n+1}}{-x_{-j} B_{2n+1} + B_{2n+2}}, \\ x_{(k+1)(2n)-j} = \dfrac{-y_{-j} B_{2n-1} + B_{2n}}{-y_{-j} B_{2n} + B_{2n+1}}, \\ y_{(k+1)(2n+1)-j} = \dfrac{-y_{-j} B_{2n} + B_{2n+1}}{-y_{-j} B_{2n+1} + B_{2n+2}}, \\ y_{(k+1)(2n)-j} = \dfrac{-x_{-j} B_{2n-1} + B_{2n}}{-x_{-j} B_{2n} + B_{2n+1}}. \end{cases}$$

Proof. We have

$$x_n^{(j)} = x_{(k+1)n-j}, \quad j \in \{0, 1, 2 \ldots, k\}.$$

So

$$x_{2n+1}^{(j)} = x_{(k+1)(2n+1)-j},$$

and

$$x_0^{(j)} = x_{-j}.$$

Then

$$x_{2n+1}^{(j)} = x_{(k+1)(2n+1)-j} = \frac{-x_{-j}B_{2n} + B_{2n+1}}{-x_{-j}B_{2n+1} + B_{2n+2}},$$

and

$$x_{2n}^{(j)} = x_{(k+1)(2n)-j} = \frac{-y_{-j}B_{2n-1} + B_{2n}}{-y_{-j}B_{2n} + B_{2n+1}}.$$

We have

$$y_n^{(j)} = y_{(k+1)n-j}.$$

So

$$y_{2n+1}^{(j)} = y_{(k+1)(2n+1)-j},$$

and

$$y_0^{(j)} = y_{-j}.$$

Hence

$$y_{2n+1}^{(j)} = y_{(k+1)(2n+1)-j} = \frac{-y_{-j}B_{2n} + B_{2n+1}}{-y_{-j}B_{2n+1} + B_{2n+2}},$$

and

$$y_{2n}^{(j)} = y_{(k+1)(2n)-j} = \frac{-x_{-j}B_{2n-1} + B_{2n}}{-x_{-j}B_{2n} + B_{2n+1}}.$$

□

4. GLOBAL STABILITY OF THE SOLUTIONS TO SYSTEM (1.1)

In this section we study the global stability character of the solutions of system (1.1). It is easy to show that (1.1) has two real equilibrium points given by

$$\overline{M} = (\overline{x}, \overline{y}) = \left(3+2\sqrt{2}, 3+2\sqrt{2}\right), \quad \overline{M'} = (\overline{x'}, \overline{y'}) = \left(3-2\sqrt{2}, 3-2\sqrt{2}\right).$$

Theorem 2. *The equilibrium point* $\overline{M} = \left(3+2\sqrt{2}, 3+2\sqrt{2}\right)$ *is locally asymptotically stable.*

Proof. The linearized system about the equilibrium point $\overline{M} = \left(3+2\sqrt{2}, 3+2\sqrt{2}\right)$ is given by

$$X_{n+1} = JX_n,$$

where

$$X_n = (x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k})^t$$

and

$$J = \begin{pmatrix} 0 & 0 & \dots & 0 & 17-12\sqrt{2} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & \dots & 17-12\sqrt{2} & \dots & 0\dots \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial of J is

$$P(\lambda) = (-\lambda)^{2k+2} - \left(17-12\sqrt{2}\right)^2.$$

Now, consider the two functions defined by

$$\varphi(\lambda) = (-\lambda)^{2k+2}, \quad \phi(\lambda) = \left(17-12\sqrt{2}\right)^2.$$

We have

$$|\phi(\lambda)| < |\varphi(\lambda)|, \quad \forall\lambda \in C : |\lambda| = 1$$

So, according to Rouche's Theorem φ and $P = \varphi + \phi$ have the same number of zeros in the unit disc $|\lambda| < 1$, and since φ admits as root $\lambda = 0$ of multiplicity $2(k+1)$, then all the roots of P are in the disc $|\lambda| < 1$. Thus, the equilibrium point is locally asymptotically stable. □

Corollary 2. *The equilibrium point $\overline{M}$ is globally asymptotically stable.*

Proof. Theorem 2 states that, $\overline{M}$ is locally asymptotically stable therefore, it must be demonstrated that this point is globally attractive. To do this we use the Corollary 1.

We have

$$\lim_{n\to\infty} x_{(k+1)(2n)-j} = \lim_{n\to\infty} \frac{-y_{-j}B_{2n-1}+B_{2n}}{-y_{-j}B_{2n}+B_{2n+1}} = \lim_{n\to\infty} \frac{-y_{-j}\dfrac{B_{2n-1}}{B_{2n}}+1}{-y_{-j}+\dfrac{B_{2n+1}}{B_{2n}}}.$$

Using the following two limits

$$\lim_{n\to\infty}\left(\frac{B^{2n+1}}{B_{2n}}\right) = \alpha, \quad \lim_{n\to\infty}\left(\frac{B^{2n-1}}{B_{2n}}\right) = \frac{1}{\alpha} = \beta,$$

we get

$$\begin{aligned} \lim_{n\to\infty} x_{(k+1)(2n)-j} &= \frac{-y_{-j}\beta+1}{-y_{-j}+\alpha} = \frac{-y_{-j}(3-2\sqrt{2})+1}{-y_{-j}+3+2\sqrt{2}} \\ &= \frac{-3y_{-j}+2\sqrt{2}y_{-j}+1}{-y_{-j}+3+2\sqrt{2}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\left(-3y_{-j}+2\sqrt{2}y_{-j}+1\right)\left(-y_{-j}+3-2\sqrt{2}\right)}{(y_{-j})^2-6y_{-j}+1} \\
&= \frac{3\left((y_{-j})^2-6y_{-j}+1\right)-2\sqrt{2}\left((y_{-j})^2-6y_{-j}+1\right)}{(y_{-j})^2-6y_{-j}+1} \\
&= 3-2\sqrt{2}=\bar{x}.
\end{aligned}
$$

However, we have

$$
\lim_{n\to\infty} x_{(k+1)(2n+1)-j} = \lim_{n\to\infty} \frac{-x_{-j}B_{2n}+B_{2n+1}}{-x_{-j}B_{2n+1}+B_{2n+2}} = \lim_{n\to\infty} \frac{-x_{-j}\dfrac{B_{2n}}{B_{2n+1}}+1}{-x_{-j}+\dfrac{B_{2n+2}}{B_{2n+1}}} = \frac{-x_{-j}\beta+1}{-x_{-j}+\alpha},
$$

Using the following two limits

$$
\lim_{n\to\infty}\left(\frac{B^{2n+1}}{B_{2n}}\right)=\alpha, \quad \lim_{n\to\infty}\left(\frac{B^{2n-1}}{B_{2n}}\right)=\frac{1}{\alpha}=\beta,
$$

we get

$$
\lim_{n\to\infty} x_{(k+1)(2n+1)-j} = 3-2\sqrt{2}=\bar{x}.
$$

So $\lim\limits_{n\to\infty} x_{(k+1)n-j}=\bar{x}$. By a similar argument, it can be shown that $\lim\limits_{n\to\infty} y_{(k+1)n-j}=\bar{y}$. Hence

$$
\lim_{n\to\infty}\left(x_{(k+1)n-j}, y_{(k+1)n-j}\right)=(\bar{x},\bar{y}).
$$

□

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