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SECOND HANKEL DETERMINANT OF THE LOGARITHMIC COEFFICIENTS FOR A SUBCLASS OF UNIVALENT FUNCTIONS

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Abstract. In the present paper, we give the bounds for the second Hankel determinant of the logarithmic coefficients of a certain subclass of normalized univalent functions, which we have introduced here. Relevant connections of the results, which we have presented here, with those available in the existing literature are also described briefly.

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1. Introduction

Let $\mathcal{A}$ stand for the normalized class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in \mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}). \tag{1.1}$$

Also let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

A function $f(z) \in \mathcal{A}$ is called starlike if $f(\mathbb{U})$ is starlike with respect to origin. Let $\mathcal{S}^*(\alpha)$ denote the class of functions in $\mathcal{A}$, which are starlike of order α $(0 \leq \alpha < 1)$ in $\mathcal{U}$. It is well known that

$$f \in \mathcal{S}^*(\alpha) \iff \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U};\ 0 \leq \alpha < 1).$$

A function $f \in \mathcal{A}$ is called convex if $f(\mathbb{U})$ is a convex domain. Let $\mathcal{K}(\alpha)$ denote the class of all functions in $\mathcal{A}$, which are convex of order α $(0 \leq \alpha < 1)$ in $\mathcal{A}$. It is also known that $f \in \mathcal{K}(\alpha)$ if and only if

$$f \in \mathcal{K}(\alpha) \iff \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U};\ 0 \leq \alpha < 1).$$

The function classes

$$\mathcal{S}^*(0) =: \mathcal{S}^* \qquad \text{and} \qquad \mathcal{K}(0) =: \mathcal{K}$$

consist of starlike and convex functions in $\mathbb{U}$, respectively. Moreover, the function f is convex if and only if $zf'(z)$ is starlike.

Let $\mathcal{P}$ denote the class of analytic functions $p(z)$ in $\mathbb{U}$ satisfying the following conditions:

$$p(0) = 1 \qquad \text{and} \qquad \Re\big(p(z)\big) > 0.$$

Thus, if $p \in \mathcal{P}$, then the function $p(z)$ has the following form:

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \qquad (z \in \mathbb{U}). \tag{1.2}$$

Functions in the class $\mathcal{P}$ are known as Carathédory functions. In fact, the coefficients of functions in $\mathcal{S}^*$ and $\mathcal{K}$ are known to have suitable representations in terms of the coefficients of functions in the Carathéodory class $\mathcal{P}$.

Associated with each function $f \in \mathcal{S}$ are its logarithmic coefficients γ_k defined by

$$\mathcal{F}_f := \log\left(\frac{f(z)}{z}\right) = 2\sum_{k=1}^{\infty} \gamma_k z^k \qquad (z \in \mathbb{U}). \tag{1.3}$$

In particular, the logarithmic coefficients of the Koebe function $\mathfrak{K}(z) = z(1-z)^{-2}$ are $\gamma_k = \frac{1}{k}$. Because of the extremal properties of the Koebe function $\mathfrak{K}(z)$, one could expect that $\gamma_k \leq \frac{1}{k}$ for each $f \in \mathcal{S}$. But this conjecture is false even in the case when $k = 2$. For the whole class $\mathcal{S}$, the sharp estimates of single logarithmic coefficients are known only for

$$|\gamma_1| \leq 1 \qquad \text{and} \qquad |\gamma_2| \leq \frac{1}{2} + \frac{1}{e^2} = 0.6353...$$

and are not known for $k \geq 3$.

Recently, logarithmic coefficients were studied by many authors and the upper bounds of the logarithmic coefficients of functions in some important subclasses of the univalent function class $\mathcal{S}$ were found (see, for example, [1,2,4,6,16,22,25]). For a summary of some of the significant results concerning the bounds of the logarithmic coefficients for univalent functions, we refer to [24].

Next, for $q,n \in \mathbb{N} = \{1,2,3,\cdots\}$, the Hankel determinant $H_{q,n}(f)$ of $f \in \mathcal{A}$ of form (1.1) is defined as follows.

Definition 1. The Hankel determinant $H_{q,n}(f)$ $(q,n \in \mathbb{N})$ is defined for a function $f \in \mathcal{A}$ of form (1.1) by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

In particular, the Hankel determinant $H_{2,1}(f) = a_3 - a_2^2$ is the well-known Fekete-Szegö functional. The second Hankel determinant $H_{2,2}(f)$ is given by $H_{2,2}(f) = a_2a_4 - a_3^2$.

The problem of computing the upper bound of $H_{q,n}$ for functions in various subfamilies of the analytic function class $\mathcal{A}$ is interesting and widely studied in the literature on Geometric Function Theory of Complex Analysis. Sharp upper bounds of $H_{2,2}$ and $H_{3,1}$ for subclasses of analytic functions were obtained by several authors (see, for details, [3, 7, 8, 11–13, 18]). Srivastava *et al.* [21] considered a family of normalized analytic functions with bounded turnings in the open unit disk $\mathbb{U}$ and obtained the estimates of the fourth Hankel determinant. On the other hand, Srivastava *et al.* [19] investigated the upper bound of the third Hankel determinant for the subclass of close-to-convex functions related to the right half of the lemniscate of Bernoulli. Moreover, the bounds of the Hankel determinant of order three for the certain subfamilies of univalent functions associated with exponential functions, which are symmetric along the real axis in the region of open unit disk, were considered by Shi *et al.* [17] (see also the developments reported by Srivastava *et al.* [20] on the Fekete-Szegö type and other coefficient estimates for subclasses of analytic functions which satisfy some subordination conditions and are associated with the Gegenbauer polynomials).

Quite recently, Kowalczyk and Lecko [9] introduced the following Hankel determinant $H_{q,n}\left(\frac{F_f}{2}\right)$ in which the entries are logarithmic coefficients of f.

Definition 2. The Hankel determinant $H_{q,n}\left(\frac{F_f}{2}\right)$ involving the logarithmic coefficients of f is defined by

$$H_{q,n}\left(\frac{F_f}{2}\right) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2(q-1)} \end{vmatrix}.$$

For a function $f \in \mathcal{S}$ given by (1.1), by differentiating (1.3) one can obtain the following logarithmic coefficients:

$$\gamma_1 = \frac{1}{2}a_2,\ \gamma_2 = \frac{1}{2}\big(a_3 - \frac{1}{2}a_2^2\big) \quad \text{and} \quad \gamma_3 = \frac{1}{2}\big(a_4 - a_2a_3 + \frac{1}{3}a_2^3\big). \tag{1.4}$$

Therefore, the second Hankel determinant of $F_f/2$ can be obtained by

$$H_{2,1}\left(\frac{F_f}{2}\right) = \gamma_1\gamma_3 - \gamma_2^2 = \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right). \tag{1.5}$$

Furthermore, if $f \in \mathcal{S}$, then we find for

$$f_\theta(z) = e^{-i\theta} f(e^{i\theta}z) \qquad (\theta \in \mathbb{R})$$

that (see [10])

$$H_{2,1}\left(\frac{F_{f_\theta}}{2}\right) = e^{4i\theta} H_{2,1}\left(\frac{F_{f_\theta}}{2}\right).$$

In their sequel to [9], Kowalczyk and Lecko [10] derived sharp bounds for $H_{2,1}\left(\frac{F_f}{2}\right)$ for the classes of starlike and convex functions of order α in $\mathbb{U}$. The problem of computing the sharp bounds of $H_{2,1}\left(\frac{F_f}{2}\right)$ for starlike and convex functions with respect to symmetric points in the open unit disk $\mathbb{U}$ was considered by Allu and Arora [23].

Motivated essentially by the above-mentioned recent developments, we continue here to deal further with

$$H_{2,1}\left(\frac{F_f}{2}\right) = \gamma_1\gamma_3 - \gamma_2^2$$

for functions in the class $\mathcal{R}(\lambda,\alpha)$ defined as follows.

Definition 3. For $0 \leq \alpha < 1$, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}(\lambda,\alpha)$ $(\lambda \geq 0)$ if it satisfies the following condition:

$$\Re\left(\frac{zf'(z) + \lambda z^2 f''(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}). \tag{1.6}$$

2. A SET OF LEMMAS

In order to establish our main results, we will require the following lemmas.

Lemma 1. (see [5]) *If the function $p \in \mathcal{P}$ is given by the series (1.2), then*

$$|c_k| \leq 2 \qquad (k \in \mathbb{N}). \tag{2.1}$$

This inequality is sharp for each k.

Lemma 2. (see [14, 15]) *Let the function $p \in \mathcal{P}$ be given by the series (1.2). Then there exist $x, z \in \mathbb{C}$ with $|x| \leq 1$ and $|z| \leq 1$ such that*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right) \tag{2.2}$$

and

$$\begin{aligned} 4c_3 = c_1^3 &+ 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 \\ &+ 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)z. \end{aligned} \tag{2.3}$$

3. RESULTS

Our main results on the bounds involving the logarithmic coefficients are stated as the following theorem.

Theorem 1. *Let $f(z) \in \mathcal{R}(\lambda,\alpha)$. Then*

$$|\gamma_1\gamma_3-\gamma_2^2| \leq \begin{cases} 12T(1-\alpha)^2(1+2\lambda)^3(8\lambda^2+6\lambda+1) \\ \qquad\qquad \left(0 \leq \lambda \leq \frac{-3+\sqrt{9+24(1-\alpha)}}{12}\right) \\ 4T(1-\alpha)^2\left[3(1+2\lambda)^4(1+4\lambda)-\frac{2B^2}{A}\right] \\ \qquad\qquad \left(\lambda > \frac{-3+\sqrt{9+24(1-\alpha)}}{12}\right), \end{cases}$$

where

$$A := 2(1-\alpha)^2\lambda^2(5+12\lambda)-(1+2\lambda)^3(12\lambda^2+6\lambda+1), \tag{3.1}$$

$$B := (1+2\lambda)^2\lambda(6\lambda^2+3\lambda+\alpha-1) \tag{3.2}$$

and

$$T = \left[48(1+2\lambda)^4(1+3\lambda)^2(1+4\lambda)\right]^{-1}. \tag{3.3}$$

Proof. Let $f(z) \in \mathcal{R}(\lambda,\alpha)$. Then, in view of (1.6), there exists a $p \in \mathcal{P}$ such that

$$zf'(z)+\lambda z^2 f''(z) = \big((1-\alpha)p(z)+\alpha\big)f(z) \qquad (z \in \mathbb{U}). \tag{3.4}$$

Thus, by using the representations (1.1) and (1.2) in (3.4), we obtain

$$z+\sum_{k=2}^{\infty} k\left[1+\lambda(k-1)\right]a_k z^k = \left(1+(1-\alpha)\sum_{k=1}^{\infty} c_k z^k\right)\left(z+\sum_{k=2}^{\infty} a_k z^k\right) \tag{3.5}$$

Upon comparing the coefficients on both sides of (3.5), we find that

$$a_2 = \frac{1-\alpha}{1+2\lambda}c_1, \tag{3.6}$$

$$a_3 = \frac{(1-\alpha)^2c_1^2+(1-\alpha)(1+2\lambda)c_2}{2(1+3\lambda)(1+2\lambda)} \tag{3.7}$$

and

$$a_4 = \left[(1-\alpha)^3c_1^3+(1-\alpha)^2(3+8\lambda)c_1c_2+2(1-\alpha)(1+3\lambda)(1+2\lambda)c_3\right] \cdot \left[6(1+4\lambda)(1+3\lambda)(1+2\lambda)\right]^{-1}. \tag{3.8}$$

We now substitute the above expressions for the coefficients a_2, a_3 and a_4 into the equation (1.5). After further simplification, we have

$$H_{2,1}\left(\frac{F_f}{2}\right) = \gamma_1\gamma_3-\gamma_2^2 = \frac{1}{4}\left(a_2a_4-a_3^2+\frac{1}{12}a_2^4\right) = T\left[d_1c_1^4+d_2c_1^2c_2+d_3c_1c_3+d_4c_2^2\right], \tag{3.9}$$

where T is given in (3.3) and

$$d_1 = (1-\alpha)^4\lambda^2(5+12\lambda), \tag{3.10}$$

$$d_2 = -2(1-\alpha)^3\lambda(1+2\lambda)^2, \tag{3.11}$$

$$d_3 = 4(1-\alpha)^2(1+2\lambda)^3(1+3\lambda)^2 \tag{3.12}$$

and

$$d_4 = -3(1-\alpha)^2(1+2\lambda)^4(1+4\lambda). \tag{3.13}$$

Since $p(z) \in \mathcal{P}$, by Lemma 1, we have $|c_1| \leq 2$. By letting $c_1 = c$, we may assume without loss of generality that $c \in [0,2]$. Also, upon substituting the values of c_2 and c_3 from (2.2) and (2.3) into the equation (3.9), we have

$$\begin{aligned}\left|H_{2,1}\left(\frac{F_f}{2}\right)\right| = \frac{T}{4}\Big| & c^4\left(4d_1+2d_2+d_3+d_4\right)+2c^2x(4-c^2)\left(d_2+d_3+d_4\right) \\ & +x^2(4-c^2)\left(-d_3c^2+d_4(4-c^2)\right)+2d_3c(4-c^2)\left(1-|x|^2\right)z\Big|.\end{aligned}$$

Applying the triangle inequality along with $|z| \leq 1$, replacing $|x|$ by μ and substituting the values of d_1, d_2, d_3 and d_4 from (3.10) to (3.13), we get

$$\begin{aligned}\left|H_{2,1}\left(\frac{F_f}{2}\right)\right| \leq \frac{T}{4}\Big(& Kc^4+2c^2\mu(4-c^2)(1-\alpha)^2(1+2\lambda)^2 \\ & \cdot\left[2\alpha\lambda+1+6\lambda(1+2\lambda)^2\right]+\mu^2(4-c^2)(1-\alpha)^2(1+2\lambda)^3 \\ & \cdot\left[c^2\left(12\lambda^2+6\lambda+1\right)-8c(1+3\lambda)^2+12\left(8\lambda^2+6\lambda+1\right)\right] \\ & +8c(4-c^2)(1-\alpha)^2(1+2\lambda)^3(1+3\lambda)^2\Big) \\ =: \mathcal{F}_\lambda(c,\mu), &\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}K := & 4(1-\alpha)^4\lambda^2(5+12\lambda)-4(1-\alpha)^3\lambda(1+2\lambda)^2 \\ & +4(1-\alpha)^2(1+2\lambda)^3(1+3\lambda)^2-3(1-\alpha)^2(1+2\lambda)^4(1+4\lambda).\end{aligned}$$

We next maximize the function $\mathcal{F}_\lambda(c,\mu)$ on the closed region $[0,2]\times[0,1]$. Differentiating $\mathcal{F}_\lambda(c,\mu)$ defined in (3.14) partially with respect to μ, we have

$$\begin{aligned}\frac{\partial\mathcal{F}_\lambda(c,\mu)}{\partial\mu} = \frac{T}{2}(4-c^2)(1-\alpha)^2(1+2\lambda)^2\Big(& c^2\left[2\alpha\lambda+1+6\lambda(1+2\lambda)^2\right] \\ & +\mu(1+2\lambda)\left[c^2\left(12\lambda^2+6\lambda+1\right)-8c(1+3\lambda)^2+12\left(8\lambda^2+6\lambda+1\right)\right]\Big).\end{aligned} \tag{3.15}$$

For $0 < \mu < 1$, and for a fixed c with $0 < c < 2$ and $\lambda \geq 0$, we observe from (3.15) that

$$\frac{\partial \mathcal{F}_\lambda(c,\mu)}{\partial \mu} > 0 \qquad (0 < \mu < 1;\ 0 < c < 2;\ \lambda \geq 0).$$

Consequently, $\mathcal{F}_\lambda(c,\mu)$ is an increasing function of μ and hence it cannot have a maximum value at any point in the interior on the closed region $[0,2] \times [0,1]$. Further, for fixed $c \in [0,2]$, we have

$$\max_{0 \leq \mu \leq 1} \{\mathcal{F}_\lambda(c,\mu)\} = \mathcal{F}_\lambda(c,1) = \mathcal{G}_\lambda(c). \tag{3.16}$$

Now, upon simplifying the relations (3.14) and (3.16), we obtain

$$\mathcal{G}_\lambda(c) = \frac{T(1-\alpha)^2}{2}\Big(Ac^4 + 8Bc^2 + 24(1+2\lambda)^3(8\lambda^2+6\lambda+1)\Big), \tag{3.17}$$

$$\mathcal{G}'_\lambda(c) = 2Tc(1-\alpha)^2\big(Ac^2+4B\big) \tag{3.18}$$

and

$$\mathcal{G}''_\lambda(c) = 2T(1-\alpha)^2\big(3Ac^2+4B\big), \tag{3.19}$$

where A, B and T are given in (3.1), (3.2) and (3.3), respectively. We consider the following cases.

Case 1. If $\lambda = 0$, $\mathcal{G}_\lambda(c)$ is a constant and its value is given by

$$\mathcal{G}_0(c) = \frac{(1-\alpha)^2}{4}.$$

Case 2. Since $0 \leq \alpha < 1$, if we take

$$0 < \lambda \leq \frac{-3+\sqrt{9+24(1-\alpha)}}{12},$$

then we conclude that $A < 0$ and $B \leq 0$. Therefore, we observe that $\mathcal{G}'_\lambda(c) < 0$ for $0 < c < 2$. Therefore, in this case, $\mathcal{G}_\lambda(c)$ is a decreasing function of c on $[0,2]$, and we have

$$\max\{\mathcal{G}_\lambda(c)\} = \mathcal{G}_\lambda(0) = 12T(1-\alpha)^2(1+2\lambda)^3(8\lambda^2+6\lambda+1). \tag{3.20}$$

Case 3. Since $0 \leq \alpha < 1$, if we choose

$$\lambda > \frac{-3+\sqrt{9+24(1-\alpha)}}{12},$$

then $A < 0$ and $B > 0$. From the equation $\mathcal{G}'_\lambda(c) = 0$, it is easy to verify that the function $\mathcal{G}_\lambda(c)$ has only one critical point in the open interval $(0,2)$ given by

$$c = 2\sqrt{-\frac{B}{A}}. \tag{3.21}$$

Substituting the value of c from (3.21) into (3.19), and after simplifying, we get

$$\mathcal{G}_\lambda''\left(2\sqrt{-\frac{B}{A}}\right) = -16T(1-\alpha)^2B < 0.$$

Therefore, by the second derivative test, $\mathcal{G}_\lambda(c)$ has its maximum value at c, where c is given by (3.21). Substituting the value of c in (3.17), and upon simplification, we obtain $\max\{\mathcal{G}_\lambda(c)\}$, given by

$$\max\{\mathcal{G}_\lambda(c)\} = 4T(1-\alpha)^2\left[3(1+2\lambda)^4(1+4\lambda) - \frac{2B^2}{A}\right]. \tag{3.22}$$

By summarizing the conclusions in Cases 1, 2 and 3, we arrive at the following inequality:

$$|\gamma_1\gamma_3 - \gamma_2^2| \leq \begin{cases} 12T(1-\alpha)^2(1+2\lambda)^3(8\lambda^2+6\lambda+1) \\ \qquad \left(0 \leq \lambda \leq \frac{-3+\sqrt{9+24(1-\alpha)}}{12}\right) \\ 4T(1-\alpha)^2\left[3(1+2\lambda)^4(1+4\lambda) - \frac{2B^2}{A}\right] \\ \qquad \left(\lambda > \frac{-3+\sqrt{9+24(1-\alpha)}}{12}\right). \end{cases}$$

This completes the proof of the theorem. □

In its special case when $\lambda = 0$, the above theorem would yield the corresponding estimate for functions in the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$

Corollary 1. (see [10]) *Let $f(z) \in \mathcal{S}^*(\alpha)$. Then*

$$|\gamma_1\gamma_3 - \gamma_2^2| \leq \frac{(1-\alpha)^2}{4}.$$

4. Conclusion

The present investigation is motivated essentially by several recent developments. We have given the bounds for the second Hankel determinant:

$$H_{2,1}\left(\frac{F_f}{2}\right) = \gamma_1\gamma_3 - \gamma_2^2$$

of the logarithmic coefficients of functions belonging to a certain subclass $\mathcal{R}(\lambda,\alpha)$ of normalized univalent functions, which we have introduced here. We have also briefly described relevant connections of the results, which we have presented here, with those available in the existing literature in Geometric Function Theory of Complex Analysis.

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