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ON GENERALIZED M-PROJECTIVE CURVATURE TENSOR OF PARA-SASAKIAN MANIFOLD

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Abstract. The purpose of the present paper is to study some properties of generalized M-projective curvature tensor of a para-Sasakian manifold admitting Zamkovoy connection. The generalized M-projective is obtained with the help of a new generalized (0,2) symmetric tensor $\mathcal{Z}$ introduced by Mantica and Suh [10]. It is shown that para-Sasakian manifold satisfying the condition $R(X,Y)\cdot\tilde{M}^{**}=0$ is an η-Einstein manifold. Also, we found that a para-Sasakian manifold satisfying $\tilde{M}^{**}(X,Y)\cdot S=0$ is either an Einstein manifold or $\psi=1$ on it.

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1. Introduction

The notion of the almost para-contact structure on a differentiable manifold is defined by I. Sato [14, 15]. The para-contact metric manifolds have been studied by many authors in recent years. The structure is an analogue of the almost contact structure [5, 13] and is closely related to almost product structure (in contrast to almost contact structure, which is related to almost complex structure). Every differentiable manifold with almost para-contact structure defined by I. Sato has a compatible Riemannian metric.

An almost para-contact structure on a pseudo-Riemannian manifold M of dimension $(2n+1)$ defined is by S. Kaneyuki and M. Konzai [18] and they constructed the almost paracomplex structure on $M\times R$. Recently, S. Zamkovoy [22] has associated the almost para-contact structure given in [18] to a pseudo-Riemannian metric of signature $(n+1,n)$ and showed that any almost para-contact structure admits such a pseudo-Riemannian metric.

The study of M-projective curvature tensor has been a very attractive field for investigations in the past many decades. M-projective curvature tensor was introduced by G. P. Pokhariyal and R. S. Mishra [12] in 1971. Also, in 1986, R. H. Ojha [11] extended some properties of the M-projective curvature tensor in Sasakian and Kähler manifolds. In 2010, the study of M-projective curvature tensor in Riemannian manifolds and also in Kenmotsu manifolds was resumed by S. K. Chaubey and R. H. Ojha [6]. Further, R. N. Singh and S. K. Pandey [19] have studied various geometric properties of M-projective curvature tensor on $N(k)$-contact metric manifolds. In 2020, A. Mandal and A. Das [8] studied some properties of the M-projective curvature tensor in Sasakian manifolds. Afterwards, several researchers have carried out the study of M-projective curvature tensor in a variety of directions such as [9, 16–18]. The M-projective curvature tensor defined by G. P. Pokhariyal and R. S. Mishra [12] is given as below;

$$M^*(X,Y,U) = R(X,Y,U) - \frac{1}{2(n-1)}[S(Y,U)X - S(X,U)Y + g(Y,U)QX - g(X,U)QY],$$

for all $X,Y,U \in \chi(M)$, where $\chi(M)$ is the set of all vector field of manifold M, $R(X,Y)U$ is the Riemannian curvature tensor of type $(0,3)$ and S is the Ricci tensor, i.e,

$$S(X,Y) = g(QX,Y),$$

where Q is a Ricci operator of type $(1,1)$.

Also, the type $(0,4)$ M-projective curvature tensor field $'M^*$ is given by

$$'M^*(X,Y,U,V) = {}'R(X,Y,U,V) - \frac{1}{2(n-1)}[S(Y,U)g(X,V) - S(X,U)g(Y,V) + g(Y,U)S(X,V) - g(X,U)S(Y,V)], \tag{1.1}$$

where

$$'M^*(X,Y,U,V) = g(M^*(X,Y,U),V)$$

and

$$'R(X,Y,U,V) = g(R(X,Y,U),V)$$

for the arbitrary vector fields $X,Y,U,V \in \chi(M)$.

A new generalized $(0,2)$ symmetric tensor $\mathcal{Z}$, defined by Mantica and Suh [10], is given by the following relation

$$\mathcal{Z}(X,Y) = S(X,Y) + \psi g(X,Y), \tag{1.2}$$

where ψ is an arbitrary scalar function.

From equation (1.2), we have

$$\mathcal{Z}(\phi X,\phi Y) = S(\phi X,\phi Y) + \psi g(\phi X,\phi Y),$$

By using equation (1.2) in equation (1.1), we get

$$\begin{aligned} {}'M^{*}(X,Y,U,V) = {}&{}'R(X,Y,U,V) \\ &-\frac{1}{2(n-1)}[\mathcal{Z}(Y,U)g(X,V)-\mathcal{Z}(X,U)g(Y,V) \\ &\qquad +g(Y,U)\mathcal{Z}(X,V)-g(X,U)\mathcal{Z}(Y,V)] \\ &-\frac{\psi}{(n-1)}[g(Y,V)g(X,U)-g(Y,U)g(X,V)]. \end{aligned} \tag{1.3}$$

If we denote the first five terms of above equation by

$$\begin{aligned} {}'M^{**}(X,Y,U,V) = {}&{}'R(X,Y,U,V) \\ &-\frac{1}{2(n-1)}[\mathcal{Z}(Y,U)g(X,V)-\mathcal{Z}(X,U)g(Y,V) \\ &\qquad +g(Y,U)\mathcal{Z}(X,V)-g(X,U)\mathcal{Z}(Y,V)], \end{aligned} \tag{1.4}$$

then the equation (1.3) reduces to

$$\begin{aligned} {}'M^{**}(X,Y,U,V) = {}&({}'M^{*})(X,Y,U,V)+\frac{\psi}{(n-1)}[g(Y,V)g(X,U) \\ &-g(X,V)g(Y,U)]. \end{aligned}$$

We call this new tensor field ${}'M^{**}$ defined by equation (1.4), generalized M-projective curvature tensor of para-Sasakian manifold.

In 2008, The notion of Zamkovoy connection was introduced by S. Zamkovoy [22] for a para-contact manifold. And this connection is defined as a canonical para-contact connection whose torsion is the obstruction of para-contact manifold to be a para-Sasakian manifold [1]. For an n-dimensional almost contact metric manifold M equipped with an almost contact metric structure (ϕ,ξ,η,g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, the Zamkovoy connection is defined by [22]

$$\tilde{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y \tag{1.5}$$

for all $X,Y,U \in \chi(M)$.

This connection was further studied by A. M. Blaga in para Kenmotsu manifolds [4] and A. Biswas, K. K. Baishya in Sasakian manifolds [2, 3].

In this paper, we study some properties of the generalized M-projective curvature tensor of para-Sasakian manifold with respect to the Zamkovoy connection. The present paper is organized as follows: Section 2 is devoted to preliminaries and we give some relations between curvature tensor (resp. Ricci tensor) with respect to Zamkovoy connection and curvature tensor (resp. Ricci tensor) with respect to Levi-Civita connection. In Section 3, we describe briefly the generalized M-projective curvature tensor on para-Sasakian manifold with respect to the connection. In Section 4, we show that a generalized M-projectively semi-symmetric para-Sasakian manifold is an η-Einstein manifold. Further in Section 5, the goal is to examine implication of the condition $\tilde{M}^{**}(X,Y)\cdot S=0$ and we show that the para-Sasakian manifold is either an Einstein manifold or $\psi=1$ on it. In the last section, we show that $\phi^2((\nabla_V\tilde{M}^{**})(X,Y,U))=0$ is an η-Einstein manifold.

2. Preliminaries

An $(2n+1)$-dimensional differentiable manifold M is said to have almost para-contact structure (ϕ,ξ,η), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field known as characteristic vector field and η is a 1-form on M satisfying the following relations [18]

$$\phi^2=I-\eta\otimes\xi, \tag{2.1}$$

$$\eta(\xi)=1, \tag{2.2}$$

$$\phi(\xi)=0, \qquad \eta(\phi X)=0,$$

and

$$\text{rank}(\phi)=2n,$$

where I denotes the identity transformation, a differentiable manifold with almost para-contact structure (ϕ,ξ,η) is called an almost para-contact manifold [1].

Moreover, the tensor field ϕ induces an almost paracomplex structure on the para-contact distribution D = ker (η), i.e, the eigendistributions $D^{\pm}$ corresponding to the eigenvalues ±1 of ϕ are both n-dimensional.

If an almost para-contact manifold M with an almost para-contact structure (ϕ,ξ,η) admits a pseudo-Riemannian metric g such that [22]

$$g(\phi X,\phi Y)=-g(X,Y)+\eta(X)\eta(Y), \tag{2.3}$$

for all $X,Y\in\chi(M)$, then we say that M is an almost para-contact metric manifold with an almost para-contact metric structure (ϕ,ξ,η,g) and such metric g is called compatible metric. Any compatible metric g is necessarily of signature $(n+1,n)$.

From (2.3) one can see that [22]

$$g(X,\phi Y)=-g(\phi X,Y),$$

and also we take

$$\eta(X)=g(X,\xi) \tag{2.4}$$

for any $X,Y\in\chi(M)$. The fundamental 2-form of M is defined by

$$\alpha(X,Y)=g(X,\phi Y).$$

The structure (ϕ,ξ,η,g) satisfying conditions (2.1) to (2.4) is called an almost para-contact metric structure and the manifold M with such a structure is called an almost para-contact Riemannian manifold [14].

An almost para-contact metric structure becomes a para-contact metric structure [22] if

$$g(X,\phi Y) = d\eta(X,Y),$$

for all vector field $X,Y \in \chi(M)$, where

$$d\eta(X,Y) = \frac{1}{2}[X\eta(Y) - Y\eta(X) - \eta([X,Y])].$$

For a $(2n+1)$ dimensional manifold M with the structure (ϕ,ξ,η,g), one can also construct a local orthonormal basis which is called a ϕ-basis $(X_i,\phi X_i,\xi)$, $(i = 1,2,\dots,n)$ [22]

An almost para-contact metric manifold structure (ϕ,ξ,η,g) is para-Sasakian manifold if and only if the Levi-Civita connection ∇ of g satisfies [22]

$$(\nabla_X\phi)Y = -g(X,Y)\xi + \eta(Y)X, \tag{2.5}$$

for any $X,Y \in \chi(M)$.

From (2.5), it can be seen that

$$(\nabla_X\xi) = -\phi X. \tag{2.6}$$

Example 1. [1]. Let $M = R^{2n+1}$ be the $(2n+1)$- dimensional real number space with $(x_1,y_1,x_2,y_2,\dots,x_n,y_n,z)$ standard coordinate system. Defining

$$\begin{aligned}
&\phi\frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial y_\alpha}, \quad \phi\frac{\partial}{\partial y_\alpha} = \frac{\partial}{\partial x_\alpha}, \quad \phi\frac{\partial}{\partial z} = 0,\\
&\xi = \frac{\partial}{\partial z}, \quad \eta = dz,\\
&g = \eta\otimes\eta + \sum_{\alpha=1}^{n} dx_\alpha\otimes dx_\alpha - \sum_{\alpha=1}^{n} dy_\alpha\otimes dy_\alpha,
\end{aligned}$$

where $\alpha = 1,2,\dots,n$, then the set (M,ϕ,ξ,η,g) is an almost para-contact metric manifold.

In a para-Sasakian manifold, the following relations also hold [22]:

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.7}$$

$$\begin{aligned}
&R(X,Y,\xi) = \eta(X)Y - \eta(Y)X,\\
&R(X,\xi,Y) = -R(\xi,X,Y) = -g(X,Y)\xi + \eta(Y)X, \qquad (2.8)\\
&R(\xi,X,\xi) = X - \eta(X)\xi,\\
&S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y),
\end{aligned}$$

and

$$S(X,\xi) = -2n\eta(X), \tag{2.9}$$

for any vector fields $X,Y,Z \in \chi(M)$. Here, R is Riemannian curvature tensor and S is the Ricci tensor defined by $g(QX,Y) = S(X,Y)$, where Q is the Ricci operator.

In view of (2.6), the equation (1.5) becomes

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + \eta(Y)\phi X + g(X,\phi Y)\xi. \tag{2.10}$$

On a para-Sasakian manifold, the connection $\tilde{\nabla}$ has the following properties [1]:

$$\tilde{\nabla}\eta = 0, \quad \tilde{\nabla} g = 0, \quad \tilde{\nabla}\xi = 0,$$

and

$$(\tilde{\nabla}_X \phi)Y = (\nabla_X \phi)Y + g(X,Y)\xi - \eta(Y)X.$$

for any vector fields $X,Y, \in \chi(M)$.

It is known that the curvature tensor $\tilde{R}$ of a para-Sasakian manifold M with respect to the Zamkovoy connection $\tilde{\nabla}$ defined by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z$$

satisfies the following [1]

$$\begin{aligned}\tilde{R}(X,Y)Z = {} & R(X,Y)Z + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X \\ & - \eta(X)\eta(Z)Y + 2g(X,\phi Y)\phi Z + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X, \\ \tilde{S}(X,Y) = {} & S(X,Y) - 2g(X,Y) + (2n+2)\eta(X)\eta(Y),\end{aligned} \tag{2.11}$$

and

$$\tilde{r} = r - 2n$$

for any $X,Y,Z \in \chi(M)$, where R, S and r are curvature tensor, Ricci tensor and scalar curvature relative to ∇ respectively and $\tilde{R}$, $\tilde{S}$ and $\tilde{r}$ are curvature tensor, Ricci tensor and scalar curvature relative to $\tilde{\nabla}$. From (2.11), it is easy to note that $\tilde{S}$ is symmetric.

Further, it is known that [1] on a para-Sasakian manifold, the following relations hold

$$g(\tilde{R}(X,Y)Z,\xi) = \eta(\tilde{R}(X,Y)Z) = 0, \tag{2.12}$$

$$\tilde{R}(X,Y)\xi = \tilde{R}(\xi,X)Y = \tilde{R}(\xi,X)\xi = 0, \tag{2.13}$$

and

$$\tilde{S}(X,\xi) = 0$$

for any $X,Y,U \in \chi(M)$.

Definition 1. An $(2n+1)$-dimensional para-Sasakian manifold M is said to be η-Einstein manifold if the Ricci tensor of type $(0,2)$ is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$$

for any $X,Y \in \chi(M)$ where a and b are scalars.

From (2.11), it can also be noted that if $\tilde{S}(X,Y)=0$. then

$$S(X,Y)=2g(X,Y)+(-2n-2)\eta(X)\eta(Y).$$

which proves that if a para- Sasakian manifold M is Ricci-flat with respect to the Zamkovoy connection, then it is an η-Einstein manifold.

3. Generalized M-projective curvature tensor of para-Sasakian manifold

In this section, we study generalized M-projective curvature tensor of the para-Sasakian manifold with respect to the Zamkovoy connection and state some of its properties. The M-projective curvature tensor $\tilde{M}$ with respect to the Zamkovoy connection $\tilde{\nabla}$ is given by

$$\begin{aligned}\tilde{M}^*(X,Y,U)= {} & \tilde{R}(X,Y,U)\\ & -\frac{1}{2(n-1)}[\tilde{S}(Y,U)X-\tilde{S}(X,U)Y\\ & +g(Y,U)\tilde{Q}X-g(X,U)\tilde{Q}Y],\end{aligned} \tag{3.1}$$

Also, the type $(0,4)$ M-projective curvature tensor field $'\tilde{M}^*$ is given by

$$\begin{aligned}'\tilde{M}^*(X,Y,U,V)= {} & '\tilde{R}(X,Y,U,V)\\ & -\frac{1}{2(n-1)}[\tilde{S}(Y,U)g(X,V)-\tilde{S}(X,U)g(Y,V)\\ & +g(Y,U)\tilde{S}(X,V)-g(X,U)\tilde{S}(Y,V)],\end{aligned} \tag{3.2}$$

where

$$'\tilde{M}^*(X,Y,U,V)=g(\tilde{M}^*(X,Y,U),V)$$

and

$$'\tilde{R}(X,Y,U,V)=g(\tilde{R}(X,Y,U),V)$$

for the arbitrary vector fields $X,Y,U,V\in\chi(M)$.

Now, differentiating covariantly equation (3.1) with respect to V, we get

$$\begin{aligned}(\nabla_V\tilde{M}^*)(X,Y)U= {} & (\nabla_V\tilde{R})(X,Y)U\\ & -\frac{1}{2(n-1)}[(\nabla_V\tilde{S})(Y,U)X-(\nabla_V\tilde{S})(X,U)Y\\ & +g(Y,U)(\nabla_V\tilde{Q})X-g(X,U)(\nabla_V\tilde{Q})Y].\end{aligned} \tag{3.3}$$

The generalized M-projective curvature tensor of para-Sasakian manifold with respect to the Zamkovoy connection is defined by,

$$\begin{aligned}{}'\tilde{M}^{*}(X,Y,U,V) = {}'\tilde{R}(X,Y,U,V) \\ &- \frac{1}{2(n-1)}[\mathcal{Z}(Y,U)g(X,V) - \mathcal{Z}(X,U)g(Y,V) \\ &\quad + g(Y,U)\mathcal{Z}(X,V) - g(X,U)\mathcal{Z}(Y,V)] \\ &- \frac{\psi}{(n-1)}[g(Y,V)g(X,U) - g(Y,U)g(X,V)].\end{aligned}$$

If we denote the first five terms of above equation by

$$\begin{aligned}{}'\tilde{M}^{**}(X,Y,U,V) = {}'\tilde{R}(X,Y,U,V) \\ &- \frac{1}{2(n-1)}[\mathcal{Z}(Y,U)g(X,V) - \mathcal{Z}(X,U)g(Y,V) \\ &\quad + g(Y,U)\mathcal{Z}(X,V) - g(X,U)\mathcal{Z}(Y,V)],\end{aligned} \tag{3.4}$$

then the equation (3.4) reduces to

$$\begin{aligned}{}'\tilde{M}^{**}(X,Y,U,V) = {}'\tilde{M}^{*}(X,Y,U,V) + \frac{\psi}{(n-1)}[g(Y,V)g(X,U) \\ - g(X,V)g(Y,U)].\end{aligned} \tag{3.5}$$

We call this new tensor field ${}'\tilde{M}^{**}$ defined by equation (3.4), generalized M-projective curvature tensor of para-Sasakian manifold with respect to the Zamkovoy connection.

If ψ=0, then from equation (3.5), we have

$${}'\tilde{M}^{**}(X,Y,U,V) = {}'\tilde{M}^{*}(X,Y,U,V).$$

Lemma 1. *If the scalar function ψ vanishes on para-Sasakian manifold, then the M-projective curvature tensor and generalized M-projective curvature tensor are identical.*

Lemma 2. *Generalized M-projective curvature tensor of para-Sasakian manifold with respect to the Zamkovoy connection satisfies Bianchi's first identity.*

Remark 1. Generalized M-projective curvature tensor ${}'\tilde{M}^{**}$ of para-Sasakian manifold with respect to the Zamkovoy connection is:

(a) skew-symmetric in the first two slots,
(b) skew-symmetric in the last two slots

and

(c) symmetric in pair of slots.

Proposition 1. *Generalized M-projective curvature tensor of para-Sasakian manifold satisfies the following identities:*

$$\begin{aligned}(a)\ \tilde{M}^{**}(\xi,Y,U) = -\tilde{M}^{**}(Y,\xi,U) &= \left[\frac{(1-\psi)}{(n-1)}\right][g(Y,U)\xi-\eta(U)Y] \\ &\quad -\left[\frac{1}{2(n-1)}\right][S(Y,U)\xi-\eta(U)QY],\end{aligned} \tag{3.6}$$

$$\begin{aligned}(b)\ \tilde{M}^{**}(X,Y,\xi) = &-\left[\frac{(1-\psi)}{(n-1)}\right][\eta(X)Y-\eta(Y)X] \\ &+\left[\frac{1}{2(n-1)}\right][\eta(Y)QX-\eta(X)QY],\end{aligned} \tag{3.7}$$

$$\begin{aligned}(c)\ \eta(\tilde{M}^{**}(U,V,Y)) = &\left[\frac{(1-\psi)}{(n-1)}\right][g(V,Y)\eta(U)-g(U,Y)\eta(V)] \\ &+\left[\frac{1}{2(n-1)}\right][S(U,Y)\eta(V)-S(V,Y)\eta(U)].\end{aligned} \tag{3.8}$$

4. GENERALIZED M-PROJECTIVELY SEMI-SYMMETRIC PARA-SASAKIAN MANIFOLD WITH RESPECT TO THE ZAMKOVOY CONNECTION

Definition 2. A para-Sasakian manifold is said to be semi-symmetric [20] if it satisfies the condition

$$R(X,Y)\cdot R = 0,$$

where $R(X,Y)$ is considered as the derivation of the tensor algebra at each point of the manifold.

Definition 3. A para-Sasakian manifold is said to be generalized M-projectively semi-symmetric if it satisfies the condition

$$R(X,Y)\cdot \tilde{M}^{**} = 0,$$

where $\tilde{M}^{**}$ is generalized M-projective curvature tensor relative to $\tilde{\nabla}$ and $R(X,Y)$ is considered as the derivation of the tensor algebra at each point of the manifold.

Theorem 1. *A generalized M-projectively semi-symmetric para-Sasakian manifold with respect to the Zamkovoy connection is an η-Einstein manifold.*

Proof. Consider

$$R(X,Y)\cdot \tilde{M}^{**} = 0,$$

Now, we put $X=\xi$ in above equation, we get

$$(R(\xi,X)\cdot \tilde{M}^{**})(U,V,Y) = 0,$$

for any $X,Y,U,V \in \chi(M)$, where $\tilde{M}^{**}$ is generalized M-projective curvature tensor, which gives

$$\begin{aligned}0 = R(\xi,X,\tilde{M}^{**}(U,V,Y)) - \tilde{M}^{**}(R(\xi,X,U),V,Y)\\ - \tilde{M}^{**}(U,R(\xi,X,V),Y) - \tilde{M}^{**}(U,V,R(\xi,X,Y).\end{aligned}$$

In view of the equation (2.8), the above equation takes the form

$$\begin{aligned}0 = \eta(\tilde{M}^{**}(U,V,Y))X -' \tilde{M}^{**}(U,V,Y,X)\xi\\ + g(X,U)\eta(\tilde{M}^{**}(\xi,V,Y) - \eta(V)\eta(\tilde{M}^{**}(U,X,Y)\\ + g(X,V)\eta(\tilde{M}^{**}(U,\xi,Y) - \eta(Y)\eta(\tilde{M}^{**}(U,V,X)\\ + g(X,Y)\eta(\tilde{M}^{**}(U,V,\xi) - \eta(U)\eta(\tilde{M}^{**}(X,V,Y).\end{aligned}$$

Taking inner product of above equation with ξ and using equations (2.2), (2.3), (2.12), (2.13), (3.5), (3.6), (3.7) and (3.8), we get

$$\begin{aligned}0 = &-{}'\tilde{M}^{**}(U,V,Y,X)\\ &- \frac{1}{2(n-1)}[S(U,X)\eta(Y)\eta(V) - S(V,X)\eta(U)\eta(Y)]\\ &+ \frac{(1-\psi)}{(n-1)}[g(X,U)g(Y,V) - g(X,V)g(Y,U)]\\ &- \frac{1}{2(n-1)}[S(V,Y)g(X,U) - S(Y,U)g(X,V)]\\ &- \frac{n}{(n-1)}[g(X,U)\eta(Y)\eta(V) - g(X,V)\eta(U)\eta(Y)].\end{aligned}$$

By virtue of the equations (2.11) and (3.2), the above equation reduces to

$$\begin{aligned}{}'\tilde{R}(U,V,Y,X) = &\frac{1}{2(n-1)}[\tilde{S}(V,Y)g(U,X) - \tilde{S}(U,Y)g(V,X)\\ &+ \tilde{S}(U,X)g(V,Y) - \tilde{S}(V,X)g(U,Y)]\\ &- \frac{1}{2(n-1)}[S(U,X)\eta(Y)\eta(V) - S(V,X)\eta(U)\eta(Y)]\\ &+ \frac{(1-\psi)}{(n-1)}[g(X,U)g(Y,V) - g(X,V)g(Y,U)]\\ &- \frac{1}{2(n-1)}[S(V,Y)g(X,U) - S(Y,U)g(X,V)]\\ &- \frac{n}{(n-1)}[g(X,U)\eta(Y)\eta(V) - g(X,V)\eta(U)\eta(Y)].\end{aligned}$$

Let $\{e_i : i = 1, 2 \ldots, n\}$ be an orthonormal basis. Putting $X = U = e_i$ in above equation and taking summation over i, we get

$$S(Y,V) = \left[\frac{r+2n-4n\psi-2}{(2n-1)}\right] g(Y,V) + \left[\frac{r+4n^2-2}{(2n-1)}\right] \eta(Y)\eta(V).$$

This shows that generalized M-projectively semi-symmetric para-Sasakian manifold is an η-Einstein manifold. □

5. PARA-SASAKIAN MANIFOLD SATISFYING $\tilde{M}^{**}(X,Y) \cdot S = 0$

In this section, we consider para-Sasakian manifold with Zamkovoy connection [7] satisfying the condition

$$\tilde{M}^{**}(X,Y) \cdot S = 0,$$

for all $X, Y \in \chi(M)$, where $\tilde{M}^{**}$ is generalized M-projective curvature tensor of para-Sasakian manifold.

Theorem 2. *A para-Sasakian manifold admitting Zamkovoy connection satisfying* $\tilde{M}^{**}(X,Y) \cdot S = 0$ *is either an Einstein manifold or* $\psi = 1$.

Proof. Consider

$$(\tilde{M}^{**}(\xi,X) \cdot S)(U,V) = 0,$$

which gives

$$S(\tilde{M}^{**}(\xi,X,U),V) + S(U,\tilde{M}^{**}(\xi,X,V)) = 0.$$

Using equations (2.9), (2.10) and (3.6) in above equation, we get

$$\begin{aligned} 0 = & \left[\frac{-2n(\psi-1)}{(n-1)}\right] [g(X,U)\eta(V) + g(X,V)\eta(U)] \\ & - \left[\frac{1}{(n-1)}\right] [S(X,V)\eta(U) + S(X,U)\eta(V)] \\ & + \left[\frac{\psi}{(n-1)}\right] [S(X,V)\eta(U) + S(X,U)\eta(V)]. \end{aligned}$$

Putting $U = \xi$ in the above equation and using the equations (2.3), (2.4) and (2.9), we get

$$\left[\frac{(\psi-1)}{(n-1)}\right] [S(X,V) - 2ng(X,V)] = 0,$$

which gives either $\psi = 1$ or

$$S(X,V) = 2ng(X,V).$$

This shows that generalized M-projectively Ricci semi-symmetric para-Sasakian manifold with respect to the Zamkovoy connection is either an Einstein manifold or $\psi = 1$ on it. □

6. A PARA-SASAKIAN MANIFOLD SATISFYING $\phi^2((\nabla_V R)(X,Y,U)) = 0$

Below we present the definition given by Takahashi [21]

Definition 4. A para-Sasakian manifold is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_V R)(X,Y,U)) = 0, \tag{6.1}$$

for all vector fields X,Y,U,V orthogonal to ξ.

Definition 5. A para-Sasakian manifold is said to be ϕ-symmetric if

$$\phi^2((\nabla_V R)(X,Y,U)) = 0, \tag{6.2}$$

for arbitrary vector fields X,Y,U,V.

Analogous to the conditions (6.1) and (6.2), we consider a para-Sasakian manifold satisfying

$$\phi^2((\nabla_V \tilde{M}^{**})(X,Y,U)) = 0, \tag{6.3}$$

for arbitary vector fields X,Y,U,V.

Theorem 3. *A para-Sasakian manifold admitting Zamkovoy connection satisfying $\phi^2((\nabla_V \tilde{M}^{**})(X,Y,U)) = 0$ is an Einstein manifold.*

Proof. Taking covariant derivative of equation (3.5) with respect to vector field V, we obtain

$$(\nabla_V \tilde{M}^{**})(X,Y,U) = (\nabla_V \tilde{M}^{*})(X,Y,U) + \frac{dr(\psi)}{(n-1)}[g(X,U)Y - g(Y,U)X)].$$

Using equation (3.3) in the above equation, we get

$$\begin{aligned}(\nabla_V \tilde{M}^{**})(X,Y,U) &= (\nabla_V \tilde{R})(X,Y,U) + \frac{dr(\psi)}{(n-1)}[g(X,U)Y - g(Y,U)X)] \\ &\quad - \frac{1}{2(n-1)}[(\nabla_V \tilde{S})(Y,U)X - (\nabla_V \tilde{S})(X,U)Y \\ &\qquad + g(Y,U)(\nabla_V \tilde{Q})X - g(X,U)(\nabla_V \tilde{Q})Y].\end{aligned} \tag{6.4}$$

Assume that the manifold is generalized M-projectively ϕ-symmetric, then from equation (6.3), we have

$$\phi^2((\nabla_V \tilde{M}^{**})(X,Y,U)) = 0,$$

which on using equation (2.1), gives

$$(\nabla_V \tilde{M}^{**})(X,Y,U) = \eta((\nabla_V \tilde{M}^{**})(X,Y,U))\xi.$$

Using equation (6.4) in above equation, we get

$$\begin{aligned}
&(\nabla_V \tilde{R})(X,Y,U) + \frac{dr(\psi)}{(n-1)}[g(X,U)Y - g(Y,U)X)] \\
&- \frac{1}{2(n-1)}[(\nabla_V \tilde{S})(Y,U)X - (\nabla_V \tilde{S})(X,U)Y \\
&\qquad + g(Y,U)(\nabla_V \tilde{Q})X - g(X,U)(\nabla_V \tilde{Q})Y] \\
&= \eta((\nabla_V \tilde{R})(X,Y,U))\xi + \frac{dr(\psi)}{(n-1)}[g(X,U)\eta(Y) - g(Y,U)\eta(X))]\xi \\
&- \frac{1}{2(n-1)}[(\nabla_V \tilde{S})(Y,U)\eta(X) - (\nabla_V \tilde{S})(X,U)\eta(Y) \\
&\qquad + g(Y,U)\eta((\nabla_V \tilde{Q})X) - g(X,U)\eta((\nabla_V \tilde{Q})Y)]\xi.
\end{aligned}$$

Taking inner product of the above equation with W, we get

$$\begin{aligned}
&g((\nabla_V \tilde{R})(X,Y,U),W) + \frac{dr(\psi)}{(n-1)}[g(X,U)g(Y,W) - g(Y,U)g(X,W)] \\
&- \frac{1}{2(n-1)}[(\nabla_V \tilde{S})(Y,U)g(X,W) - (\nabla_V \tilde{S})(X,U)g(Y,W) \\
&\qquad + g(Y,U)g((\nabla_V \tilde{Q})X,W) - g(X,U)g((\nabla_V \tilde{Q})Y,W)] \\
&= \eta((\nabla_V \tilde{R})(X,Y,U))\eta(W) + \frac{dr(\psi)}{(n-1)}[g(X,U)\eta(Y)\eta(W) \\
&\qquad\qquad - g(Y,U)\eta(X)\eta(W)] \\
&- \frac{1}{2(n-1)}[(\nabla_V \tilde{S})(Y,U)\eta(X)\eta(W) - (\nabla_V \tilde{S})(X,U)\eta(Y)\eta(W) \\
&\qquad + g(Y,U)\eta((\nabla_V \tilde{Q})X)\eta(W) - g(X,U)\eta((\nabla_V \tilde{Q})Y)\eta(W)].
\end{aligned}$$

Putting $X = W = e_i$ and taking summation over i, we obtain

$$\begin{aligned}
0 = &-\frac{1}{(n-1)}(\nabla_V \tilde{S})(Y,U) - \frac{dr(\psi)}{(n-1)}[\eta(Y)\eta(U) - g(Y,U)] \\
&- \frac{1}{2(n-1)}[g(Y,U)g((\nabla_V \tilde{Q})e_i,e_i) - g((\nabla_V \tilde{Q})Y,U)] \\
&- \frac{2n}{(n-1)}dr(\psi)g(Y,U) - \eta((\nabla_V \tilde{R})(e_i,Y,U))\eta(e_i) \\
&+ \frac{1}{2(n-1)}[(\nabla_V \tilde{S})(Y,U) - (\nabla_V \tilde{S})(e_i,U)\eta(Y)\eta(e_i) \\
&\qquad + g(Y,U)\eta((\nabla_V \tilde{Q})e_i)\eta(e_i) - \eta((\nabla_V \tilde{Q})Y)\eta(U)].
\end{aligned}$$

Taking $U=\xi$ in the above equation, we have

$$\begin{aligned}0=&-\frac{1}{2(n-1)}(\nabla_V\tilde{S})(Y,\xi)-\eta((\nabla_V\tilde{R})(e_i,Y,\xi))\eta(e_i)\\&-\frac{1}{2(n-1)}[dr(\tilde{V})\eta(Y)-(\nabla_V\tilde{S})(e_i,\xi)\eta(e_i)\eta(Y)\\&\qquad+\eta((\nabla_V\tilde{Q})e_i)\eta(e_i)\eta(Y)]\\&-\frac{2n}{(n-1)}dr(\psi)\eta(Y).\end{aligned} \tag{6.5}$$

Now

$$\eta((\nabla_V\tilde{R})(e_i,Y,\xi)\eta(e_i)=g((\nabla_V\tilde{R})(e_i,Y,\xi),\xi)g(e_i,\xi).$$

Also

$$\begin{aligned}g((\nabla_V\tilde{R})(e_i,Y,\xi),\xi)=&\,g(\nabla_V\tilde{R}(e_i,Y,\xi),\xi)-g(\tilde{R}(\nabla_Ve_i,Y,\xi),\xi)\\&-g(\tilde{R}(e_i,\nabla_VY,\xi),\xi)-g(\tilde{R}(e_i,Y,\nabla_V\xi),\xi).\end{aligned} \tag{6.6}$$

Since $\{e_i\}$ is an orthonormal basis, so $\nabla_Xe_i=0$ and using equation (2.8), we get

$$g(\tilde{R}(e_i,\nabla_VY,\xi),\xi)=0,$$

Since

$$g(\tilde{R}(e_i,Y,\xi),\xi)+g(\tilde{R}(\xi,\xi,Y),e_i)=0.$$

Therefore, we have

$$g(\nabla_V\tilde{R}(e_i,Y,\xi),\xi)+g(\tilde{R}e_i,Y,\xi),\nabla_V\xi)=0.$$

Using this fact in equation (6.6), we get

$$g((\nabla_V\tilde{R})(e_i,Y,\xi),\xi)=0. \tag{6.7}$$

Also

$$\eta((\nabla_V\tilde{Q})e_i)\eta(e_i)=g((\nabla_V\tilde{Q})e_i,\xi)g(e_i,\xi)=g((\nabla_V\tilde{Q})\xi,\xi).$$

Using equations (2.6) and (2.9), we get

$$\eta((\nabla_V\tilde{Q})e_i)\eta(e_i)=0. \tag{6.8}$$

Using equations (6.7) and (6.8) in (6.5), we have

$$(\nabla_V\tilde{S})(Y,\xi)=-4ndr(\psi)\eta(Y)+dr(\tilde{V})\eta(Y). \tag{6.9}$$

Taking $Y=\xi$ in above equation and using equations (2.7) and (2.10), we get

$$dr(\psi)=-\frac{dr(\tilde{V})}{4n}, \tag{6.10}$$

which shows that r is constant.

Now, we have

$$(\nabla_V\tilde{S})(Y,\xi)=\nabla_V\tilde{S}(Y,\xi)-\tilde{S}(\nabla_VY,\xi)-\tilde{S}(Y,\nabla_V\xi),$$

then by using (2.5), (2.6), (2.10) in the above equation, it follows that

$$\tilde{S}(Y,\phi V) = 0. \tag{6.11}$$

Putting $Y = \phi Y$ in above equation and using (2.11), (6.9), (6.10) and (6.11), we obtain

$$S(Y,V) = -2g(Y,V) - 2n\eta(V)\eta(Y).$$

which shows that M^{2n+1} is an η-Einstein manifold. □

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