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NUMERICAL SOLUTION OF FRACTIONAL VOLTERRA INTEGRAL EQUATIONS BASED ON RATIONAL CHEBYSHEV APPROXIMATION

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Abstract. We aim to give the numerical method for solving the fractional Volterra integral equations of first and second kinds. We here use the techniques based upon rational Chebyshev functions and Riemann-Liouville fractional integrals. Some illustrative experiments with a view of estimating error and graphics are given in order to show the validity and applicability of the technique. Our experiments show that the new technique has high accuracy and is very efficient when compare to the other approaches existing in literature.

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Keywords: fractional integral equations, rational Chebyshev functions, numerical scheme

1. INTRODUCTION

The equation in which the unknown function $u(x)$ appears within an integral sign is called an integral equation. The linear integral equations

$$\lambda \int_a^b K(x,\rho)u(\rho)d\rho = f(x) \tag{1.1}$$

and

$$u(x) - \lambda \int_a^b K(x,\rho)u(\rho)d\rho = f(x) \tag{1.2}$$

of first and second kinds, respectively, play a particularly important role in the study of pure and applied mathematics. Here, the kernel $K(x,\rho)$ and a function $f(x)$ are given, and the function $u(x)$ is to be determined. Note that, in (1.1) and (1.2), the interval of integration (a,b) can be extended to infinity in one or both directions.

An Italian mathematician Volterra introduced an integral equation and later called his integral equation "Volterra integral equation", briefly, VIE. His work is summarized in his book [33] and it is one of the widely-studied integral equation in several

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branches of mathematics. A Volterra integral equation is one in which one of the integral's limits is variable. For example, the VIE corresponding to (1.2) is given by

$$u(x) - \lambda \int_a^x K(x,\rho)u(\rho)d\rho = f(x). \tag{1.3}$$

Several numerical and theoretical methods have been demonstrated by several researchers to solve Volterra integral equations such as (i) Adomian decomposition method [6], (ii) Taylor-series expansion method [17], (iii) Sinc-collocation method [22], (iv) quadrature method with variable step [24], (v) collocation method [16]. Moreover, Sezer et al. have developed powerful techniques for different kinds of VIE [3, 34]. Recently, a numerical solution was obtained based on Bernstein approximation method to solve VIE of first and second kind as well as their singular form [15].

Fractional operators have been used by many researchers to describe some complex phenomena. Many differential equations have been generalized and revisited with the aid of new fractional definitions [1, 2, 5, 8, 9, 23, 30, 32]. Correlatively, the theory of fractional integral equations (FIE) can be counted as a generalization of the classical idea of integral equations wherein calculus of integrals of any arbitrary real or complex order [25]. In the last three decades, this theory has got attention from several researchers because of its widespread application in engineering and science.

Consider $I = [a,b]$ $(-\infty < a < b < \infty)$ is an interval on the real line $\mathbb{R}$. Recall as in [25] that the left and right Riemann-Liouville fractional integrals ${}^{\alpha}\mathcal{F}^{x}_{a_+}u$ and ${}^{\alpha}\mathcal{F}^{x}_{b_-}u$, respectively, of order α, $0 < \alpha < 1$, for an absolutely continuous function $u : I \to \mathbb{R}$ are defined by

$$ {}^{\alpha}\mathcal{F}^{x}_{a_+}u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1}u(s)ds \qquad (x > a)$$

and

$$ {}^{\alpha}\mathcal{F}^{x}_{b_-}u(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-s)^{\alpha-1}u(s)ds \qquad (x < b),$$

where $\Gamma(\alpha)$ is a Gamma function given by

$$\Gamma(\alpha) = \int_0^\infty s^{\alpha-1}e^{-s}ds.$$

In this study, we apply the rational Chebyshev functions to solve the following fractional Volterra integral equations (FVIE) of first and second kinds as follows:

$$\upsilon(x) = {}^{\alpha}\mathcal{F}^{x}_{a_+}(\kappa(x)u(x)) \tag{1.4}$$

and

$$u(x) = \upsilon(x) + \beta(x)\left[{}^{\alpha}\mathcal{F}^{x}_{a_+}(\kappa(x)u(x))\right], \tag{1.5}$$

where $\alpha \in (0,1)$ and $\kappa, \beta, u : [a,b] \to \mathbb{R}$ are absolutely continuous functions. Most recently, Usta [31] has obtained the numerical method for solving the fractional Volterra integral equations (1.4) and (1.5) which is based on Bernstein approximation method. We now show that more efficient and accurate results are obtained by using the rational Chebyshev functions even for the lower degrees of iterations.

1.1. *Rational Chebyshev functions*

Orthogonal polynomials are very important and serve to approximate functions. Classical orthogonal polynomials were developed by P. L. Chebyshev in the late 19th century. Significance of the orthogonal polynomials can be seen especially in the solution of systems of linear equations and in the least-squares approximations.

With respect to the weighted function

$$w(x) = \frac{1}{\sqrt{1-x^2}},$$

the Chebyshev polynomials are orthogonal in $[-1,1]$ which can be formulated by the recurrence formula:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1} \quad (n \geq 1).$$

Guo et al. [10] demonstrated the rational Chebyshev functions, in short, we will write RC functions, by

$$R_n(x) = T_n\left(\frac{x-1}{x+1}\right). \tag{1.6}$$

Consequently, one can write

$$R_0(x) = 1, \quad R_1(x) = \frac{x-1}{x+1}, \quad R_{n+1}(x) = 2\left(\frac{x-1}{x+1}\right)R_n(x) - R_{n-1}(x) \quad (n \geq 1). \tag{1.7}$$

We remark that RC functions are orthogonal with respect to the weight function

$$w(x) = \frac{1}{\sqrt{x}(x+1)}$$

in the interval $[0,\infty)$. In this case, the orthogonal property is

$$\int_0^\infty R_n(x)R_m(x)w(x)dx = \frac{s_m\pi}{2}\delta_{mn}; \; s_m = \begin{cases} 2 & if \;\; m=0 \\ 1 & if \;\; m \neq 1 \end{cases},$$

where δ_{mn} is the well-known Kronecher function. Parand and Razzaghi [21] used the technique based on RC functions with a view of solving higher-order ordinary differential equations. For more information about RC functions, we refer to the articles [7, 21].

2. Numerical schemes

We apply rational Chebyshev approximation method to fractional Volterra integral equations of first and second kinds in order to establish numerical schemes for numerical related solutions.

2.1. *Numerical scheme for FVIE of first kind*

In order to solve the first kind FVIE (1.4) numerically, we approximate the unknown function as

$$u(x) \simeq u_n(x) = \sum_{k=0}^{n} c_k R_k(x). \tag{2.1}$$

Substituting (2.1) in

$$\upsilon(x) = {}^{\alpha}\mathcal{F}_{a_+}^{x}(\kappa(x)u(x)) \tag{2.2}$$

we obtain

$$\upsilon(x) = \left[{}^{\alpha}\mathcal{F}_{a_+}^{x} \left(\kappa(x) \sum_{k=0}^{n} c_k R_k(x) \right) \right].$$

Let ε be any arbitrary small number. Replacing x by $x_j = j/n + \varepsilon$ and reorganizing the above equation, we achieve that

$$\sum_{k=0}^{n} c_k \left[{}^{\alpha}\mathcal{F}_{a_+}^{x_j} (\kappa(x_j) R_k(x_j)) \right] = \upsilon(x_j),$$

which yields $[\mathbf{R}][\mathbf{C}] = [\mathbf{V}]$, where

$$[R] = \left[{}^{\alpha}\mathcal{F}_{a_+}^{x_j} (\kappa(x_j) R_k) \right]_{(n+1)\times(n+1)}, \quad j,k = 0,1,\cdots,n \tag{2.3}$$

and

$$[V] = \begin{bmatrix} v(x_0) \\ v(x_1) \\ \vdots \\ \vdots \\ v(x_n) \end{bmatrix}_{(n+1)\times 1}, \quad [\mathbf{C}] = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ c_n \end{bmatrix}_{(n+1)\times 1}.$$

2.2. *Numerical scheme for FVIE of second kind*

In order to solve the second kind FVIE (1.5) numerically, we substitute (2.1) in

$$u(x) = \upsilon(x) + \beta(x) \left[{}^{\alpha}\mathcal{F}_{a_+}^{x} (\kappa(x) u(x)) \right]. \tag{2.4}$$

Thus, we have the following equation

$$\sum_{k=0}^{n} c_k R_k(x) = \upsilon(x) + \beta(x) {}^{\alpha}\mathcal{F}_{a_+}^{x} \left(\kappa(x) \sum_{k=0}^{n} c_k R_k(x) \right). \tag{2.5}$$

We have following relation if we rearrange the above equation.

$$\sum_{k=0}^{n} c_k \left\{ R_k(x) - \beta(x) {}^{\alpha}\mathcal{F}_{a_+}^{x} (\kappa(x) R_k(x)) \right\} = v(x). \tag{2.6}$$

Let ε be any arbitrary small number. Replacing x by $x_j = j/n+\varepsilon$, we obtain

$$\sum_{k=0}^{n} c_k \left\{R_k(x_j) - \beta(x_j)\, {}^{\alpha}\mathcal{F}_{a_+}^{x_j}\left(\kappa(x_j)R_k(x_j)\right)\right\} = v(x_j), \tag{2.7}$$

which yields $[\mathbf{R}][\mathbf{C}] = [\mathbf{V}]$, where $[\mathbf{R}]$ is an $(n+1)\times(n+1)$ matrix and given by

$$[\mathbf{R}] = \left[R_k(x_j) - \beta(x_j)\, {}^{\alpha}\mathcal{F}_{a_+}^{x_j}\left(\kappa(x_j)R_k(x_j)\right)\right]_{(n+1)\times(n+1)}, \quad j,k = 0,1,\cdots,n,$$

and $[\mathbf{C}]$ and $[\mathbf{V}]$ are $((n+1)\times 1)$ vectors given in (2.4). The system (2.2) is stable if the matrix $[\mathbf{R}]$ is invertible. Hence we can have the solution matrix $[\mathbf{C}]$ by the help of $[\mathbf{C}] = \left[\mathbf{R}^{-1}\right][\mathbf{V}]$. Finally, the approximate solution of 2nd FVIEs can be computed by substituting the matrix $[\mathbf{C}]$ in equation (1.5).

3. Numerical results and analysis

Since the outcomes of these experiments are computed in different references, they can be compared to those obtained using other computational technique. We provide the following notations for analysing error of the implemented procedure:

$$e_n(x) = |u(x) - u_n(x)| \text{ and } \|e_n\|_\infty = \max\left\{e_n(x_j), j = 0,1,2,\cdots,n\right\},$$

where $u(x)$ and $u_n(x)$ are exact solution and approximate solution, respectively, of the test problems and x_j is the uniform grid on $[0,1]$. Also the root mean square error can calculated by

$$RmsE = \sqrt{\frac{\sum_{j=1}^{n}\left(u(x_j) - u_n(x_j)\right)^2}{n}}.$$

Experiment 1. Let us consider the Abel integral equation (see [12, 35]) as follows:

$$e^x - 1 = \int_0^x \frac{g(t)}{(x-t)^{1/2}}dt \tag{3.1}$$

with the exact solution

$$g(x) = \frac{e^x}{\sqrt{\pi}} erf(\sqrt{x});\ \ erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}ds.$$

Equation (3.1) can be considered as FVIE of first kind as follows:

$$e^x - 1 = \Gamma(1/2)\left[{}^{1/2}\mathcal{F}_{0_+}^{x} u(x)\right], \quad x \in [0,1].$$

Therefore, we have

$$\upsilon(x) = e^x - 1,\ \kappa(x) = \Gamma(1/2),\ a_+ = 0_+ \text{ and } \alpha = \frac{1}{2}.$$

For $n = 3$, we have the following matrices:

$$\mathbf{R} = \begin{bmatrix} 0.2 & -0.197354 & 0.189502 & -0.176691 \\ 1.17189 & -0.749508 & -0.163557 & 0.819708 \\ 1.6452 & -0.672325 & -0.935528 & 1.10479 \\ 2.00998 & -0.486655 & -1.49239 & 0.767512 \end{bmatrix}, \mathbf{V} = \begin{bmatrix} 0.0100502 \\ 0.409639 \\ 0.967309 \\ 1.7456 \end{bmatrix}$$

and therefore the matrix $\mathbf{C}$ can be computed as

$$\mathbf{R}^{-1} \cdot \mathbf{V} = \mathbf{C} = \begin{bmatrix} 2.86872 \\ 4.02452 \\ 1.5845 \\ 0.394493 \end{bmatrix}.$$

Thus, third order approximation will be

$$u_3(x) = \frac{8.87223x^3 - 1.20921x^2 + 2.57654x + 0.0342116}{(x+1)^3}$$

Similarly, for $n = 4$, one can get the following matrices:

$$\mathbf{R} = \begin{bmatrix} 0.2 & -0.197354 & 0.189502 & -0.176691 & 0.159327 \\ 1.0198 & -0.726484 & 0.0440107 & 0.576421 & -0.74932 \\ 1.42829 & -0.733834 & -0.572921 & 1.07927 & -0.372143 \\ 1.74356 & -0.631576 & -1.09533 & 1.0569 & 0.381453 \\ 2.00998 & -0.486655 & -1.49239 & 0.767512 & 0.974483 \end{bmatrix}, \mathbf{V} = \begin{bmatrix} 0.0100502 \\ 0.29693 \\ 0.665291 \\ 1.13828 \\ 1.7456 \end{bmatrix},$$

and the desired matrix can be calculated as:

$$\mathbf{R}^{-1} \cdot \mathbf{V} = \mathbf{C} = \begin{bmatrix} 3.76836 \\ 5.59245 \\ 2.59438 \\ 0.849141 \\ 0.115932 \end{bmatrix}.$$

According to these results, the fourth order approximation becomes

$$u_4(x) = \frac{12.9203x^4 + 0.746768x^3 + 4.78162x^2 + 2.15289x + 0.0370701}{(x+1)^4}.$$

This problem has been considered in [12, 35]. According to the Tables 1, 2 and 3, our new results are slightly better than those obtained by the methods used in [12, 35]. Figure 1 validates the theoretical results given in the problem. Figure 1 proves that new solutions are valid also for larger region.

TABLE 1. Estimated and exact solution of Experiment 1 [12].

x	$n=1$	$n=10$	$n=100$	Exact value
0.1	0.2154319668	0.2152921762	0.2152904646	0.2152905021
0.2	0.3267280013	0.3258941876	0.3258841023	0.3258840762
0.3	0.4300194238	0.4275954299	0.4275658716	0.4275656575

TABLE 2. Approximated and exact values for Experiment 1 [35].

x	$n=10$	$n=20$	$n=30$	Exact value
0.1	0.2152897706	0.2152904948	0.2152905016	0.2152905021
0.2	0.3258833448	0.325884069	0.3258840758	0.3258840763
0.3	0.427564926	0.4275656502	0.427565657	0.4275656576
0.4	0.5293323416	0.5293330658	0.5293330726	0.5293330731
0.5	0.6350311405	0.6350318647	0.6350318715	0.635031872
0.6	0.7470394417	0.7470401659	0.7470401727	0.7470401732
0.7	0.8671868544	0.8671875786	0.8671875854	0.867187586
0.8	0.9970886453	0.9970893695	0.9970893763	0.9970893769
0.9	1.138297847	1.138298572	1.138298578	1.138298579
1.	1.292387361	1.292388086	1.292388092	1.292388093

TABLE 3. Absolute errors of the present method for Experiment 1.

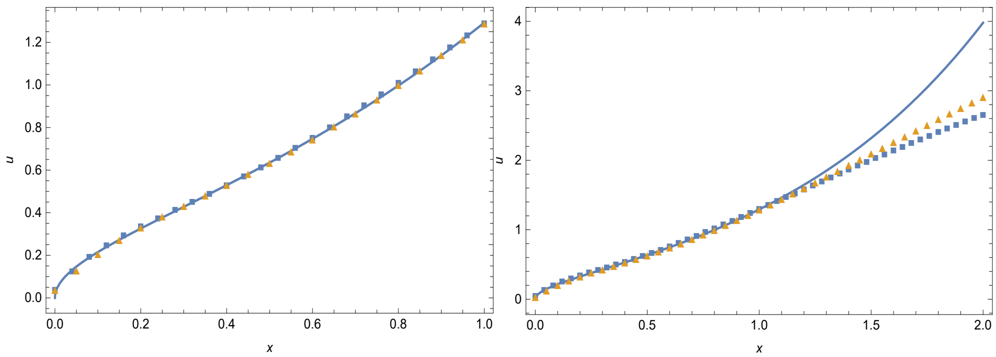
x	$n=10$	$n=20$	$n=30$
0.1	1.1714×10^{-5}	9.0681×10^{-7}	6.0623×10^{-9}
0.2	1.392×10^{-5}	9.2251×10^{-7}	6.2612×10^{-9}
0.3	1.6727×10^{-5}	9.5101×10^{-7}	6.6218×10^{-9}
0.4	2.0269×10^{-5}	9.9437×10^{-7}	7.1813×10^{-9}
0.5	2.4709×10^{-5}	1.0547×10^{-6}	7.9944×10^{-9}
0.6	3.0246×10^{-5}	1.1342×10^{-6}	9.1416×10^{-9}
0.7	3.7123×10^{-5}	1.2353×10^{-6}	1.0744×10^{-8}
0.8	4.5636×10^{-5}	1.3604×10^{-6}	1.2988×10^{-8}
0.9	5.6147×10^{-5}	1.512×10^{-6}	1.618×10^{-8}
1.	6.9096×10^{-5}	1.6929×10^{-6}	2.0842×10^{-8}

Experiment 2. Consider the FVIE of second kind as

$$u(x)=\sqrt{\pi}(1+x)^{-3/2}-0.02\frac{x^3}{1+x}+0.01x^{5/2}\left[{}^{1/2}\mathcal{F}_{0_+}^{x}u(x)\right],\quad x\in[0,1]$$

with the exact solution $u(x)=\sqrt{\pi}(1+x)^{-3/2}$. In this experiment, we have

$$\upsilon(x)=\sqrt{\pi}(1+x)^{-3/2}-0.02\frac{x^3}{1+x},\ \beta(x)=0.01x^{5/2},\ \kappa(x)=1,\ a_+=0_+ \text{ and } \alpha=\frac{1}{2}.$$



(A) Approximations for $n = 3$ (■) and $n = 4$ (▲) with the exact solution (−)

(B) Approximations for $n = 3$ (■) and $n = 4$ (▲) with the exact solution (−) for larger domain

FIGURE 1. Approximations in $x \in [0,1]$ and in $x \in [0,2]$.

For $n = 4$, we have the following matrices:

$$\mathbf{R} = \begin{bmatrix} 1. & -0.980198 & 0.921576 & -0.826457 & 0.698606 \\ 0.999802 & -0.58716 & -0.310162 & 0.951497 & -0.807464 \\ 0.998503 & -0.323734 & -0.788795 & 0.835695 & 0.24668 \\ 0.995047 & -0.134569 & -0.959698 & 0.395946 & 0.852922 \\ 0.988374 & 0.00778994 & -0.991319 & -0.0193642 & 0.994166 \end{bmatrix}, \mathbf{V} = \begin{bmatrix} 1.7462 \\ 1.25292 \\ 0.953476 \\ 0.754124 \\ 0.611735 \end{bmatrix},$$

and therefore the matrix $\mathbf{C}$ can be computed as

$$\mathbf{R}^{-1} \cdot \mathbf{V} = \mathbf{C} = \begin{bmatrix} 0.74613 \\ -0.913927 \\ 0.120294 \\ 0.00870883 \\ 0.000820654 \end{bmatrix}.$$

By proceeding as mentioned in Section 2, the fourth order RC approximation to this problem is obtained as:

$$u_4(x) = \frac{1.77246 + 4.43015x + 3.33129x^2 + 0.530591x^3 - 0.0379734x^4}{(1+x)^4}.$$

Similarly, for $n = 6$, one can get the following matrices:

$$\mathbf{R} = \begin{bmatrix} 1. & -0.980198 & 0.921576 & -0.826457 & 0.698606 & -0.543088 & 0.366061 \\ 0.999938 & -0.699667 & -0.0208101 & 0.728796 & -0.999093 & 0.669365 & 0.0623669 \\ 0.999543 & -0.488542 & -0.522019 & 0.998938 & -0.454573 & -0.554589 & 0.996835 \\ 0.998503 & -0.323734 & -0.788795 & 0.835695 & 0.24668 & -0.996071 & 0.399885 \\ 0.996504 & -0.191414 & -0.923635 & 0.547495 & 0.713269 & -0.823165 & -0.395785 \\ 0.993232 & -0.0828303 & -0.98098 & 0.248905 & 0.940395 & -0.409556 & -0.871125 \\ 0.988374 & 0.00778994 & -0.991319 & -0.0193642 & 0.994166 & 0.0284576 & -0.994792 \end{bmatrix},$$

$$\mathbf{V} = \begin{bmatrix} 1.7462 \\ 1.38856 \\ 1.13781 \\ 0.953476 \\ 0.812708 \\ 0.701714 \\ 0.611735 \end{bmatrix}. \text{ Hence } \mathbf{R^{-1}} \cdot \mathbf{V} = \mathbf{C} = \begin{bmatrix} 0.749586 \\ -0.907714 \\ 0.124798 \\ 0.0112702 \\ 0.00190131 \\ 0.000307628 \\ 0.000032497 \end{bmatrix}.$$

By using this matrix and RC functions, one can find the sixth order approximation as:

$$u_6(x) = \frac{1.77245 + 7.97598x + 13.9589x^2 + 11.6271x^3 + 4.37473x^4 + 0.416709x^5 - 0.0198182x^6}{(1+x)^6}.$$

Even for $n = 6$, we have $\|e_6\|_\infty = \max\{e_6(x_i), i = 0,1,2,3\} = 3.28592 \times 10^{-10}$, where the method in [18] gets

$$\|e_{24}\|_\infty = \max\{e_{24}(x_i), i = 0,1,2,\cdots,24\} = 2.507237 \times 10^{-6}$$

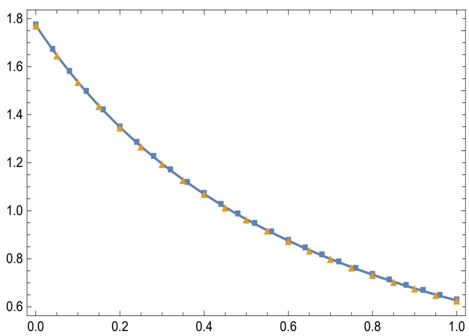
with ten iterations. Table 4 shows the errors obtained by proposed technique. One can deduce that, this method is more accurate than other techniques for solving Experiment 1. Figures are also given to show the efficiency of the technique. Our results are valid also for larger domain as it can be seen from Figure 2. All figures and tables demonstrate that the approximate solutions have high accuracy even for smaller n.

TABLE 4. Comparison of maximum errors $\|e_n\|_\infty$ obtained by present method and the results obtained in [18, 31] for Experiment 1.

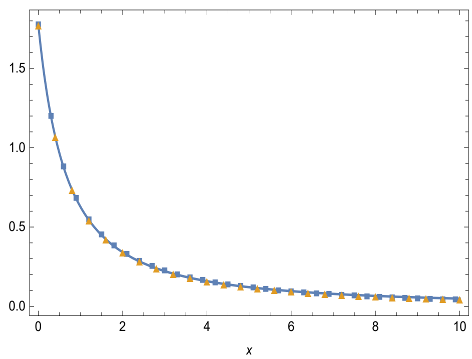
	Present method	Method in [31]	Method in [18] with 10 iterations
$n = 12$	1.81437×10^{-13}	1.046918×10^{-11}	9.966408×10^{-5}
$n = 18$	1.10668×10^{-15}	1.221245×10^{-15}	4.446827×10^{-5}
$n = 24$	5.05891×10^{-18}	3.774758×10^{-14}	2.507237×10^{-6}

TABLE 5. Absolute and relative errors of approximations for Experiment 1.

	Absolute errors		Relative Errors	
x	$n=3$	$n=6$	$n=3$	$n=6$
0.1	5.00546×10^{-3}	7.82261×10^{-6}	3.25805×10^{-3}	5.09173×10^{-6}
0.2	4.54269×10^{-3}	3.37788×10^{-5}	3.36907×10^{-3}	2.50519×10^{-5}
0.3	3.7144×10^{-4}	8.17478×10^{-6}	3.1062×10^{-4}	6.83623×10^{-6}
0.4	3.14638×10^{-3}	2.891×10^{-5}	2.94055×10^{-3}	2.70187×10^{-5}
0.5	4.21141×10^{-3}	9.99757×10^{-9}	4.36505×10^{-3}	1.03623×10^{-8}
0.6	2.72061×10^{-3}	2.50915×10^{-5}	3.1065×10^{-3}	2.86505×10^{-5}
0.7	2.72982×10^{-4}	6.16972×10^{-6}	3.41375×10^{-4}	7.7155×10^{-6}
0.8	2.90836×10^{-3}	2.1963×10^{-5}	3.96261×10^{-3}	2.99244×10^{-5}
0.9	2.74717×10^{-3}	4.34802×10^{-6}	4.0592×10^{-3}	6.42461×10^{-6}
1	3.06959×10^{-3}	2.00571×10^{-5}	4.89835×10^{-3}	3.20065×10^{-5}

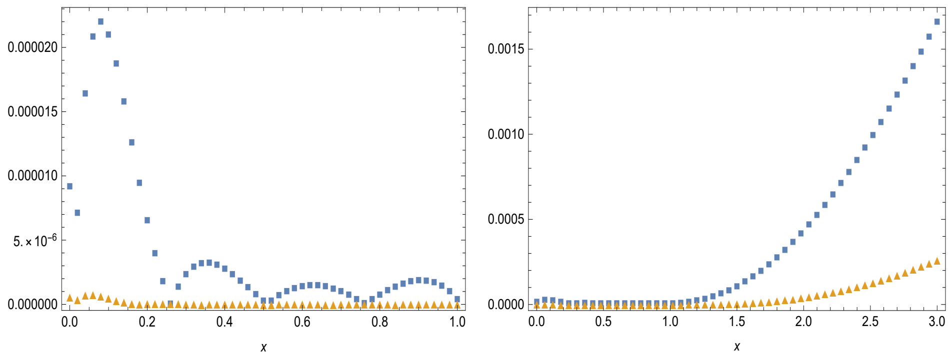


(A) Approximations for $n=4$ (■) and $n=6$ (▲) with the exact solution (−)



(B) Approximations for $n=4$ (■) and $n=6$ (▲) with the exact solution (−) for larger domain

FIGURE 2. Approximations in $x\in[0,1]$ and in $x\in[0,3]$.



(A) Absolute errors for $n = 4$ (■) and $n = 6$ (▲)

(B) Absolute errors for $n = 4$ (■) and $n = 6$ (▲) for larger domain

FIGURE 3. Absolute errors in $x \in [0,1]$ and in $x \in [0,3]$.

Experiment 3. Consider the FVIE of second kind as

$$u(x) = \Gamma\left(\frac{2}{3}\right)x - \frac{1}{40}x^{8/3} + \frac{1}{27}\left[{}^{2/3}\mathcal{J}^{x}_{0_+} xu(x)\right], \quad x \in [0,1]$$

with the exact solution $u(x) = \Gamma\left(\frac{2}{3}\right)x$. In this experiment, we have

$$\upsilon(x) = \Gamma\left(\frac{2}{3}\right)x - \frac{1}{40}x^{8/3},\ \beta(x) = \frac{1}{27},\ \kappa(x) = x,\ a_+ = 0_+ \text{ and } \alpha = \frac{2}{3}.$$

For $n = 4$, we have the following matrices:

$$\mathbf{R} = \begin{bmatrix} 0.999989 & -0.980187 & 0.921566 & -0.826447 & 0.698597 \\ 0.997393 & -0.585534 & -0.309982 & 0.949724 & -0.80536 \\ 0.991986 & -0.320835 & -0.785028 & 0.829785 & 0.248151 \\ 0.98442 & -0.131823 & -0.950695 & 0.388677 & 0.84844 \\ 0.974972 & 0.00900327 & -0.977827 & -0.0241485 & 0.983044 \end{bmatrix}, \mathbf{V} = \begin{bmatrix} 0.0135411 \\ 0.351382 \\ 0.686449 \\ 1.0171 \\ 1.34199 \end{bmatrix}$$

and therefore the matrix $\mathbf{C}$ can be computed as

$$\mathbf{R^{-1}} \cdot \mathbf{V} = \mathbf{C} = \begin{bmatrix} 2.8232 \\ 3.99973 \\ 1.54097 \\ 0.435018 \\ 0.0719605 \end{bmatrix}.$$

By proceeding as mentioned in Section 2, the fourth order RC approximation to this problem is obtained as:

$$u_4(x) = \frac{8.87087x^4 + 5.02323x^3 + 6.56676x^2 + 1.20484x + 0.00138086}{(x+1)^4}$$

Similarly, for $n=6$, one can get the following matrices:

$$\mathbf{R}=\begin{bmatrix} 0.999989 & -0.980187 & 0.921566 & -0.826447 & 0.698597 & -0.54308 & 0.366055 \\ 0.998631 & -0.698665 & -0.0210517 & 0.728197 & -0.997982 & 0.6683 & 0.0628871 \\ 0.995856 & -0.48636 & -0.520985 & 0.995716 & -0.451944 & -0.554561 & 0.994521 \\ 0.991986 & -0.320835 & -0.785028 & 0.829785 & 0.248151 & -0.992085 & 0.395664 \\ 0.987162 & -0.188449 & -0.916403 & 0.540234 & 0.711094 & -0.815699 & -0.397755 \\ 0.981469 & -0.0804581 & -0.970295 & 0.242074 & 0.933545 & -0.401143 & -0.868101 \\ 0.974972 & 0.00900327 & -0.977827 & -0.0241485 & 0.983044 & 0.0350695 & -0.986688 \end{bmatrix}$$

$$\mathbf{V}=\begin{bmatrix} 0.0135411 \\ 0.238982 \\ 0.463469 \\ 0.686449 \\ 0.907464 \\ 1.1261 \\ 1.34199 \end{bmatrix} \text{ and } \mathbf{R}^{-1}\cdot\mathbf{V}=\mathbf{C}=\begin{bmatrix} 3.59335 \\ 5.3876 \\ 2.55307 \\ 1.01763 \\ 0.323744 \\ 0.0746041 \\ 0.00990615 \end{bmatrix}.$$

Using the above information, the sixth order RC approximation to fractional Volterra integral equation (FVIE) can be computed as:

$$u_6(x)=\frac{12.9599x^6+13.4391x^5+32.0296x^4+18.4223x^3+8.48603x^2+1.32651x+0.000241776}{(x+1)^6}.$$

Even for $n=6$, we have $\|e_6\|_\infty=\max\{e_6(x_i), i=0,1,2,3\}=1.40454\times10^{-7}$, where the methods in [31] and [18] have found

$$\|e_{24}\|_\infty=\max\{e_{24}(x_i), i=0,1,2,\cdots,24\}=1.233508\times10^{-7}$$

and

$$\|e_{24}\|_\infty=\max\{e_{24}(x_i), i=0,1,2,\cdots,24\}=6.045722\times10^{-6}$$

with 24 iterations, respectively. Table 6 shows the errors obtained by proposed technique. One can deduce that this method is more accurate than other techniques for solving Experiment 2. Figures are also given to show the efficiency of the technique. Our results are valid also for larger domain as can be seen from Figure 4. All figures prove that the proposed method is very powerful even for smaller n.

TABLE 6. Comparison of maximum errors $\|e_n\|_\infty$ obtained by present method and the results obtained in [18,31] for Experiment 2.

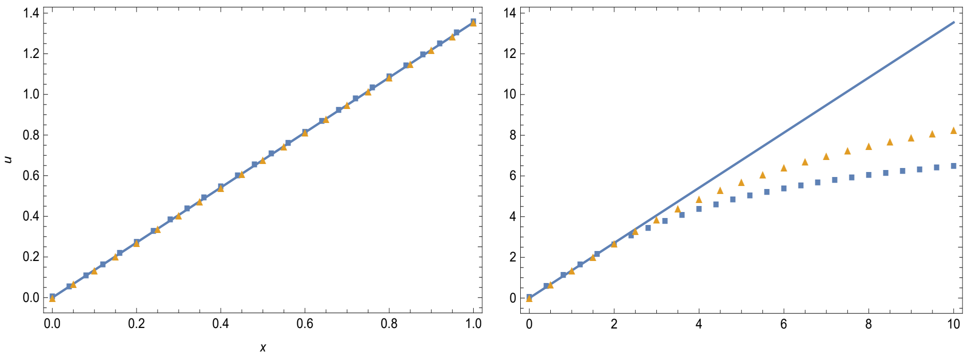
	Present method	Method in [31]	Method in [18] with 10 iterations
$n=12$	9.089011×10^{-11}	1.714861×10^{-8}	2.492919×10^{-5}
$n=18$	7.066315×10^{-11}	9.354121×10^{-8}	1.304662×10^{-5}
$n=24$	7.770991×10^{-13}	1.233508×10^{-7}	6.045722×10^{-6}

Experiment 4. For the last experiment, we assume the FVIE of second kind as follows:

$$u(x)=2\sqrt{\frac{x}{\pi}}-\left[{}^{1/2}\mathcal{J}_{0_+}^{x}u(x)\right],\quad x\in[0,1]$$

TABLE 7. Absolute and relative errors of approximations for Experiment 1.

	Absolute errors		Relative Errors	
x	$n=4$	$n=6$	$n=4$	$n=6$
0.1	3.288×10^{-3}	2.0572×10^{-4}	2.4282×10^{-2}	1.5192×10^{-3}
0.2	1.0523×10^{-3}	2.2107×10^{-5}	3.8857×10^{-3}	8.1628×10^{-5}
0.3	3.7546×10^{-4}	1.3241×10^{-5}	9.2423×10^{-4}	3.2594×10^{-5}
0.4	4.6486×10^{-4}	7.3716×10^{-6}	8.5823×10^{-4}	1.361×10^{-5}
0.5	4.1574×10^{-5}	9.823×10^{-7}	6.1403×10^{-5}	1.4508×10^{-6}
0.6	2.4577×10^{-4}	3.5124×10^{-6}	3.0249×10^{-4}	4.3231×10^{-6}
0.7	1.7586×10^{-4}	1.4662×10^{-6}	1.8552×10^{-4}	1.5468×10^{-6}
0.8	1.3202×10^{-4}	2.971×10^{-6}	1.2187×10^{-4}	2.7426×10^{-6}
0.9	3.5444×10^{-4}	5.5372×10^{-6}	2.9084×10^{-4}	4.5435×10^{-6}
1	7.4223×10^{-5}	2.8036×10^{-6}	5.4813×10^{-5}	2.0704×10^{-6}



(A) Approximations for $N=4$ (■) and $N=6$ (▲) with the exact solution (−)

(B) Approximations for $N=4$ (■) and $N=6$ (▲) with the exact solution (−) for larger domain.

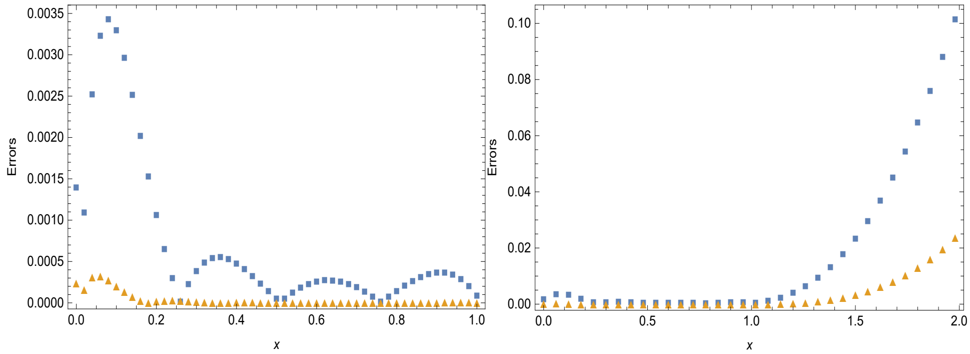
FIGURE 4. Approximations in $x\in[0,1]$ and in $x\in[0,10]$.

with the exact solution

$$u(x)=1-e^{x}\operatorname{erfc}(\sqrt{x});\operatorname{erfc}(z)=1-\frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds.$$

In this experiment, we have

$$\upsilon(x)=2\sqrt{\frac{x}{\pi}},\ \beta(x)=-1,\ \kappa(x)=1,\ a_+=0_+ \text{ and } \alpha=\frac{1}{2}.$$



(A) Absolute errors for $n = 4$ (■) and $n = 6$ (▲)

(B) Absolute errors for $n = 4$ (■) and $n = 6$ (▲)

FIGURE 5. Absolute errors

For $n = 4$, we have the following matrices:

$$\mathbf{R} = \begin{bmatrix} 1.11284 & -1.09154 & 1.02849 & -0.926144 & 0.788496 \\ 1.57536 & -0.997176 & -0.285323 & 1.27682 & -1.23037 \\ 1.80582 & -0.738525 & -1.11263 & 1.44574 & 0.0363306 \\ 1.9837 & -0.492692 & -1.58078 & 0.995238 & 1.06922 \\ 2.13401 & -0.26959 & -1.84194 & 0.418097 & 1.5496 \end{bmatrix}, \mathbf{V} = \begin{bmatrix} 0.112838 \\ 0.575363 \\ 0.805824 \\ 0.983698 \\ 1.13401 \end{bmatrix}$$

and the desired matrix can be calculated as:

$$\mathbf{R}^{-1} \cdot \mathbf{V} = \mathbf{C} = \begin{bmatrix} 0.283994 \\ -0.0961519 \\ -0.355155 \\ -0.118124 \\ -0.0663045 \end{bmatrix}.$$

Thus, we can get the following approximation

$$u_4(x) = \frac{-0.351742x^4 + 5.87456x^3 + 0.614194x^2 + 2.95169x + 0.07681}{(x+1)^4}.$$

By doing the similar calculations for $n = 6$, we have

$$\mathbf{R} = \begin{bmatrix} 1.11284 & -1.09154 & 1.02849 & -0.926144 & 0.788496 & -0.620922 & 0.429959 \\ 1.47428 & -1.076 & 0.109234 & 0.874858 & -1.32292 & 0.999436 & -0.118471 \\ 1.66117 & -0.911698 & -0.61436 & 1.46173 & -0.868575 & -0.508846 & 1.28017 \\ 1.80582 & -0.738525 & -1.11263 & 1.44574 & 0.0363306 & -1.31907 & 0.761709 \\ 1.9282 & -0.572162 & -1.45344 & 1.17315 & 0.789894 & -1.32601 & -0.278787 \\ 2.03623 & -0.415809 & -1.68568 & 0.805609 & 1.28484 & -0.90136 & -1.0784 \\ 2.13401 & -0.26959 & -1.84194 & 0.418097 & 1.5496 & -0.324764 & -1.46415 \end{bmatrix},$$

$$\mathbf{V}=\begin{bmatrix}0.112838\\0.474277\\0.66117\\0.805824\\0.928202\\1.03623\\1.13401\end{bmatrix}.\text{ Hence }\mathbf{C}=\begin{bmatrix}-0.285357\\-1.15588\\-1.19127\\-0.674898\\-0.368269\\-0.119576\\-0.0349047\end{bmatrix}.$$

Substituting the above matrix as coefficients of appropriate RC functions, we get

$$u_6(x)=\frac{-3.83015x^6+21.2857x^5-14.1179x^4+28.9658x^3+0.45665x^2+3.81248x+0.0705518}{(x+1)^6}.$$

Even for $n=6$, we have $\|e_6\|_\infty=\max\{e_6(x_i),i=0,1,2,3\}=1.09464\times10^{-3}$, where the methods in [31] and [18] have found

$$\|e_{12}\|_\infty=\max\{e_{12}(x_i),i=0,1,2,\cdots,24\}=1.543002\times10^{-3}$$

and $\|e_{18}\|_\infty=\max\{e_{18}(x_i),i=0,1,2,\cdots,18\}=6.999936\times10^{-3}$ respectively. Table 8 shows the errors obtained by proposed technique. One can deduce that, this method is more accurate than other techniques for solving Experiment 4. Figures are also given to show the efficiency of the technique. Our results are valid also for larger domain as can be seen from Figure 6. All figures prove that the proposed method gives effective results even for smaller n.

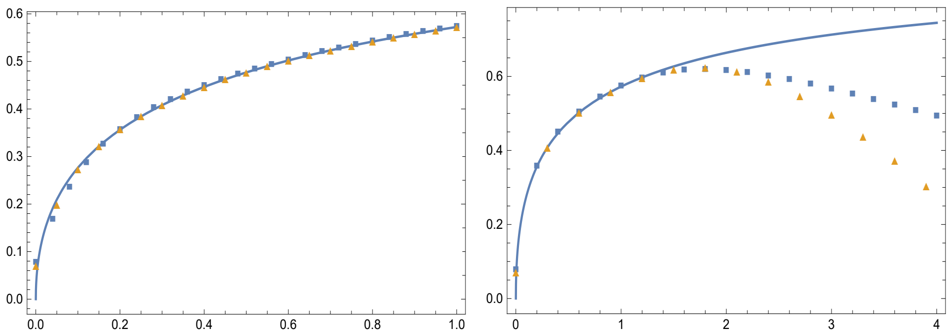
TABLE 8. Comparison of maximum errors $\|e_n\|_\infty$ obtained by present method and the results obtained in [18,31] for Experiment 4.

	Present method	Method in [31]	Method in [18] with 10 iterations
$n=12$	2.121911×10^{-4}	1.543002×10^{-3}	1.007444×10^{-2}
$n=18$	4.099874×10^{-5}	7.725839×10^{-4}	6.999936×10^{-3}
$n=24$	2.106335×10^{-7}	4.577828×10^{-4}	2.381096×10^{-3}

Figures and tables are given to show the efficiency of the technique.

TABLE 9. Absolute and relative errors of approximations for Experiment 4.

	Absolute errors		Relative Errors	
x	For $n=4$	For $n=6$	For $n=4$	For $n=6$
0.1	0.0141718	0.00314341	0.0512688	0.0113718
0.2	0.000237314	0.0011302	0.000666215	0.00317283
0.3	0.00284334	0.000579788	0.00696929	0.00142111
0.4	0.00204502	0.000284363	0.00458121	0.000637024
0.5	0.000892315	0.000272262	0.0018713	0.000570966
0.6	0.000382037	0.000244394	0.000761067	0.000486864
0.7	0.000459034	0.000181115	0.000877195	0.000346103
0.8	0.000745512	0.000152942	0.00137611	0.000282308
0.9	0.00084146	0.000168345	0.00150805	0.000301704
1.	0.00042765	0.000133346	0.000747096	0.000232953



(A) Approximations for $n=4$ (■) and $n=6$ (▲) with the exact solution (−)

(B) Approximations for $n=4$ (■) and $n=6$ (▲) with the exact solution (−) for larger domain

FIGURE 6. Approximations in $x \in [0,1]$ and in $x \in [0,4]$.

4. CONCLUSION

In our present discussion, we have presented the numerical method to solve the fractional Volterra integral equations ($x > a, 0 < \alpha < 1$) of first and second kinds as

$$\upsilon(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\kappa(x)u(s)}{(x-s)^{1-\alpha}} ds \quad \text{and} \quad u(x) = \upsilon(x) + \frac{\beta(x)}{\Gamma(\alpha)} \int_a^x \frac{\kappa(x)u(s)}{(x-s)^{1-\alpha}} ds.$$

Our technique is based on approximating unknown function with rational Chebyshev functions. We have provided numerical experiments in order to demonstrate the significance of our numerical schemes. Moreover, we have compared our illustrative experiments with the existing approximate methods in the literature and seen that our method has less error and hence the results based on our new technique has high accuracy and efficiency. For further studies, one can also investigate (i) some substantially general forms of the Riemann-Liouville fractional integrals [27, 28]; (ii) collocation methods based upon various orthogonal polynomials [11, 14, 26, 29]; (iii) Volterra and related integro-differential equations [4]. Our next aim is to combine rational Chebyshev functions with approximation theory [13, 19, 20] to solve fractional integral equation.

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