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ON IDEAL CONVERGENCE OF SEQUENCES VIA CATALAN MATRIX OPERATOR

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Abstract. Catalan sequence initially introduced by Euler but named after Eugene Catalan has found abundant applications in the fields of probability theory, number theory, analysis, combinatorics etc. Recently İlkhan formulated the matrix associated with Catalan numbers and studied its domain in $c_0(\tilde{C})$ and $c(\tilde{C})$. In this paper, by employing Catalan matrix defined by Catalan sequence and convergence via ideal, we establish Catalan sequence spaces $c_0^I(\tilde{C}), c^I(\tilde{C})$ and $\ell_\infty^I(\tilde{C})$, study the properties of these corresponding spaces and establish inclusion relations.

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1. Introduction

A linear subspace of the set of all real or complex sequences denoted by ω is termed as a sequence space. The standard sequence spaces namely null, convergent and bounded sequence spaces are denoted by c_0, c and ℓ_∞ respectively. Establishment of sequence spaces to study their algebraic and topological properties have a substantial relevance in summability theory. Every so often when the sequences fail to approach a finite limit, matrix transformation comes in handy by acting as a linear operator between two sequence spaces and assigining a limit after the transformation. If γ and Υ are two sequence spaces and $M = (m_{nk})$ is an inifinite matrix such that $(m_{nk}) \in \mathbb{R}$ then M defines a matrix transformation from γ into Υ denoted by $M\colon \gamma \to \Upsilon$ if for every sequence $\mathsf{u} = (\mathsf{u}_k) \in \gamma$, the sequence $M\mathsf{u} = \{M_n(\mathsf{u})\}$, the M transform of u is in Υ, where

$$M_n(\mathsf{u}) = \sum_{k=0}^{\infty} m_{nk}\mathsf{u}_k \quad \text{for all } n \in \mathbb{N}. \tag{1.1}$$

The classification of all matrices M such that $M\colon \gamma \to \Upsilon$ is denoted by (γ, Υ). Therefore $M \in (\gamma, \Upsilon)$ iff the series in equation (1.1) on the right hand side converges $\forall\, n \in \mathbb{N}$, every $\mathsf{u} \in \gamma$ and we obtain $M\mathsf{u} = \{M_n(\mathsf{u})\} \in \Upsilon$.

Over the course of time the domains of various triangular matrices have been investigated as a means of assigning a finite number to the sequences' limit. A handful of them have been explored in these literatures, see [1, 4, 5, 11–13]. In addition to the mentioned references, more can be found on applications of summability theory and matrix transformations in [2, 3, 6, 9, 17].

Classifed among such pre–existing notions of matrix domain, a novel matrix namely Catalan matrix was defined by İlkhan in [10] whose terms are comprised of Catalan numbers. It is an integer sequence with its first few terms as 1, 1, 2, 5, 14, 42, 132, 429 .... The sequence named after Catalan was originally found in Euler's work, which he encountered while attempting to find the number of ways a convex polygon can be divided into triangles with no diagonals intersecting each other. It was Catalan who found an analogy between this sequence and the parantheses problem, see [8]. The answer to innumerable problems from every field of mathematics, one of them being able to find all possible arrangements which allow $2n$ number of people sitting across a round table to shake hands with each other without crossing hands; was this interesting sequence of numbers, see [15]. The standard formula for finding nth Catalan number was given by Catalan as

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

and the recurrence with the initial condition given as

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad C_0 = 1 \text{ is satisfied by Catalan numbers.}$$

Recently the Catalan matrix associated with the Catalan numbers $\tilde{\mathcal{C}} = (\tilde{c}_{nk})$ was defined by Merve İlkhan in [10] as,

$$(\tilde{c}_{nk}) = \begin{cases} \frac{C_k C_{n-k}}{C_{n+1}} & \text{if } 0 \le k \le n, \\ 0 & \text{if } k > n. \end{cases} \tag{1.2}$$

Equivalently

$$\tilde{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 & \dots \\ \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} & 0 & 0 & 0 & \dots \\ \frac{14}{42} & \frac{5}{42} & \frac{4}{42} & \frac{5}{42} & \frac{14}{42} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and its corresponding inverse $\tilde{\mathcal{C}}^{-1} = (\tilde{c}_{nk}^{-1})$ is given by,

$$\tilde{c}_{nk}^{-1} = \begin{cases} (-1)^{n-k} \frac{C_{k+1}}{C_0^{n-k+1} C_n} \mathcal{H}_{n-k} & \text{if } 0 \le k \le n, \\ 0 & \text{if } k > n. \end{cases}$$

where $\mathcal{H}_0 = 1$ and $\mathcal{H}_n$ is given by

$$\mathcal{H}_n = \begin{vmatrix} C_1 & C_0 & 0 & 0 & 0 & \dots & 0 \\ C_2 & C_1 & C_0 & 0 & 0 & \dots & 0 \\ C_3 & C_2 & C_1 & C_0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ C_n & C_{n-1} & C_{n-2} & C_{n-3} & C_{n-4} & \dots & C_1 \end{vmatrix}$$

$\forall\, n \in \mathbb{N}\backslash 0$.

İlkhan in [10] defined new Banach sequence spaces using the Catalan matrix, namely $c_0(\tilde{\mathcal{C}})$ and $c(\tilde{\mathcal{C}})$ as following,

$$c_0(\tilde{\mathcal{C}}) = \{u = (u_k) \in \omega : \lim_{n\to\infty} \frac{1}{C_{n+1}} \sum_{k=0}^{n} C_k C_{n-k} u_k = 0\},$$

$$c(\tilde{\mathcal{C}}) = \{u = (u_k) \in \omega : \lim_{n\to\infty} \frac{1}{C_{n+1}} \sum_{k=0}^{n} C_k C_{n-k} u_k \text{ exists}\}.$$

and showed some promising results of these spaces such as linearity, *BK* spaces with respect to norms defined on them, linear isomorphism and moved on to define compact operators too.

İlkhan and Kara later in [14] came up with two novel spaces defined manoeuvering the Catalan matrix as,

$$\ell_p(\tilde{\mathcal{C}}) = \{u = (u_k) \in \omega : \sum_{n=0}^{\infty} \left| \frac{1}{C_{n+1}} \sum_{k=0}^{n} C_k C_{n-k} u_k \right|^p < \infty\},$$

$$\ell_\infty(\tilde{\mathcal{C}}) = \{u = (u_k) \in \omega : \sup_n \left| \frac{1}{C_{n+1}} \sum_{k=0}^{n} C_k C_{n-k} u_k \right| < \infty\}.$$

In the entire paper the $\tilde{\mathcal{C}}$–transform of a sequence $u = (u_k)$ will be denoted and defined as,

$$v = \tilde{\mathcal{C}}_n(u) = \frac{1}{C_{n+1}} \sum_{k=0}^{n} C_k C_{n-k} u_k \quad \text{for all } n \in \mathbb{N}.$$

Inquisitive readers about Catalan sequences and their properties can look further in the literature [15, 18, 22].

2. Preliminaries

The problem of assigning a novel version of limit to sequences whose limits did not exist in the usual sense was addressed by Steinhaus [19]. He came up with the idea of defining density of the set $\mathcal{L}_\varepsilon(\mathsf{c}) = \{\mathsf{i} \in \mathbb{N}; |\mathsf{c}_\mathsf{i} - \mathsf{c}| \geq \varepsilon\}$ to be 0 for the sequence (c_i) to be statistically convergent. Later Kostyrko et al. [16] put forward the notion of ideal convergence of a sequence as a sequel to statistical convergence. He stated that for a sequence (c_i) to be ideal convergent to c, the set $\{i \in \mathbb{N} : |\mathsf{c}_\mathsf{i} - \mathsf{c}| \geq \varepsilon\}$ belongs to I, where I is a family of subsets of $\mathbb{N}$ such that $\phi \in I, A$ and B in $I \implies A \cup B \in I$ and $A \subseteq B, B \in I \implies A \in I$. A sequence $\mathsf{c} = (\mathsf{c}_\mathsf{i}) \in \omega$ is I–bounded if there exists $L > 0$ such that $\{i \in \mathbb{N} : |\mathsf{c}_\mathsf{i}| > L\} \in I$. The concept of a Cauchy sequence was generalised to I-Cauchy sequence in [20] stating if for a sequence $\mathsf{c} = (\mathsf{c}_\mathsf{i}) \in \omega$ and $\forall\, \varepsilon > 0$, $\exists\, K = K(\varepsilon) \in \mathbb{N}$ such that the set $\{k \in \mathbb{N} : |\mathsf{c}_\mathsf{i} - \mathsf{c}_K| \geq \varepsilon\}$ belongs to I, then the sequence is I–Cauchy.

On parallel grounds of being motivated to initiate different other forms of convergence method; summability method using infinite matrix as an operator was defined to study the sequences' convergence. The idea was to use the matrix with certain conditions in a way that it sums convergent series to its Cauchy sums and eventually would appoint a limit to the diverging ones too. In [21], an infinite matrix $M = (m_{nk})_{n,k\in\mathbb{N}}$ was deemed to be regular iff all the terms of the sequence after operating matrix is bounded, $\lim_{n\to\infty} m_{nk} = 0 \;\forall\, k \in \mathbb{N}$ and $\lim_{n\to\infty} \sum_k m_{nk} = 1$.

Following are certain terms and definitions that will be required further in the literature.

Definition 1 ([1, Definition 8.3.1])**.** Two sequences $\mathsf{c} = (\mathsf{c}_\mathsf{i})$ and $\mathsf{d} = (\mathsf{d}_\mathsf{i})$ are said to be equal for almost all i relative to I, if the set $\{\mathsf{i} \in \mathbb{N} : \mathsf{c}_\mathsf{i} \neq \mathsf{d}_\mathsf{i}\} \in I$.

Definition 2 ([20, Definition 2.1])**.** A sequence space S is said to be solid (or normal), if $(\beta_\mathsf{i}\mathsf{c}_\mathsf{i}) \in S$ whenever $(\mathsf{c}_\mathsf{i}) \in S$ and (β_i) be a sequence of scalar in ω such that $|\beta_\mathsf{i}| < 1$ and $\mathsf{i} \in \mathbb{N}$.

Definition 3 ([20, Definition 2.3])**.** A sequence space is said to be monotone, if for all infinite $K \subseteq \mathbb{N}$ and $(\mathsf{c_i})$ belonging to the space, the sequence $(\alpha_\mathsf{i}\mathsf{c_i})$, where

$$\alpha_\mathsf{i} = \begin{cases} 1, & \mathsf{i} \in K, \\ 0 & \text{otherwise}; \end{cases}$$

also lies in the same sequence space.

Lemma 1 ([20, Lemma 2.6])**.** *Let $K \in \mathcal{F}(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$.*

Definition 4 ([1, Definition 4.1.1])**.** Matrix domain of an infinite matrix $M = (m_{nk})$ is the sequence space defined as $\mathcal{X}_M = \{\mathsf{c} = (\mathsf{c_i}) \in \omega : M\mathsf{c} \in \mathcal{X}\}$.

3. Catalan statistical convergence

Definition 5. A sequence $\mathsf{u} = (\mathsf{u}_k)$ is defined to be Catalan statistically convergent (or $\tilde{\mathcal{C}}$- statistically convergent), if for every $\varepsilon > 0$, $\exists$ a number u_0, such that the density of the following set,

$$\mathcal{K}_\varepsilon(\tilde{\mathcal{C}}) = \left\{k \in \mathbb{N} : |\tilde{\mathcal{C}}(\mathsf{u}_k) - \mathsf{u}_0| \geq \varepsilon\right\},$$

is 0. The space of all $\tilde{\mathcal{C}}$-statistically convergent sequences will be designated by $\mathcal{S}(\tilde{\mathcal{C}})$.

Example 1. Classic example of $\mathcal{S}(\tilde{\mathcal{C}})$ is the space of all Catalan convergent sequences $c(\tilde{\mathcal{C}})$.

Example 2.

$$\tilde{\mathcal{C}}(\mathsf{u}_k) = \begin{cases} k & \text{if } k = 3^n \text{ or } 5^n \\ \frac{1}{k} & \text{otherwise.} \end{cases}$$

The above mentioned sequence $(\mathsf{u}_k) \in \mathcal{S}(\tilde{\mathcal{C}})$ which statiscally converges to 0.

Definition 6. A sequence $\mathsf{u} = (\mathsf{u}_k)$ is defined to be $\tilde{\mathcal{C}}$- statistically Cauchy, if $\exists$ $\mathcal{N} = \mathcal{N}(\varepsilon)$, such that the density of the set,

$$\left\{k \in \mathbb{N} : |\tilde{\mathcal{C}}\mathsf{u}_k - \tilde{\mathcal{C}}\mathsf{u}_\mathcal{N}| \geq \varepsilon\right\},$$

is $0 \;\forall\; \varepsilon > 0$.

Theorem 1. *If u is a $\tilde{\mathcal{C}}$–statistically convergent sequence, then u is $\tilde{\mathcal{C}}$–statistically Cauchy.*

Proof. Let $\varepsilon > 0$. Assume that $\mathsf{u}_k \xrightarrow{\mathcal{S}(\tilde{\mathcal{C}})} \mathsf{u}_0$. This implies $|\tilde{\mathcal{C}}u_k - \mathsf{u}_0| < \frac{\varepsilon}{2}$ for almost all k. Choose $\mathcal{N}$ such that $|\tilde{\mathcal{C}}\mathsf{u}_\mathcal{N} - \mathsf{u}_0| < \frac{\varepsilon}{2}$ then

$$|\tilde{\mathcal{C}}\mathsf{u}_k - \tilde{\mathcal{C}}\mathsf{u}_\mathcal{N}| < |\tilde{\mathcal{C}}\mathsf{u}_k - \mathsf{u}_0| + |\tilde{\mathcal{C}}\mathsf{u}_\mathcal{N} - \mathsf{u}_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for almost all } k.$$

Thus u is a $\tilde{\mathcal{C}}$–statistically Cauchy sequence. □

Theorem 2. *If u is a sequence for which there is a $\tilde{C}$ –statistically convergent sequence v such that $\tilde{C}\mathsf{u}_k = \tilde{C}\mathsf{v}_k$ for almost all k, then u is a $\tilde{C}$–statistically convergent sequence.*

Proof. Assume that $\tilde{C}\mathsf{u}_k = \tilde{C}\mathsf{v}_k$ for almost all k and $\mathsf{v}_k \xrightarrow{S(\tilde{C})} l$. Then for $\varepsilon > 0$ and for each k,

$$\{k \leq n : |\tilde{C}u_k - l| \geq \varepsilon\} \subseteq \{k \leq n : \tilde{C}u_k \neq \tilde{C}v_k\} \cup \{k \leq n : |\tilde{C}v_k - l| \leq \varepsilon\}.$$

Since $\tilde{C}u_k = \tilde{C}v_k$, the set on the right hand side of the equation contains a finite number of integers, say $h = h(\varepsilon)$. Thus for $\tilde{C}u_k = \tilde{C}v_k$ for almost all k,

$$\lim_n \frac{|\{k \leq n : |\tilde{C}u_k - l| \geq \varepsilon\}|}{n} \leq \lim_k |n \leq k : \tilde{C}u_k \neq \tilde{C}v_k| + \lim_n \frac{h}{n} = 0$$

Hence $u_k \xrightarrow{S(\tilde{C})} l$. □

Definition 7. A sequence (u_k) is affirmed to be $\tilde{C}$-stastically bounded provided the existence of some number $\mathsf{u}_0 \geq 0$ guarantees the density of the set $\{k : |\mathsf{u}_k| > \mathsf{u}_0\}$ is null.

Corollary 1. *$\tilde{C}$–statistically convergence $\implies$ $\tilde{C}$–statistically boundedness.*

Corollary 2. *The set of $\tilde{C}$– statistically convergent sequences is dense in ω.*

4. I–CONVERGENT CATALAN SEQUENCE SPACE

In the current section we put forward some novel sequence spaces as an assembly of sequences whose $\tilde{C}$ transforms lie in the spaces c_0^I, c^I and ℓ_∞. Advancing, we study some compelling results of the spaces and derive a handful of inclusion relations too. Throughout the paper we take I to be an admissible ideal of subset of $\mathbb{N}$. We delineate

$$c_0^I(\tilde{C}) = \left\{\mathsf{u} = (\mathsf{u}_k) \in \omega : \left\{n \in \mathbb{N} : |\tilde{C}_n(\mathsf{u}_k)| \geq \varepsilon\right\} \in I\right\},$$
$$c^I(\tilde{C}) = \left\{\mathsf{u} = (\mathsf{u}_k) \in \omega : \left\{n \in \mathbb{N} : |\tilde{C}_n(\mathsf{u}_k) - l| \geq \varepsilon\right\} \in I\right\},$$
$$\ell_\infty^I(\tilde{C}) = \left\{\mathsf{u} = (\mathsf{u}_k) \in \omega : \exists\, M > 0 \text{ s.t. } \left\{n \in \mathbb{N} : |\tilde{C}_n(u_k)| > M\right\} \in I\right\}.$$

We set down the following spaces too,

$$m_0^I(\tilde{C}) := c_0^I(\tilde{C}) \cap \ell_\infty(\tilde{C}),$$

and

$$m^I(\tilde{C}) := c^I(\tilde{C}) \cap \ell_\infty(\tilde{C}).$$

A general Catalan sequence space denoted by $\mathcal{X}(\tilde{C})$ where $\mathcal{X}$ is any sequence space can be given by

$$\mathcal{X}(\tilde{C}) = \{\mathsf{u} = (\mathsf{u}_k) \in \omega : (\tilde{C}\mathsf{u}_k) \in \mathcal{X}\}.$$

Definition 8. If $\mathsf{u} = (\mathsf{u}_k)$ is a sequence in ω then u is said to be Catalan I–Cauchy if, for every $\varepsilon > 0, \exists$ a number $N = N(\varepsilon) \in \mathbb{N}$ such that $\left\{k \in \mathbb{N} : |\tilde{C}_n(\mathsf{u}) - \tilde{C}_N(\mathsf{u})| \geq \varepsilon\right\} \in I$.

Corollary 3. *Given I a finite admissible ideal of subset of $\mathbb{N}$, the space $c^{I_f}(\tilde{C})$ coincides with $c(\tilde{C})$ i.e. for $I = I_f$, $c^{I_f}(\tilde{C}) = c(\tilde{C})$.*

Corollary 4. *For a non-trivial ideal of subset of $\mathbb{N}$ such that $I_d = \{K \subseteq \mathbb{N} : d(K) = 0\}$, $c^{I_d}(\tilde{C}) = \mathcal{S}(\tilde{C})$.*

Theorem 3. *The sequence spaces $c_0^I(\tilde{C}), c^I(\tilde{C})$ and $\ell_\infty^I(\tilde{C})$ are linear spaces over $\mathbb{R}$.*

Proof. Let $\mathsf{u} = (\mathsf{u}_k)$ and $\mathsf{v} = (\mathsf{v}_k)$ be random elements in $c^I(\tilde{C})$ and λ and μ be scalars. Then for a given $\varepsilon > 0$ there exist $l_1, l_2 \in \mathbb{R}$ such that

$$\left\{n \in \mathbb{N} : \left|\tilde{C}_n(\mathsf{u}) - l_1\right| \geq \frac{\varepsilon}{2\lambda}\right\} \in I,$$

$$\left\{n \in \mathbb{N} : \left|\tilde{C}_n(\mathsf{v}) - l_2\right| \geq \frac{\varepsilon}{2\mu}\right\} \in I,$$

Let

$$\mathcal{D}_1 = \left\{n \in \mathbb{N} : \left|\tilde{C}_n(\mathsf{u}) - l_1\right| < \frac{\varepsilon}{2\lambda}\right\} \in \mathcal{F}(I),$$

$$\mathcal{D}_2 = \left\{n \in \mathbb{N} : \left|\tilde{C}_n(\mathsf{v}) - l_2\right| < \frac{\varepsilon}{2\mu}\right\} \in \mathcal{F}(I).$$

be such that $\mathcal{D}_1^c, \mathcal{D}_2^c \in I$. Then

$$\begin{aligned}\mathcal{D}_3 &= \left\{n \in \mathbb{N} : \left|(\lambda\tilde{C}_n(\mathsf{u}) + \mu\tilde{C}_n(\mathsf{v})) - (\lambda l_1 + \mu l_2)\right| < \varepsilon\right\} \\ &\supseteq \left[\left\{n \in \mathbb{N} : \left|\tilde{C}_n(\mathsf{u}) - l_1\right| < \frac{\varepsilon}{2|\lambda|}\right\}\right] \left[\left\{n \in \mathbb{N} : \left|\tilde{C}_n(\mathsf{v}) - l_2\right| < \frac{\varepsilon}{2|\mu|}\right\}\right].\end{aligned}$$

Clearly $\mathcal{D}_3 \in \mathcal{F}(I)$ and hence $\mathcal{D}_3^c = \mathcal{D}_1^c \cup \mathcal{D}_2^c \in I$. Following which we can conclude that $(\lambda\mathsf{u}_k + \mu\mathsf{v}_k) \in c^I(\tilde{C})$. □

Theorem 4. *The spaces $\mathcal{X}(\tilde{C})$ are normed spaces with norm*

$$||u||_{\mathcal{X}(\tilde{C})} = \sup_n |\tilde{C}_n(u)| \text{ where } C_n(u) = \sum_{k=0}^{\infty} c_{nk} u_k \text{ and } \mathcal{X} \in \{m^I, m_0^I\}. \tag{4.1}$$

Proof. Since the above defined spaces are linear spaces and it can be easily verified that equation (4.1) defines a norm. We therefore omit the proof since it is very straight forward. □

Theorem 5. *Consider the normed space $\mathcal{X}(\tilde{C})$ and let I be a non–trivial ideal. If for a sequence (u_k) we have I–$\lim_{n\to\infty} \tilde{C}_n(u_k) = l_1$ and I–$\lim_{n\to\infty} \tilde{C}_n(u_k) = l_2$, then $l_1 = l_2$.*

Proof. Omitted. □

Example 3. Following is an example that depicts the convergence of a sequence via Catalan matrix which does not converge in the usual sense. Consider the sequence

$$\mathsf{u}_k = \begin{cases} 1 & \text{if } k = 5^n, \forall\, n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

The terms of the sequence can be written as $\mathsf{u} = (\mathsf{u}_k) = \{0,1,0,1,0,0,0,1,\dots\}$. Operating Catalan matrix we get $C_n(\mathsf{u}) = \{0,0,0,0,0.33,0,0,0,\dots\}$. Thus the sequence eventually converges to 0. Thus $(\mathsf{u}_k) \in c^I(\tilde{\mathcal{C}}) \backslash c$, where c denotes the space of all convergent sequences.

Theorem 6. *The absolute property does not hold on the spaces $c^I(\tilde{\mathcal{C}}), c_0^I(\tilde{\mathcal{C}})$ and $\ell_\infty^I(\tilde{\mathcal{C}})$, that is $||u||_{X(\tilde{\mathcal{C}})} \neq ||\,|u|\,||_{X(\tilde{\mathcal{C}})}$. Following is an example of a sequence in $c^I(\tilde{\mathcal{C}})$ in support of our claim. Consider a sequence $(\mathsf{u}_k) = (-1)^k$, then,*

$$||\,|u|\,||_{X(\tilde{\mathcal{C}})} = \sup_n \left| \sum_{k=0}^{\infty} c_{nk} |u_k| \right|$$

$$\sum_{k=0}^{\infty} c_{nk}|u_k| = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 & \dots \\ \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} & 0 & 0 & 0 & \dots \\ \frac{14}{42} & \frac{5}{42} & \frac{4}{42} & \frac{5}{42} & \frac{14}{42} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}$$

Thus $||\,|u|\,||_{X(\tilde{\mathcal{C}})} = \sup_n\{1,1,1,\dots\} = 1$ *when* $n \to \infty$ *and*

$$||\, u\, ||_{X(\tilde{\mathcal{C}})} = \sup_n \left| \sum_{k=0}^{\infty} c_{nk} u_k \right|$$

$$\sum_{k=0}^{\infty} c_{nk} u_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 & \dots \\ \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} & 0 & 0 & 0 & \dots \\ \frac{14}{42} & \frac{5}{42} & \frac{4}{42} & \frac{5}{42} & \frac{14}{42} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0.6 \\ 0 \\ 0.52 \\ \vdots \end{bmatrix}$$

Thus $\| \ u \ \|_{X(\tilde{C})} = \sup_n\{1, 0, 0.6, 0, 0.52, 0, 0.49, \dots\} = 0$ *when* $n \to \infty$. *Clearly* $\|u\|_{X(\tilde{C})} \neq \| \ |u| \ \|_{X(\tilde{C})}$.

Theorem 7. *The inclusion relations* $c_0^I(\tilde{C}) \subset c^I(\tilde{C}) \subset \ell_\infty^I(\tilde{C})$ *are strict.*

Proof. The forward relation is quite evident. The following examples are proposed to back our claim.

Example 4. Consider the sequence $(u_k) = \left(1 - \frac{1}{k}\right)$. Now, for a non-trivial finite ideal $I = I_f$, $\tilde{C}_n(u) = \{0, 0.25, 0.366, 0.452, \dots\}$. One can clearly see that the sequence eventually approaches 1. Thus $(u_k) \in c^I(\tilde{C}) \backslash c_0^I(\tilde{C})$.

To support our claim of strictness relation between $c^I(\tilde{C})$ and $\ell_\infty^I(\tilde{C})$, we put forward the following example.

Example 5. Consider the sequence

$$u_k = \begin{cases} 1 - \frac{1}{k} & \text{if } k \text{ is odd,} \\ 10\left(1 + \frac{1}{k}\right) & \text{if } k \text{ is even.} \end{cases}$$

Evidently for a non-trivial finite ideal $I = I_f$, since (u_k) is bounded therefore $(u_k) \in \ell_\infty^I(\tilde{C})$. Now, $\tilde{C}_n(u) = \{0, 7.5, 0.567, 7.56, 0.657, 7.6, \dots\}$. Therefore the odd and even subsequences converge to different limits namely 1 and 10 respectively. Thus $(u_k) \in \ell_\infty^I(\tilde{C}) \backslash c^I(\tilde{C})$.

□

Theorem 8. *The inclusion relation* $c(\tilde{C}) \subset c^I(\tilde{C})$ *is strict where* I *is a non trivial ideal.*

Proof. It is well established that for $I = I_f$, $c^I = c$ and in general $c \subset c^I$, therefore for given sequence spaces $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{X} \subseteq \mathcal{Y}$ implies $\mathcal{X}(\tilde{C}) \subseteq \mathcal{Y}(\tilde{C})$. Thus it is easily observable that $c(\tilde{C}) \subset c^I(\tilde{C})$.

In support of our claim of strictness we construct the following example.

Example 6. Define the sequence $\mathsf{u} = (\mathsf{u}_k)$ as following

$$\tilde{\mathcal{C}}_k(\mathsf{u}) = \begin{cases} k^{\frac{1}{3}} & \text{if } k \text{ is a cube root,} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the sequence $\mathsf{u} \in S(\tilde{\mathcal{C}}) \backslash c(\tilde{\mathcal{C}})$.

Example 7. Let I be a non–trivial ideal in $\mathbb{N}$. For

$$\tilde{\mathcal{C}}_k(\mathsf{u}) = (0,0,0,0,0,0,2,0,0,0,2,2,2,0,0,2,\dots),$$

we have

$$\mathcal{A}_\varepsilon = \{k \in \mathbb{N} : \tilde{\mathcal{C}}_k(\mathsf{u}) \neq 0\} = \{7,11,12,13,16,18,19,21,23,25,27,\dots\}$$

From the looks of it we can clearly conclude that the set $\mathcal{A}$ eventually contains all the odd numbers and that $\tilde{\mathcal{C}}_k(\mathsf{u}) \in c^I(\tilde{\mathcal{C}})$. But clearly it's asymptotic density is $\frac{1}{2}$ and therefore $\tilde{\mathcal{C}}_k(\mathsf{u}) \notin S(\tilde{\mathcal{C}})$.

□

Theorem 9. *The spaces $m_0^I(\tilde{\mathcal{C}})$ and $m^I(\tilde{\mathcal{C}})$ are Banach spaces normed by* (4.1)*.*

Proof. Consider a Cauchy sequence $(\mathsf{u}_k^{(l)})$ in $m^I(\tilde{\mathcal{C}}) \subset l_\infty(\tilde{\mathcal{C}})$. Then $(\mathsf{u}_k^{(l)})$ is convergent in $\ell_\infty(\tilde{\mathcal{C}})$, i.e. $\lim_{l\to\infty} \tilde{\mathcal{C}}_k^{(l)}(\mathsf{u}) = \tilde{\mathcal{C}}_k(\mathsf{u})$. Let $I\text{–}\lim \tilde{\mathcal{C}}_k^{(l)}(\mathsf{u}) = \mathsf{c}_l$, $l \in \mathbb{N}$. To conclude the result, we verify that,

(i) (c_l) is convergent to a number c.
(ii) $I\text{–}\lim \tilde{\mathcal{C}}_k(\mathsf{u}) = \mathsf{c}$.

(i) $(\mathsf{u}_k^{(l)})$ being a Cauchy sequence, for any $\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ such that

$$\left|\tilde{\mathcal{C}}_k^{(l)}(\mathsf{u}) - \tilde{\mathcal{C}}_k^{(m)}(\mathsf{u})\right| < \frac{\varepsilon}{3}, \text{ for all } l,m \geq n_0. \tag{4.2}$$

Now let $\mathcal{B}_l$ and $\mathcal{B}_m$ be two sets in I. Thus,

$$\mathcal{B}_l = \left\{k \in \mathbb{N} : |\tilde{\mathcal{C}}_k^{(l)}(\mathsf{u}) - \mathsf{c}_l| \geq \frac{\varepsilon}{3}\right\} \tag{4.3}$$

and

$$B_m = \left\{k \in \mathbb{N} : |\tilde{\mathcal{C}}_k^{(m)}(\mathsf{u}) - \mathsf{c}_m| \geq \frac{\varepsilon}{3}\right\}. \tag{4.4}$$

Now suppose that $l,m \geq n_0$ and $k \notin \mathcal{B}_l \cap \mathcal{B}_m$. Then by using (4.2), (4.3) and (4.4) we have

$$|\mathsf{c}_l - \mathsf{c}_m| \leq |\tilde{\mathcal{C}}_k^{(l)}(\mathsf{u}) - \mathsf{c}_l| + |\tilde{\mathcal{C}}_k^{(m)}(\mathsf{u}) - \mathsf{c}_m| + |\tilde{\mathcal{C}}_k^{(l)}(\mathsf{u}) - \tilde{\mathcal{C}}_k^{(m)}(\mathsf{u})| < \varepsilon.$$

Thus (c_l) is a Cauchy sequence and is thus convergent to c, hence $\lim_{l\to\infty} \mathsf{c}_l = \mathsf{c}$.

(ii) Given $\delta > 0$, we can chose j_0 as

$$|\mathsf{c}_l - \mathsf{c}| < \frac{\delta}{3} \quad \text{for each } l > j_0. \tag{4.5}$$

Since $(\mathsf{u}_k^{(l)}) \to \mathsf{u}_k$ as $l \to \infty$. Thus

$$|\tilde{\mathcal{C}}_k^{(l)}(\mathsf{u}) - \tilde{\mathcal{C}}_k(\mathsf{u})| < \frac{\delta}{3} \quad \text{for each } l > j_0. \tag{4.6}$$

Since $(\tilde{\mathcal{C}}_k^{(m)})$ is I–convergent to c_l, there exists $\mathcal{A} \in I$ such that for each $k \notin \mathcal{A}$, we have

$$|\tilde{\mathcal{C}}_k^{(m)}(\mathsf{u}) - \mathsf{c}_m| < \frac{\delta}{3}. \tag{4.7}$$

Let $m > j_0$, then with aid of (4.5), (4.6) and (4.7) for all $k \notin \mathcal{A}$, we have

$$|\tilde{\mathcal{C}}_k(\mathsf{u}) - \mathsf{c}| \leq |\tilde{\mathcal{C}}_k(\mathsf{u}) - \tilde{\mathcal{C}}_k^{(m)}(\mathsf{u})| + |\tilde{\mathcal{C}}_k^{(m)}(\mathsf{u}) - \mathsf{c}_m| + |\mathsf{c}_m - \mathsf{c}| < \delta.$$

This implies that (u_k) is Catalan I–convergent to ℓ. Thus $m^I(\tilde{\mathcal{C}})$ is a Banach space.

□

Corollary 5. *Every I-Cauchy sequence in $m_0^I(\tilde{\mathcal{C}}), m^I(\tilde{\mathcal{C}})$ is I-convergent.*

Theorem 10. *Let $\mathcal{A}$ be a closed subset of a Banach space $\mathcal{X}$. Then $\mathcal{A}(\tilde{\mathcal{C}})$ is also closed in $\mathcal{X}(\tilde{\mathcal{C}})$.*

Proof. The proof is trivial. □

Corollary 6. *$\mathcal{X}(\tilde{\mathcal{C}})$ is a separable space under the restriction of $\mathcal{X}$ being a separable space too.*

Theorem 11. *The spaces $c_0^I(\tilde{\mathcal{C}}), c^I(\tilde{\mathcal{C}})$ and $\ell_\infty^I(\tilde{\mathcal{C}})$ are BK spaces according to their norms defined by* (4.1).

Proof. Since the given matrix in (1.2) is a triangular matrix and definition (4) holds; therefore by using Wilansky Theorem 4.3.12 [21] we can conclude our claim. □

Theorem 12. *The sequence spaces $m^I(\tilde{\mathcal{C}})$ and $m_0^I(\tilde{\mathcal{C}})$ are Fréchet space defined by the metric*

$$d(u,v) = \sup_n |\tilde{\mathcal{C}}_n(u) - \tilde{\mathcal{C}}_n(v)|. \tag{4.8}$$

Proof. Following Theorem 9 we conclude that the space is a complete metric space with the induced metric via norm. The metric (4.8) is a translation invariant metric since for any u,v,w $\in m^I(\tilde{C})$

$$\begin{aligned} d(\mathsf{u}+\mathsf{w},\mathsf{v}+\mathsf{w}) &= \sup_n |\tilde{C}_n(\mathsf{u}+\mathsf{w}) - \tilde{C}_n(\mathsf{v}+\mathsf{w})| \\ &= \sup_n |\tilde{C}_n(\mathsf{u}) + \tilde{C}_n(\mathsf{w}) - \tilde{C}_n(\mathsf{v}) - \tilde{C}_n(\mathsf{w})| \\ &= d(\mathsf{u},\mathsf{v}) \end{aligned}$$

Evidently the spaces have coordinates continuity since $\mathsf{u}^{(j)} \to \mathsf{u}$ via the induced metric then $\mathsf{u}_k^{(j)} \to \mathsf{u}_k$ when $j \to \infty\ \forall\ k$. □

Since the spaces's inclusions $m^I(\tilde{C}) \subset \ell_\infty(\tilde{C})$ and $m_0^I(\tilde{C}) \subset \ell_\infty(\tilde{C})$ are strict therefore in view of Theorem 9, we conclude the following result.

Theorem 13. *The spaces $m^I(\tilde{C})$ and $m_0^I(\tilde{C})$ are nowhere dense subsets of $\ell_\infty(\tilde{C})$.*

Proof. Since $m^I(\tilde{C}) = c^I(\tilde{C}) \cap \ell_\infty(\tilde{C})$ is a closed subset of $\ell_\infty(\tilde{C})$ therefore $\overline{m^I(\tilde{C})} = m^I(\tilde{C})$. Consider an arbitrary point $(\mathsf{u}_k) \in m^I(\tilde{C})$. Thus for a given radius $\mathsf{r} > 0$ there exists (v_k) in the neighbourhood of $\mathcal{B}(\mathsf{u},\mathsf{r})$ such that $(\mathsf{v}_k) \notin c^I(\tilde{C})$ and $(\mathsf{v}_k) \notin \ell_\infty(\tilde{C})$. Hence $\mathcal{B}(\mathsf{u},\mathsf{r}) \nsubseteq m^I(\tilde{C})$. Since (u_k) was arbitrary and is not an interior point,therefore $m^I(\tilde{C})^o = \phi$. □

Lemma 2. *If all the subsequences of a sequence (u_k) are matrix summable via regular matrix $\tilde{C} = (\tilde{c}_{nk})$ then the sequence (u_k) converges.*

Lemma 3 ([7, Theorem 1])**.** *Consider the finite ideal $I = I_f$ and let (u_k) be an I_f – bounded sequence in $\mathcal{X} = m_0^{I_f}(\tilde{C}), m^{I_f}(\tilde{C})$ such that $\mathcal{X}$ satisfies Banach-Saks property with respect to the Catalan matrix summability $(\tilde{c}_{nk})$, then there exists a subsequence (u_{k_l}) of u_k whose all the subsequences will be $(\tilde{c}_{nk})$ summable.*

Theorem 14. *A sequence $\mathsf{u} = (\mathsf{u}_k) \in \omega$ is Catalan I–convergent if and only if for every $\varepsilon > 0$, there exists $\mathcal{N} = \mathcal{N}(\varepsilon) \in \mathcal{N}$, such that*

$$\left\{n \in \mathbb{N} : \left|\tilde{C}_n(\mathsf{u}) - \tilde{C}_N(\mathsf{u})\right| < \varepsilon\right\} \in \mathcal{F}(I). \tag{4.9}$$

Proof. Consider the sequence $\mathsf{u} = (\mathsf{u}_k) \in \omega$ which is Catalan I–convergent to some number $m \in \mathbb{R}$, then for any $\varepsilon > 0$, we have

$$\mathcal{B}_\varepsilon = \left\{n \in \mathbb{N} : \left|\tilde{C}_n(\mathsf{u}) - m\right| < \frac{\varepsilon}{2}\right\} \in \mathcal{F}(I).$$

For a fixed integer $\mathcal{N} = \mathcal{N}(\varepsilon) \in \mathcal{B}_\varepsilon$, the following inequality,

$$\left|\tilde{C}_n(\mathsf{u}) - \tilde{C}_{\mathcal{N}}(\mathsf{u})\right| \leq \left|\tilde{C}_n(\mathsf{u}) - m\right| + \left|m - \tilde{C}_{\mathcal{N}}(\mathsf{u})\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

holds for all $n \in B_\varepsilon$. Hence (4.9) holds. Conversely, let (4.9) be true for all $\varepsilon > 0$. Then

$$\mathcal{B}'_\varepsilon = \left\{n \in \mathbb{N} : \tilde{C}_n(\mathsf{u}) \in \mathcal{J}_\varepsilon\right\} \in \mathcal{F}(I) \quad \text{for all } \varepsilon > 0,$$

where $\mathcal{J}_\varepsilon = \left[\tilde{\mathcal{C}}_n(\mathsf{u}) - \varepsilon, \tilde{\mathcal{C}}_n(\mathsf{u}) + \varepsilon\right]$. By taking $\varepsilon > 0$, we have $\mathcal{B}'_\varepsilon \in \mathcal{F}(I)$ and $\mathcal{B}'_{\frac{\varepsilon}{2}} \in \mathcal{F}(I)$. Hence $\mathcal{B}'_\varepsilon \cap \mathcal{B}'_{\frac{\varepsilon}{2}} \in \mathcal{F}(I)$. This implies that

$$\mathcal{J} = \mathcal{J}_\varepsilon \cap \mathcal{J}_{\frac{\varepsilon}{2}} \neq \varnothing,$$

that is,

$$\left\{n \in \mathbb{N} : \tilde{\mathcal{C}}_n(\mathsf{u}) \in \mathcal{J}\right\} \in \mathcal{F}(I)$$

and thus

$$\operatorname{diam}(\mathcal{J}) \leq \frac{1}{2} \operatorname{diam}(\mathcal{J}_\varepsilon),$$

where the diam depicts length of the interval. A sequence of compact intervals are thus obtained by applying induction as follows,

$$\mathcal{J}_\varepsilon = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

such that

$$\operatorname{diam}(I_n) \leq \frac{1}{2} \operatorname{diam}(I_{n-1}) \quad \text{for } n = (2, 3, \dots)$$

and

$$\left\{n \in \mathbb{N} : \tilde{\mathcal{C}}_n(\mathsf{u}) \in I_n\right\} \in \mathcal{F}(I).$$

Then there exists a number $m \in \bigcap_{n \in \mathbb{N}} I_n$ such that $m = I\text{–}\lim \tilde{\mathcal{C}}_n(\mathsf{u})$ observing that $\mathsf{u} = (\mathsf{u}_k) \in \omega$ is Catalan I–convergent. □

Theorem 15. *Let I be an admissible ideal. The following statements are equivalent.*

(i) $(u_k) \in c^I(\tilde{\mathcal{C}})$*;*
(ii) $\exists\, (v_k) \in c^I(\tilde{\mathcal{C}})$ *so that* $u_k = v_k$ *for a.a.k.r.I;*
(iii) $\exists\, (v_k) \in c^I(\tilde{\mathcal{C}})$ *and* $w_k \in c_0^I(\tilde{\mathcal{C}})$ *such that* $u_k = v_k + w_k\ \forall\, n \in \mathbb{N}$ *and*

$$\left\{n \in \mathbb{N} : \left|\tilde{\mathcal{C}}_n(u) - l\right| \geq \varepsilon\right\} \in I;$$

(iv) *there exists a subset* $\mathcal{A} = \{n_k : k \in \mathbb{N}, n_k < n_{k+1}\}$ *of* $\mathbb{N}$ *such that* $\mathcal{A} \in \mathcal{F}(I)$ *and* $\lim_{k\to\infty} \left|\tilde{\mathcal{C}}_{n_k}(u) - l\right| = 0$*.*

Proof. Omitted. □

Theorem 16. *The spaces $c_0^I(\tilde{\mathcal{C}})$ and $m_0^I(\tilde{\mathcal{C}})$ are solid and monotone.*

Proof. Consider the sequence $\mathsf{u} = (\mathsf{u}_k) \in c_0^I(\tilde{\mathcal{C}}) \implies \left\{n \in \mathbb{N} : \left|\tilde{\mathcal{C}}_n(\mathsf{u})\right| \geq \varepsilon\right\} \in I$ for $\varepsilon > 0$. Let $\xi = (\xi_n)$ be a sequence of scalars such that $-1 \leq \xi \leq 1$ for all $n \in \mathbb{N}$. Then,

$$\left|\tilde{\mathcal{C}}_n(\xi \mathsf{u}_k)\right| = \left|\xi \tilde{\mathcal{C}}_n(\mathsf{u}_k)\right| \leq |\xi| \left|\tilde{\mathcal{C}}_n(\mathsf{u}_k)\right| \leq \left|\tilde{\mathcal{C}}_n(\mathsf{u}_k)\right| \quad \text{for all } n \in \mathbb{N}.$$

Implementing inclusion relation

$$\left\{n \in \mathbb{N} : \left|\tilde{\mathcal{C}}_n(\xi \mathsf{u}_k)\right| \geq \varepsilon\right\} \subseteq \left\{n \in \mathbb{N} : \left|\tilde{\mathcal{C}}_n(\mathsf{u}_k)\right| \geq \varepsilon\right\} \in I,$$

implies

$$\left\{n \in \mathbb{N} : \left|\tilde{\mathcal{C}}_n(\xi \mathsf{u}_k)\right| \geq \varepsilon\right\} \in I.$$

Therefore, $(\xi u_k) \in c_0^I(\tilde{\mathcal{C}}_n)$. Hence the space $c_0^I(\tilde{\mathcal{C}}_n)$ is solid and therefore monotone. □

Corollary 7. *For a non-maximal ideal $I \neq I_f$, the spaces $c_0^I(\tilde{\mathcal{C}}), m_0^I(\tilde{\mathcal{C}})$ are neither solid nor monotone.*

Theorem 17. *The spaces $c_0^I(\tilde{\mathcal{C}})$ and $c^I(\tilde{\mathcal{C}})$ are sequence algebra.*

Proof. Let $\varepsilon > 0$ be given. Then

$$\mathcal{A}\left(\frac{\varepsilon}{2}\right) = \left\{n \in \mathbb{N} : |\tilde{\mathcal{C}}_n(\mathsf{u}) - l_1)| \geq \frac{\varepsilon}{2}\right\} \in I,$$
$$\mathcal{B}\left(\frac{\varepsilon}{2}\right) = \left\{n \in \mathbb{N} : |\tilde{\mathcal{C}}_n(\mathsf{v}) - l_2)| \geq \frac{\varepsilon}{2}\right\} \in I.$$

All that is left to be shown is for the following set $\mathcal{C}$ to be contained in union of $\mathcal{A}$ and $\mathcal{B}$.

$$\mathcal{C}(\varepsilon) = \left\{n \in \mathbb{N} : |\tilde{\mathcal{C}}_n(\mathsf{u} \cdot \mathsf{v}) - l_1 l_2| \geq \varepsilon\right\}.$$

Now

$$\{n \in \mathbb{N} : |\tilde{\mathcal{C}}_n(\mathsf{u} \cdot \mathsf{v}) - l_1 l_2| \geq \varepsilon\} \subseteq \left\{n \in \mathbb{N} : |\tilde{\mathcal{C}}_n(\mathsf{u}) - l_1)| \geq \frac{\varepsilon}{2}\right\}$$
$$\cup \left\{n \in \mathbb{N} : |\tilde{\mathcal{C}}_n(\mathsf{v}) - l_2)| \geq \frac{\varepsilon}{2}\right\}.$$

Since

$$\mathcal{C}(\varepsilon) \subset \left\{n \in \mathbb{N} : |\tilde{\mathcal{C}}_n(\mathsf{u}) - l_1| \geq \frac{\varepsilon}{2}\right\} \cup \left\{n \in \mathbb{N} : |\tilde{\mathcal{C}}_n(\mathsf{v}) - l_2| \geq \frac{\varepsilon}{2}\right\}.$$

Thus $\mathcal{C}(\varepsilon) \subseteq \mathcal{A}\left(\frac{\varepsilon}{2}\right) \cup \mathcal{B}\left(\frac{\varepsilon}{2}\right) \Rightarrow (\mathsf{u} \cdot \mathsf{v}) \in c^I(\tilde{\mathcal{C}})$. □

Theorem 18. *The function $\mathtt{h} \colon m^I(\tilde{\mathcal{C}}) \to \mathbb{R}$ defined by $\mathtt{h}(u) = I\text{–}\lim \tilde{\mathcal{C}}_n(u)$, where $m^I(\tilde{\mathcal{C}}) = \ell_\infty(\tilde{\mathcal{C}}) \cap c^I(\tilde{\mathcal{C}})$, is a Lipschitz function and hence uniformly continuous.*

Proof. The function is well defined since for any two sequences $\mathsf{u},\mathsf{v} \in m^I(\tilde{\mathcal{C}})$, such that

$$\mathsf{u} = \mathsf{v} \Rightarrow I\text{–}\lim \tilde{\mathcal{C}}_n(\mathsf{u}) = I\text{–}\lim \tilde{\mathcal{C}}_n(\mathsf{v})$$
$$\Rightarrow \mathtt{h}(\mathsf{u}) = \mathtt{h}(\mathsf{v}).$$

Let $(\mathsf{u}_k), (\mathsf{v}_k) \in m^I(\tilde{\mathcal{C}})$, $\mathsf{u} \neq \mathsf{v}$. Then

$$\mathcal{A}_{\mathsf{u}} = \left\{n \in \mathbb{N} : \left|\tilde{\mathcal{C}}_n(\mathsf{u}) - \mathtt{h}(\mathsf{u})\right| \geq |\mathsf{u} - \mathsf{v}|_*\right\} \in I \tag{4.10}$$

and

$$\mathcal{A}_{\mathsf{v}} = \left\{n \in \mathbb{N} : \left|\tilde{\mathcal{C}}_n(\mathsf{v}) - \mathtt{h}(\mathsf{v})\right| \geq |\mathsf{u} - \mathsf{v}|_*\right\} \in I, \tag{4.11}$$

where $|\mathsf{u}-\mathsf{v}|_* = \sup_n \left|\tilde{\mathcal{C}}_n(\mathsf{u}) - \tilde{\mathcal{C}}_n(\mathsf{v})\right|$. Thus equations (4.10), (4.11) imply

$$\mathcal{B}_{\mathsf{u}} = \left\{n \in \mathbb{N} : \left|\tilde{\mathcal{C}}_n(\mathsf{u}) - \mathrm{h}(\mathsf{u})\right| < |\mathsf{u}\text{-}\mathsf{v}|_*\right\} \in \mathcal{F}(I)$$

and

$$\mathcal{B}_{\mathsf{v}} = \left\{n \in \mathbb{N} : \left|\tilde{\mathcal{C}}_n(\mathsf{v}) - \mathrm{h}(\mathsf{v})\right| < |\mathsf{u}\text{-}\mathsf{v}|_*\right\} \in \mathcal{F}(I).$$

Hence $\mathcal{B} = \mathcal{B}_{\mathsf{u}} \cap \mathcal{B}_{\mathsf{v}} \in \mathcal{F}(I)$, so that $\mathcal{B} \neq \phi$. Therefore, selecting $n \in \mathcal{B}$, we have

$$\begin{aligned}|\mathrm{h}(\mathsf{u}) - \mathrm{h}(\mathsf{v})| &\leq \left|\mathrm{h}(\mathsf{u}) - \tilde{\mathcal{C}}_n(\mathsf{u})\right| + \left|\tilde{\mathcal{C}}_n(\mathsf{u}) - \tilde{\mathcal{C}}_n(\mathsf{v})\right| + \left|\tilde{\mathcal{C}}_n(\mathsf{v}) - \mathrm{h}(\mathsf{v})\right| \\ &\leq 3\,|\mathsf{u}-\mathsf{v}|_*.\end{aligned}$$

Thus, h is a Lipschitz function and hence uniformly continuous. □

Theorem 19. *If $u = (u_k), v = (v_k) \in m^I(\tilde{\mathcal{C}})$ such that $\tilde{\mathcal{C}}_n(u \cdot v) = \tilde{\mathcal{C}}_n(u) \cdot \tilde{\mathcal{C}}_n(v)$, then $(u \cdot v) \in m^I(\tilde{\mathcal{C}})$ and $h(u \cdot v) = h(u) \cdot h(v)$, where $h\colon m^I(\tilde{\mathcal{C}}) \longrightarrow \mathbb{R}$ is defined by $h(u) = I\text{–}\lim \tilde{\mathcal{C}}_n(u)$.*

Proof. Since $\mathsf{u}, \mathsf{v} \in m^I(\tilde{\mathcal{C}})$, therefore taking $\varepsilon = \sup_n \left|\tilde{\mathcal{C}}_n(\mathsf{u}) - \tilde{\mathcal{C}}_n(\mathsf{v})\right| > 0$,

$$\mathcal{B}_{\mathsf{u}} = \left\{n \in \mathbb{N} : \left|\tilde{\mathcal{C}}_n(\mathsf{u}) - \mathrm{h}(\mathsf{u})\right| < \varepsilon\right\} \in \mathcal{F}(I) \tag{4.12}$$

and

$$\mathcal{B}_{\mathsf{v}} = \left\{n \in \mathbb{N} : \left|\tilde{\mathcal{C}}_n(\mathsf{v}) - \mathrm{h}(\mathsf{v})\right| < \varepsilon\right\} \in \mathcal{F}(I), \tag{4.13}$$

Now, we have

$$\begin{aligned}\left|\tilde{\mathcal{C}}_n(\mathsf{u}\cdot\mathsf{v}) - \mathrm{h}(\mathsf{u})\mathrm{h}(\mathsf{v})\right| &= \left|\tilde{\mathcal{C}}_n(\mathsf{u})\tilde{\mathcal{C}}_n(\mathsf{v}) - \tilde{\mathcal{C}}_n(\mathsf{u})\mathrm{h}(\mathsf{v}) + \tilde{\mathcal{C}}_n(\mathsf{u})\mathrm{h}(\mathsf{v}) - \mathrm{h}(\mathsf{u})\mathrm{h}(\mathsf{v})\right| \\ &\leq \left|\tilde{\mathcal{C}}_n(\mathsf{u})\right|\left|\tilde{\mathcal{C}}_n(\mathsf{v}) - \mathrm{h}(\mathsf{v})\right| + |\mathrm{h}(\mathsf{v})|\left|\tilde{\mathcal{C}}_n(\mathsf{u}) - \mathrm{h}(\mathsf{u})\right|.\end{aligned} \tag{4.14}$$

As $m^I(\tilde{\mathcal{C}}) \subseteq \ell_\infty(\tilde{\mathcal{C}})$, there exists $\mathcal{M} \in \mathbb{R}$ such that $|\tilde{\mathcal{C}}_n(\mathsf{u})| < \mathcal{M}$. Therefore, from the equations (4.12), (4.13) and (4.14) we have

$$\begin{aligned}\left|\tilde{\mathcal{C}}_n(\mathsf{u}\cdot\mathsf{v}) - \mathrm{h}(\mathsf{u})\mathrm{h}(\mathsf{v})\right| &= \left|\tilde{\mathcal{C}}_n(\mathsf{u}) \cdot \tilde{\mathcal{C}}_n(\mathsf{v}) - \mathrm{h}(\mathsf{u})\mathrm{h}(\mathsf{v})\right| \\ &\leq \mathcal{M}\varepsilon + |\mathrm{h}(\mathsf{v})|\varepsilon = \varepsilon_1, \text{ (say)}\end{aligned}$$

for all $n \in \mathcal{B}_{\mathsf{u}} \cap \mathcal{B}_{\mathsf{v}} \in \mathcal{F}(I)$. Hence $(\mathsf{u}\cdot\mathsf{v}) \in m^I(\tilde{\mathcal{C}})$ and $\mathrm{h}(\mathsf{u}\cdot\mathsf{v}) = \mathrm{h}(\mathsf{u}) \cdot \mathrm{h}(\mathsf{v})$.

Example 8. Consider the sequences,

$$\mathsf{u}_k = \begin{cases} 2 & \text{if } k = 2^k, \\ 0 & \text{if } k \neq 2^k; \end{cases} \qquad \mathsf{v}_k = \begin{cases} 0 & \text{if } k = 2^k, \\ 1 & \text{if } k \neq 2^k. \end{cases}$$

Clearly the sequences do not converge in the usual sense but the sequence $\mathsf{w} = (\mathsf{u}\cdot\mathsf{v}) = \mathsf{u}_k.\mathsf{v}_k = \{0,0,0,0,\dots\}$ converges to 0 which further implies $\tilde{\mathcal{C}}_n(\mathsf{u}\cdot\mathsf{u}) \to 0$. Now

$$\tilde{\mathcal{C}}_n(\mathsf{u}) = \{0,1,0.4,1,0.48,0.36,0.31,\dots\} \text{ and } \tilde{\mathcal{C}}_n(\mathsf{u}) = \{1,0,1,0,1,1,1,0,1,\dots\}$$

which converge to 0 and 1 respectively therefore $\tilde{\mathcal{C}}_n(\mathsf{u}).\tilde{\mathcal{C}}_n(\mathsf{u}) = 0 = \tilde{\mathcal{C}}_n(\mathsf{u}\cdot\mathsf{v}) \implies \mathrm{h}(\mathsf{u}\cdot\mathsf{v}) = 0.$ □

Theorem 20. *For a non–trivial admissible ideal I and a sequence $u = (u_k) \in \omega$, if there exists a sequence $v = (v_k) \in c^I(\tilde{C})$ such that $\tilde{C}_n(u) = \tilde{C}_n(v)$, then $u \in c^I(\tilde{C})$.*

Proof. Omitted. □

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