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CONVERGENCE, OPTIMAL POINTS AND APPLICATIONS

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Abstract. In this paper, we focus on the existence of the best proximity points in binormed linear spaces. As a consequence, we obtain some fixed point results. We also provide some illustrations to support our claims. As applications, we obtain the existence of a solution to split feasible and variational inequality problems.

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1. INTRODUCTION AND PRELIMINARIES

In the fixed point approach, for two given non-empty subsets $\mathcal{E}_1$ and $\mathcal{E}_2$ of a metric space (χ, d), a non-self mapping $\Im : \mathcal{E}_1 \to \mathcal{E}_2$ the idea of a fixed point is not appropriate when the intersection of $\mathcal{E}_1$ and $\mathcal{E}_2$ is empty. If a mapping $\Im$ has a solution and the intersection of $\mathcal{E}_1$ and $\mathcal{E}_2$ is non-empty, then $\Im$ has a fixed point. Banach contraction theorem (BCT) plays an important role in nonlinear analysis. Due to its simplicity and applicability, it helps solve many kinds of nonlinear problems. This fact motivated researchers to try to extend and generalize BCT so that its area of applications should be as vast as possible. In 1968, Maia established a very interesting and beautiful generalization of BCT using assumptions on two comparable metrics defined on the set χ. Consider the case when the fixed point equation $\Im \grave{u} = \grave{u}$ has no solution in this case $d(\mathcal{E}_1, \mathcal{E}_2) > 0$. In this affair, it is interesting to find an approximate solution $\grave{u}$ such that the error $d(\grave{u}, \Im \grave{u})$ is minimum in some sense. For a nonself mapping $\Im : \mathcal{E}_1 \to \mathcal{E}_2$ a point $\grave{u}$, known as a best proximity point if satisfies the following condition

$$d(\grave{u}, \Im \grave{u}) = d(\mathcal{E}_1, \mathcal{E}_2) = \inf\{d(\grave{u}, \grave{v}) : \grave{u} \in \mathcal{E}_1, \grave{v} \in \mathcal{E}_2\}.$$

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In 1969, Fan [8] gave the classical best approximation theorem in the context of a Hausdorff locally convex topological vector space χ. After that, many authors studied the best proximity point problems in metric space or normed space (see [16–19] and references cited therein). In 2006, Eldred and Veeramani [7] proved the existence of a best proximity point for cyclic contraction mappings. After that, many authors extend the Eldred and Veeramani [7] result. In 2009, Suzuki et al. [20] proved the best proximity result using the property UC. In 2010, Kosuru et al. [12] investigated the best proximity pair result for cyclic maps using weak proximal normal structure. In 2013, Gabeleh and Abkar [2] proved the best proximity pair result for cyclic maps using proximal quasi normal structure. In 2019, Petruşel and Petruşel [15] obtained some coupled fixed point and best proximity point results satisfying orbital contraction condition. Recently, Hafshejani [9] study the existence and uniqueness of best proximity points for the generalized cyclic quasi-contraction mappings using a geometrical concept of ultrametric property.

In this paper, we investigate the existence of best proximity points in the context of binormed linear spaces. We also provide some illustrations to back up our work. As an application of our obtained results, we find the solution of split feasible and variational inequality problems.

To prove the main result of this paper, we need the following definition and lemmas in the sequel:

Definition 1. A normed vector space χ is said to be a uniformly convex Banach space [6], if for every $0 < \varepsilon \leq 2$ there is some $\delta > 0$ such that for any two vectors with $||\grave{u}|| = 1$ and $||\grave{v}|| = 1$, the condition

$$||\grave{u} - \grave{v}|| \geq \varepsilon,$$

implies

$$\left|\left|\frac{\grave{u} + \grave{v}}{2}\right|\right| \leq 1 - \delta.$$

Lemma 1 ([7]). *Let $\mathcal{E}_2$ be a nonempty closed subset and $\mathcal{E}_1$ be a nonempty convex and closed subset of a uniformly convex Banach space (UCBS) χ. Let $\{\grave{u}_n\}$ and $\{\grave{z}_n\}$ be sequences in $\mathcal{E}_1$ and $\{\grave{v}_n\}$ be a sequence in $\mathcal{E}_2$ satisfying*

(i) $||\grave{z}_n - \grave{v}_n|| \to d(\mathcal{E}_1, \mathcal{E}_2)$,

(ii) *for every $\varepsilon > 0$ there exists N_0 such that for all $m > n \geq N_0$,*

$$||\grave{u}_n - \grave{v}_n|| \leq d(\mathcal{E}_1, \mathcal{E}_2) + \varepsilon.$$

Then, for every $\varepsilon > 0$ there exists N_1 such that for all $m > n \geq N_1$, $||\grave{u}_n - \grave{z}_n|| \leq \varepsilon$.

Lemma 2 ([7]). *Let $\mathcal{E}_2$ be a nonempty closed subset and $\mathcal{E}_1$ be a nonempty convex and closed subset of a uniformly convex Banach space χ. Let $\{\grave{u}_n\}$ and $\{\grave{z}_n\}$ be sequences in $\mathcal{E}_1$ and $\{\grave{v}_n\}$ be a sequence in $\mathcal{E}_2$ satisfying*

(i) $||\grave{z}_n - \grave{v}_n|| \to d(\mathcal{E}_1, \mathcal{E}_2)$,

(ii) $||\grave{u}_n - \grave{v}_n|| \to d(\mathcal{E}_1, \mathcal{E}_2)$.

Then $||\grave{u}_n - \grave{z}_n||$ *converges to zero.*

2. MAIN RESULTS

First, we prove a very useful approximation result.

Proposition 1. *Let* $\mathcal{E}_2$ *and* $\mathcal{E}_1$ *be nonempty subsets of a metric space* (χ, d). *Suppose that* $\Im : \mathcal{E}_1 \cup \mathcal{E}_2 \to \mathcal{E}_1 \cup \mathcal{E}_2$ *is an operator fulfilling the following hypotheses:*

($\mathcal{T}_1$) $\Im(\mathcal{E}_1) \subseteq \mathcal{E}_2$ *and* $\Im(\mathcal{E}_2) \subseteq \mathcal{E}_1$,

($\mathcal{T}_2$) *there exist* $\alpha', \beta', \gamma', \delta'$ *are nonnegative real numbers with* $\alpha' + \beta' + \gamma' + 2\delta' < 1$ *such that*

$$\begin{aligned} d(\Im\grave{u}, \Im\grave{v}) \leq\ & \alpha' d(\grave{u}, \Im\grave{u}) + \beta' d(\grave{v}, \Im\grave{v}) + \gamma' d(\grave{u}, \grave{v}) + \delta'(d(\grave{u}, \Im\grave{v}) + d(\grave{v}, \Im\grave{u})) \\ & + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d(\mathcal{E}_1, \mathcal{E}_2), \end{aligned} \tag{2.1}$$

for all $\grave{u} \in \mathcal{E}_1$ *and* $\grave{v} \in \mathcal{E}_2$. *If* $\grave{u}_0 \in \mathcal{E}_1$ *and* $\grave{u}_{n+1} = \Im\grave{u}_n$ *where* $n \in \mathbb{N} \cup \{0\}$, *then* $d(\grave{u}_n, \Im\grave{u}_n) \to d(\mathcal{E}_1, \mathcal{E}_2)$.

Proof. Consider

$$\begin{aligned} d(\grave{u}_n, \grave{u}_{n+1}) =\ & d(\Im\grave{u}_{n-1}, \Im\grave{u}_n) \\ \leq\ & \alpha' d(\grave{u}_{n-1}, \Im\grave{u}_{n-1}) + \beta' d(\grave{u}_n, \Im\grave{u}_n) + \gamma' d(\grave{u}_{n-1}, \grave{u}_n) \\ & + \delta' d(\grave{u}_{n-1}, \Im\grave{u}_n) + \delta' d(\grave{u}_n, \Im\grave{u}_{n-1}) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d(\mathcal{E}_1, \mathcal{E}_2) \\ =\ & \alpha' d(\grave{u}_{n-1}, \grave{u}_n) + \beta' d(\grave{u}_n, \grave{u}_{n+1}) + \gamma' d(\grave{u}_{n-1}, \grave{u}_n) + \delta' d(\grave{u}_{n-1}, \grave{u}_{n+1}) \\ & + \delta' d(\grave{u}_n, \grave{u}_n) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d(\mathcal{E}_1, \mathcal{E}_2) \\ \leq\ & \alpha' d(\grave{u}_{n-1}, \grave{u}_n) + \beta' d(\grave{u}_n, \grave{u}_{n+1}) + \gamma' d(\grave{u}_{n-1}, \grave{u}_n) + \delta' d(\grave{u}_{n-1}, \grave{u}_n) \\ & + \delta' d(\grave{u}_n, \grave{u}_{n+1}) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d(\mathcal{E}_1, \mathcal{E}_2). \end{aligned} \tag{2.2}$$

Rewriting equation (2.2), we have

$$\begin{aligned} d(\grave{u}_n, \grave{u}_{n+1}) \leq\ & \frac{\alpha' + \gamma' + \delta'}{(1 - \beta' - \delta')} d(\grave{u}_{n-1}, \grave{u}_n) + \frac{(1 - \alpha' - \beta' - \gamma' - 2\delta')}{(1 - \beta' - \delta')}\, d(\mathcal{E}_1, \mathcal{E}_2) \\ d(\grave{u}_n, \grave{u}_{n+1}) =\ & k d(\grave{u}_{n-1}, \grave{u}_n) + (1 - k)\, d(\mathcal{E}_1, \mathcal{E}_2)\ ;\ \ k = \frac{\alpha' + \gamma' + \delta'}{1 - \beta' - \delta'} < 1, \\ d(\grave{u}_n, \grave{u}_{n+1}) \leq\ & k d(\grave{u}_{n-1}, \grave{u}_n) + (1 - k) d(\grave{u}_{n-1}, \grave{u}_n) \qquad (2.3) \\ d(\grave{u}_n, \grave{u}_{n+1}) \leq\ & d(\grave{u}_{n-1}, \grave{u}_n), \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, $\{d(\grave{u}_n, \grave{u}_{n+1})\}$ is a bounded below and decreasing sequence, so there exists $r \geq 0$ such that

$$r = \lim_{n\to\infty} d(\grave{u}_n, \grave{u}_{n+1}).$$

Again consider

$$\begin{aligned}
d(\grave{u}_n,\grave{u}_{n+1}) =\ & d(\Im\grave{u}_{n-1},\Im\grave{u}_n)\\
\leq\ & \alpha' d(\grave{u}_{n-1},\Im\grave{u}_{n-1})+\beta' d(\grave{u}_n,\Im\grave{u}_n)+\gamma'(\grave{u}_{n-1},\grave{u}_n)+\\
& +\delta'(\grave{u}_{n-1},\Im\grave{u}_n)+\delta'(\grave{u}_n,\Im\grave{u}_{n-1})+(1-\alpha'-\beta'-\gamma'-2\delta')\ d(\mathcal{E}_1,\mathcal{E}_2)\\
=\ & \alpha' d(\grave{u}_{n-1},\grave{u}_n)+\beta' d(\grave{u}_n,\grave{u}_{n+1})+\gamma' d(\grave{u}_{n-1},\grave{u}_n)+\delta' d(\grave{u}_{n-1},\grave{u}_{n+1})\\
& +\delta' d(\grave{u}_n,\grave{u}_n)+(1-\alpha'-\beta'-\gamma'-2\delta')\ d(\mathcal{E}_1,\mathcal{E}_2)\\
\leq\ & \alpha' d(\grave{u}_{n-1},\grave{u}_n)+\beta' d(\grave{u}_n,\grave{u}_{n+1})+\gamma' d(\grave{u}_{n-1},\grave{u}_n)+\delta' d(\grave{u}_{n-1},\grave{u}_n)\\
& +\delta' d(\grave{u}_n,\grave{u}_{n+1})+(1-\alpha'-\beta'-\gamma'-2\delta')\ d(\mathcal{E}_1,\mathcal{E}_2)\\
\leq\ & \frac{\alpha'+\gamma'+\delta'}{(1-\beta'-\delta')}d(\grave{u}_{n-1},\grave{u}_n)+\frac{(1-\alpha'-\beta'-\gamma'-2\delta')}{(1-\beta'-\delta')}\ d(\mathcal{E}_1,\mathcal{E}_2)\\
=\ & kd(\grave{u}_{n-1},\grave{u}_n)+(1-k)\ d(\mathcal{E}_1,\mathcal{E}_2)\\
=\ & k^2 d(\grave{u}_{n-2},\grave{u}_{n-1})+(1-k^2)\ (\mathcal{E}_1,\mathcal{E}_2)\\
\leq\ & k^3 d(\grave{u}_{n-2},\grave{u}_{n-1})+(1-k^3)\ d(\mathcal{E}_1,\mathcal{E}_2)\\
& \vdots\\
\leq\ & k^n d(\grave{u}_0,\grave{u}_1)+(1-k^n)\ d(\mathcal{E}_1,\mathcal{E}_2). \qquad (2.4)
\end{aligned}$$

Since $k<1$, $k^n \to 0$ as $n \to +\infty$, we have

$$d(\grave{u}_n,\grave{u}_{n+1}) \to d(\mathcal{E}_1,\mathcal{E}_2). \qquad (2.5)$$

□

Next, we prove an existence result for a best proximity point.

Theorem 1. *Let $\mathcal{E}_2$ and $\mathcal{E}_1$ be nonempty subsets of a metric space (χ,d). Suppose that $\Im:\mathcal{E}_1\cup\mathcal{E}_2\to\mathcal{E}_1\cup\mathcal{E}_2$ is an operator fulfilling the following hypotheses:*

($\mathcal{T}_1$) *$\Im(\mathcal{E}_1)\subseteq\mathcal{E}_2$ and $\Im(\mathcal{E}_2)\subseteq\mathcal{E}_1$,*

($\mathcal{T}_2$) *χ is complete,*

($\mathcal{T}_3$) *there exist $\alpha',\beta',\gamma',\delta'$ are nonnegative real numbers with $\alpha'+\beta'+\gamma'+2\delta'<1$ such that*

$$\begin{aligned}
d(\Im\grave{u},\Im\grave{v}) \leq\ & \alpha' d(\grave{u},\Im\grave{u})+\beta' d(\grave{v},\Im\grave{v})+\gamma' d(\grave{u},\grave{v})+\delta'(d(\grave{u},\Im\grave{v})+d(\grave{v},\Im\grave{u}))\\
& +(1-\alpha'-\beta'-\gamma'-2\delta')\ d(\mathcal{E}_1,\mathcal{E}_2), \qquad (2.6)
\end{aligned}$$

for all $\grave{u}\in\mathcal{E}_1$ and $\grave{v}\in\mathcal{E}_2$. Suppose $\grave{u}_0\in\mathcal{E}_1$ and $\grave{u}_{n+1}=\Im\grave{u}_n$ where $n\in\mathbb{N}\cup\{0\}$. If $\{u_{2n}\}$ has a convergent subsequence in $\mathcal{E}_1$ then $\Im$ has a best proximity point.

Proof. Let $\{\grave{u}_{2n(k')}\}$ be a subsequence of $\{u_{2n}\}$ which converges to a point $\grave{u}\in\mathcal{E}_1$. Now

$$d(\grave{u},\grave{u}_{2n(k')-1}) \leq d(\grave{u},\grave{u}_{2n(k')})+d(\grave{u}_{2n(k')},\grave{u}_{2n(k')-1}). \qquad (2.7)$$

Taking $n \to \infty$ in (2.7), we get

$$d(\grave{u}, \grave{u}_{2n(k')-1}) \to d(\mathcal{E}_1, \mathcal{E}_2).$$

Since $d(\mathcal{E}_1, \mathcal{E}_2) \le d(\grave{u}_{2n(k')}, \Im\grave{u}) \le d(\grave{u}_{2n(k')-1}, u)$. Then $\Im$ has a best proximity point. □

Theorem 2. *Let $\mathcal{E}_2$ be a nonempty closed subset and $\mathcal{E}_1$ be a nonempty convex and closed subset of a uniformly convex binormed linear space $(\chi, ||.||_1, ||.||)$ with $||.||_1 \le ||.||$. Suppose that $\Im : \mathcal{E}_1 \cup \mathcal{E}_2 \to \mathcal{E}_1 \cup \mathcal{E}_2$ be an operator fulfilling the following hypotheses:*

($\mathcal{T}_1$) *$\Im(\mathcal{E}_1) \subseteq \mathcal{E}_2$ and $\Im(\mathcal{E}_2) \subseteq \mathcal{E}_1$,*
($\mathcal{T}_2$) *χ is complete with respect to $||.||_1$,*
($\mathcal{T}_3$) *there exist $\alpha', \beta', \gamma', \delta'$ are nonnegative real numbers with $\alpha' + \beta' + \gamma' + 2\delta' < 1$ such that*

$$\begin{aligned} ||\Im\grave{u} - \Im\grave{v}|| \le\ & \alpha'||\grave{u} - \Im\grave{u}|| + \beta'||\grave{v} - \Im\grave{v}|| + \gamma'||\grave{u} - \grave{v}|| + \delta'(||\grave{u} - \Im\grave{v}|| + ||\grave{v} - \Im\grave{u}||) \\ & + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d(\mathcal{E}_1, \mathcal{E}_2), \end{aligned} \tag{2.8}$$

for all $\grave{u} \in \mathcal{E}_1$ and $\grave{v} \in \mathcal{E}_2$. If $\grave{u}_0 \in \mathcal{E}_1$ and $\grave{u}_{n+1} = \Im\grave{u}_n$ where $n \in \mathbb{N}$, then $\Im$ has the best proximity point in $\mathcal{E}_1$.

Proof. By Proposition 1, we have

$$||\grave{u}_{2n} - \grave{v}_{2n+1}|| \to d(\mathcal{E}_1, \mathcal{E}_2) \text{ and } ||\grave{u}_{2n+1} - \Im\grave{u}_{2n+1}|| \to d(\mathcal{E}_1, \mathcal{E}_2). \tag{2.9}$$

Since χ is a uniformly convex Banach space by Lemma (2), we get

$$||\grave{u}_{2n} - \grave{u}_{2(n+1)}|| \to 0 \text{ and } ||\Im\grave{u}_{2n+1} - \Im\grave{u}_{2n}|| \to 0. \tag{2.10}$$

We now show that for every $\varepsilon > 0$ there exists N_0 such that for all $m > n \ge N_0$, $||\grave{u}_{2m} - \Im\grave{u}_{2n}|| < d(\mathcal{E}_1, \mathcal{E}_2) + \varepsilon$. Suppose not, then there exists $\varepsilon > 0$ such that for all $k' \in N$, there exists $m(k') > n(k') \ge k'$, for $||\grave{u}_{2m(k')} - \Im\grave{u}_{2n(k')}|| \ge d(\mathcal{E}_1, \mathcal{E}_2) + \varepsilon$, this $m(k')$ can be chosen such that it is the least integer greater than $n(k')$ to satisfy the above inequality. Now

$$\begin{aligned} d(\mathcal{E}_1, \mathcal{E}_2) + \varepsilon &\le ||\grave{u}_{2m(k')} - \Im\grave{u}_{2n(k')}|| \\ &\le ||\grave{u}_{2m(k')} - \grave{u}_{2(m(k')-1)}|| + ||\grave{u}_{2(m(k')-1)} - \Im\grave{u}_{2n(k')}|| \\ &< ||\grave{u}_{2m(k')} - \grave{u}_{2(m(k')-1)}|| + d(\mathcal{E}_1, \mathcal{E}_2) + \varepsilon. \end{aligned}$$

Using equation (2.10) and taking $k' \to \infty$ in the above inequality we have

$$||\grave{u}_{2m(k')} - \Im\grave{u}_{2n(k')}|| = d(\mathcal{E}_1, \mathcal{E}_2) + \varepsilon.$$

Consider

$$\begin{aligned} ||\grave{u}_{2m(k')} - \Im\grave{u}_{2n(k')}|| &\le ||\grave{u}_{2m(k')} - \grave{u}_{2(m(k')+1)}|| + ||\grave{u}_{2(m(k')+1)} - \Im\grave{u}_{2(n(k')+1)}|| \\ &\quad + ||\Im\grave{u}_{2(n(k')+1)} - \Im\grave{u}_{2n(k')}|| \\ &\le ||\grave{u}_{2m(k')} - \grave{u}_{2(m(k')+1)}|| + ||\Im\grave{u}_{2(n(k')+1)} - \Im\grave{u}_{2n(k')}|| \end{aligned}$$

$$+k^2||\grave{u}_{2m(k')} - \Im\grave{u}_{2n(k')}|| + (1-k^2)d(\mathcal{E}_1,\mathcal{E}_2)$$
$$\leq ||\grave{u}_{2m(k')} - \grave{u}_{2(m(k')+1)}|| + ||\Im\grave{u}_{2(n(k')+1)} - \Im\grave{u}_{2n(k')}||$$
$$+k^2||\grave{u}_{2m(k')} - \Im\grave{u}_{2n(k')}|| + (1-k^2)d(\mathcal{E}_1,\mathcal{E}_2). \tag{2.11}$$

Taking $k' \to \infty$ in equation (2.11) and using equation (2.10) we get

$$d(\mathcal{E}_1,\mathcal{E}_2) + \varepsilon \leq k^2(d(\mathcal{E}_1,\mathcal{E}_2)+\varepsilon) + (1-k^2)d(\mathcal{E}_1,\mathcal{E}_2)$$
$$= d(\mathcal{E}_1,\mathcal{E}_2) + k^2\varepsilon,$$

which is a contradiction. Therefore $\{\grave{u}_{2n}\}$ is a Cauchy sequence in $\mathcal{E}$ with respect $||.||$. Because $||.||_1 \leq ||.||$, hence $\{u_{2n}\}$ is a Cauchy sequence in $\mathcal{E}_1$ with respect $||.||_1$. Since $\mathcal{E}_1$ is a closed subset of a complete metric space χ, then it is a complete subspace. By the completeness of $\mathcal{E}_1$, $\{\grave{u}_{2n}\}$ converges to a point $\grave{u}$ in $\mathcal{E}_1$, then by Theorem 1, we get $\Im$ has a best proximity point, $||\grave{u} - \Im\grave{u}||_1 = d_1(\mathcal{E}_1,\mathcal{E}_2)$ in $\mathcal{E}_1$. $\square$

3. Consequences

Throughout in this section we use some notations which required in the sequel. Let $\mathcal{E}_2$ be a nonempty closed subset and $\mathcal{E}_1$ be a nonempty convex closed subset of a uniformly convex binormed linear space $(\chi, ||.||_1, ||.||)$ with $||.||_1 \leq ||.||$ and $\Im : \mathcal{E}_1 \cup \mathcal{E}_2 \to \mathcal{E}_1 \cup \mathcal{E}_2$ be an operator.

If $\alpha' = \beta' = \gamma' = 0$ in Theorem 2, we get the following best proximity result.

Corollary 1. *Assume that $\Im$ is fulfilling the following hypotheses:*

($\mathcal{T}_1$) *$\Im(\mathcal{E}_1) \subseteq \mathcal{E}_2$ and $\Im(\mathcal{E}_2) \subseteq \mathcal{E}_1$,*
($\mathcal{T}_2$) *χ is complete with respect to $||.||_1$,*
($\mathcal{T}_3$) *there exist a nonnegative real number δ' with $2\delta' < 1$ such that*

$$||\Im\grave{u} - \Im\grave{v}|| \leq \delta'(||\grave{u} - \Im\grave{v}|| + ||\grave{v} - \Im\grave{u}||) + (1-2\delta')\, d(\mathcal{E}_1,\mathcal{E}_2),$$

for all $\grave{u} \in \mathcal{E}_1$ and $\grave{v} \in \mathcal{E}_2$. If $\grave{u}_0 \in \mathcal{E}_1$ and $\grave{u}_{n+1} = \Im\grave{u}_n$ where $n \in \mathbb{N}$, then $\Im$ has the best proximity point in $\mathcal{E}_1$.

If $\alpha' = \beta'$ and $\gamma' = \delta' = 0$ in Theorem 2, we get the following best proximity result.

Corollary 2. *Assume that $\Im$ is fulfilling the following hypotheses:*

($\mathcal{T}_1$) *$\Im(\mathcal{E}_1) \subseteq \mathcal{E}_2$ and $\Im(\mathcal{E}_2) \subseteq \mathcal{E}_1$,*
($\mathcal{T}_2$) *χ is complete with respect to $||.||_1$,*
($\mathcal{T}_3$) *there exist a nonnegative real number α with $2\alpha' < 1$ such that*

$$||\Im\grave{u} - \Im\grave{v}|| \leq \alpha'(||\grave{u} - \Im\grave{u}|| + ||\grave{v} - \Im\grave{v}||) + (1-2\alpha')\, d(\mathcal{E}_1,\mathcal{E}_2),$$

for all $\grave{u} \in \mathcal{E}_1$ and $\grave{v} \in \mathcal{E}_2$. If $\grave{u}_0 \in \mathcal{E}_1$ and $\grave{u}_{n+1} = \Im\grave{u}_n$ where $n \in \mathbb{N}$, then $\Im$ has the best proximity point in $\mathcal{E}_1$.

If $\alpha' = \beta' = \delta' = 0$ in Theorem 2, we get the following best proximity result.

Corollary 3. *Assume that $\Im$ is fulfilling the following hypotheses:*

($\mathcal{T}_1$) *$\Im(\mathcal{E}_1) \subseteq \mathcal{E}_2$ and $\Im(\mathcal{E}_2) \subseteq \mathcal{E}_1$,*
($\mathcal{T}_2$) *χ is complete with respect to $||.||_1$,*
($\mathcal{T}_3$) *there exist a nonnegative real number γ' with $\gamma' < 1$ such that*

$$||\Im\grave{u} - \Im\grave{v}|| \leq \gamma' ||\grave{u} - \grave{v}|| + (1-\gamma')\, d(\mathcal{E}_1, \mathcal{E}_2),$$

for all $\grave{u} \in \mathcal{E}_1$ and $\grave{v} \in \mathcal{E}_2$. If $\grave{u}_0 \in \mathcal{E}_1$ and $\grave{u}_{n+1} = \Im\grave{u}_n$ where $n \in \mathbb{N}$, then $\Im$ has the best proximity point in $\mathcal{E}_1$.

If $\alpha' = \beta' = \gamma'$ and $\delta' = 0$ in Theorem 2, we get the following best proximity result.

Corollary 4. *Assume that $\Im$ is fulfilling the following hypotheses:*

($\mathcal{T}_1$) *$\Im(\mathcal{E}_1) \subseteq \mathcal{E}_2$ and $\Im(\mathcal{E}_2) \subseteq \mathcal{E}_1$,*
($\mathcal{T}_2$) *χ is complete with respect to $||.||_1$,*
($\mathcal{T}_3$) *there exist a nonnegative real number α' with $3\alpha' < 1$ such that*

$$||\Im\grave{u} - \Im\grave{v}|| \leq \alpha'(||\grave{u} - \Im\grave{u}|| + ||\grave{v} - \Im\grave{v}|| + ||\grave{u} - \grave{v}||) + (1-3\alpha')\, d(\mathcal{E}_1, \mathcal{E}_2),$$

for all $\grave{u} \in \mathcal{E}_1$ and $\grave{v} \in \mathcal{E}_2$. If $\grave{u}_0 \in \mathcal{E}_1$ and $\grave{u}_{n+1} = \Im\grave{u}_n$ where $n \in \mathbb{N}$, then $\Im$ has the best proximity point in $\mathcal{E}_1$.

Remark 1.

- By choosing different values of $\alpha', \beta', \gamma', \delta'$ in Theorem 2, we get many best proximity results.
- If we take $||.|| = ||.||_1$ in Corollaries 1, 2, 3 and 4, then we get the corresponding results of [7, 10, 14].

If $\mathcal{E}_1 = \mathcal{E}_2 = \chi$ in Theorem 2, we have the following fixed point result.

Corollary 5. *Assume that $\Im$ is fulfilling the following hypotheses:*

($\mathcal{T}_1$) *χ is complete with respect to $||.||_1$,*
($\mathcal{T}_2$) *there exist $\alpha', \beta', \gamma', \delta'$ are nonnegative real numbers with $\alpha' + \beta' + \gamma' + 2\delta' < 1$ such that*

$$||\Im\grave{u} - \Im\grave{v}|| \leq \alpha'||\grave{u} - \Im\grave{u}|| + \beta'||\grave{v} - \Im\grave{v}|| + \gamma'||\grave{u} - \grave{v}|| + \delta'(||\grave{u} - \Im\grave{v}|| + ||\grave{v} - \Im\grave{u}||), \tag{3.1}$$

for all $\grave{u}, \grave{v} \in \chi$. If $\grave{u}_0 \in \chi$ and $\grave{u}_{n+1} = \Im\grave{u}_n$ where $n \in \mathbb{N}$, then $\Im$ has a fixed point.

If we take $\mathcal{E}_1 = \mathcal{E}_2 = \chi$ in Corollaries 1, 2, 3, 4, we get the following fixed point results.

Corollary 6. *Assume that $\Im$ is fulfilling the following hypotheses:*

($\mathcal{T}_1$) *χ is complete with respect to $||.||_1$,*
($\mathcal{T}_2$) *there exist a nonnegative real number δ' with $2\delta' < 1$ such that*

$$||\Im\grave{u} - \Im\grave{v}|| \leq \delta'(||\grave{u} - \Im\grave{v}|| + ||\grave{v} - \Im\grave{u}||),$$

for all $\grave{u}, \grave{v} \in \chi$. If $\grave{u}_0 \in \chi$ and $\grave{u}_{n+1} = \Im\grave{u}_n$ where $n \in \mathbb{N}$, then $\Im$ has a fixed point.

Corollary 7. *Assume that $\Im$ is fulfilling the following hypotheses:*

($\mathcal{T}_1$) *χ is complete with respect to $||.||_1$,*
($\mathcal{T}_2$) *there exist a nonnegative real number α' with $2\alpha' < 1$ such that*

$$||\Im\grave{u} - \Im\grave{v}|| \leq \alpha'(||\grave{u} - \Im\grave{u}|| + ||\grave{v} - \Im\grave{v}||),$$

for all $\grave{u}, \grave{v} \in \chi$. If $\grave{u}_0 \in \chi$ and $\grave{u}_{n+1} = \Im\grave{u}_n$ where $n \in \mathbb{N}$, then $\Im$ has a fixed point.

Corollary 8 ([13])**.** *Assume that $\Im$ is fulfilling the following hypotheses:*

($\mathcal{T}_1$) *χ is complete with respect to $||.||_1$,*
($\mathcal{T}_2$) *there exist a nonnegative real number γ' with $\gamma' < 1$ such that*

$$||\Im\grave{u} - \Im\grave{v}|| \leq \gamma'||\grave{u} - \grave{v}||,$$

for all $\grave{u}, \grave{v} \in \chi$. If $\grave{u}_0 \in \chi$ and $\grave{u}_{n+1} = \Im\grave{u}_n$ where $n \in \mathbb{N}$, then $\Im$ has a fixed point.

Corollary 9. *Assume that $\Im$ is fulfilling the following hypotheses:*

($\mathcal{T}_1$) *χ is complete with respect to $||.||_1$,*
($\mathcal{T}_2$) *there exist a nonnegative real numbers α' with $3\alpha' < 1$ such that*

$$||\Im\grave{u} - \Im\grave{v}|| \leq \alpha'(||\grave{u} - \Im\grave{u}|| + ||\grave{v} - \Im\grave{v}|| + ||\grave{u} - \grave{v}||),$$

for all $\grave{u}, \grave{v} \in \chi$. If $\grave{u}_0 \in \chi$ and $\grave{u}_{n+1} = \Im\grave{u}_n$ where $n \in \mathbb{N}$, then $\Im$ has a fixed point.

Remark 2.

- By choosing different values of $\alpha', \beta', \gamma', \delta'$ in Corollary 5, we get many fixed point results.
- If we take $||.|| = ||.||_1$ in Corollaries 6, 7, 8 and 9, then we get the corresponding results of [3, 11].

Now, we provide several illustrations that support our findings in this section.

Example 1. Consider $\chi = \mathbb{R}$, define $||.||, ||.||_1 : \chi \to \mathbb{R}_+$ by

$$||\grave{u}|| = 2|\grave{u}| \text{ and } ||\grave{u}||_1 = |\grave{u}|$$

for all $\grave{u} \in \chi$. It is easy to see that $||\grave{u}||_1 < ||\grave{u}||$, for all $\grave{u} \in \chi$. Suppose $\mathcal{E}_1 = [\frac{1}{4}, \frac{1}{2}]$ and $\mathcal{E}_2 = [\frac{3}{4}, \frac{9}{8}]$ are two subsets of χ, then $d(\mathcal{E}_1, \mathcal{E}_2) = 0.5$ and $d_1(\mathcal{E}_1, \mathcal{E}_2) = 0.25$.

Define $\Im : \mathcal{E}_1 \cup \mathcal{E}_2 \to \mathcal{E}_1 \cup \mathcal{E}_2$ by $\Im(\grave{u}) = \begin{cases} \frac{\grave{u}}{32} + \frac{3}{4} \text{ if } \grave{u} \in [\frac{1}{4}, \frac{1}{2}) \\ \frac{3}{4} \text{ if } \grave{u} = \frac{1}{2} \\ \frac{1}{2} \text{ if } \grave{u} \in [\frac{3}{4}, \frac{9}{8}] \end{cases}$

for all $\grave{u} \in \mathcal{E}_1 \cup \mathcal{E}_2$. Next we prove that $\Im$ satisfies the following inequality,

$$\begin{aligned} ||\Im\grave{u} - \Im\grave{v}|| \leq\ & \alpha'||\grave{u} - \Im\grave{u}|| + \beta'||\grave{v} - \Im\grave{v}|| + \gamma'||\grave{u} - \grave{v}|| + \delta'(||\grave{u} - \Im\grave{v}|| + ||\grave{v} - \Im\grave{u}||) \\ & + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d(\mathcal{E}_1, \mathcal{E}_2), \end{aligned}$$

for all $\grave{u} \in \mathcal{E}_1$ and $\grave{v} \in \mathcal{E}_2$. Let $\alpha' = 0.488$, $\beta' = 0.01$, $\gamma' = \frac{1}{2}$, $\delta' = 0$ with $\alpha' + \beta' + \gamma' + 2\delta' < 1$.

$$||\Im\grave{u} - \Im\grave{v}|| = \left|\left|\frac{\grave{u}}{32} + \frac{3}{4} - \frac{1}{2}\right|\right| = \left|\left|\frac{\grave{u}}{32} + \frac{1}{4}\right|\right|.$$

If $\grave{u} \in \mathcal{E}_1$ and $\grave{v} \in \mathcal{E}_2$ then $\left|\left|\frac{\grave{u}}{32} + \frac{1}{4}\right|\right| \in [0.5156, 0.5312]$, $||\grave{u} - \grave{v}|| \in [0.5, 1.75]$, $||\grave{u} - \Im\grave{u}|| \in [0.75, 2]$ and $||\grave{v} - \Im\grave{v}|| \in [0.5, 1.25]$. This implies

$$\begin{aligned} ||\Im\grave{u} - \Im\grave{v}|| \leq\ & \alpha'||\grave{u} - \Im\grave{u}|| + \beta'||\grave{v} - \Im\grave{v}|| + \gamma'||\grave{u} - \grave{v}|| + \delta'(||\grave{u} - \Im\grave{v}|| + ||\grave{v} - \Im\grave{u}||) \\ & + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d(\mathcal{E}_1, \mathcal{E}_2), \end{aligned}$$

for all $\grave{u} \in \mathcal{E}_1$ and $\grave{v} \in \mathcal{E}_2$ and $\Im(\mathcal{E}_1) \subseteq \mathcal{E}_2$, $\Im(\mathcal{E}_2) \subseteq \mathcal{E}_1$. Since $||\grave{u}||_1 < ||\grave{u}||$, for all $\grave{u} \in \chi$, we have

$$\begin{aligned} ||\Im\grave{u} - \Im\grave{v}||_1 \leq\ & \alpha'||\grave{u} - \Im\grave{u}||_1 + \beta'||\grave{v} - \Im\grave{v}||_1 + \gamma'||\grave{u} - \grave{v}||_1 \\ & + \delta'(||\grave{u} - \Im\grave{v}||_1 + ||\grave{v} - \Im\grave{u}||_1) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d_1(\mathcal{E}_1, \mathcal{E}_2). \end{aligned}$$

Starting with point $\grave{u}_0 = \frac{1}{4} \in \mathcal{E}_1$, we construct a sequence as

$\grave{u}_{n+1}$	$\grave{u}_0$	$\grave{u}_1$	$\grave{u}_2$	$\grave{u}_3$	$\grave{u}_4$	$\grave{u}_5$	$\grave{u}_6$	$\grave{u}_7$	$\cdots$
$\Im\grave{u}_n$	0.25	0.7578	0.50	0.75	0.50	0.75	0.50	0.75	$\cdots$

We found that $\{\grave{u}_{2n}\}$ has a subsequence (0.25, 0.5, 0.5, 0.5,...), which converges to $\frac{1}{2}$. All the conditions of Theorem 2 are satisfied, and $\Im$ has a best proximity point $\frac{1}{2}$.

Example 2. Consider $\chi = \mathbb{R}^2$ with usual metric defined as

$$||\grave{z}_1|| = ||\grave{z}_1||_1 = \sqrt{\grave{u}_1^2 + \grave{v}_1^2} \tag{3.2}$$

for all $\grave{z}_1 = (\grave{u}_1, \grave{v}_1) \in \mathbb{R}^2$. Suppose

$$\mathcal{E}_1 = \{(0, \grave{u}_1) : 0 \leq \grave{u}_1 \leq 3\} \text{ and } \mathcal{E}_2 = \{(0, \grave{u}_1) : 0 \leq \grave{u}_1 \leq 2\},$$

are two subsets of $\mathbb{R}^2$, then $d(\mathcal{E}_1, \mathcal{E}_2) = 0$. Define $\Im : \mathcal{E}_1 \to \mathcal{E}_2$ by

$$\Im(\grave{z}_1) = \frac{\grave{z}_1}{3} \tag{3.3}$$

for all $\grave{z}_1 \in \mathcal{E}_1$. Let $\grave{z}_1, \grave{z}_2 \in \mathcal{E}_1$. Next we prove that $\Im$ satisfies the following inequality,

$$\begin{aligned} ||\Im\grave{z}_1 - \Im\grave{z}_2|| \leq\ & \alpha'||\grave{z}_1 - \Im\grave{z}_1|| + \beta'||\grave{z}_2 - \Im\grave{z}_2|| + \gamma'||\grave{z}_1 - \grave{z}_2|| \\ & + \delta(||\grave{z}_1 - \Im\grave{z}_2|| + ||\grave{z}_2 - \Im\grave{z}_1||) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d(\mathcal{E}_1, \mathcal{E}_2), \end{aligned}$$

for all $\grave{z}_1, \grave{z}_2 \in \mathcal{E}_1$. Let $\alpha' = \beta' = \delta' = 0$, $\gamma' = \frac{1}{2}$ with $\alpha' + \beta' + \gamma' + 2\delta' < 1$. Take $\grave{z}_1 = (0, \grave{u}_1), \grave{z}_2 = (0, \grave{u}_2)$ in $\mathcal{E}_1$ then

$$||\Im\grave{z}_1 - \Im\grave{z}_2|| = \left|\left|\frac{\grave{z}_1}{3} - \frac{\grave{z}_1}{3}\right|\right| = \left|\left|\frac{(0, \grave{u}_1)}{3} - \frac{(0, \grave{u}_2)}{3}\right|\right| = \sqrt{\left(\frac{\grave{u}_1}{3} - \frac{\grave{u}_2}{3}\right)^2}$$

$$= \sqrt{\frac{1}{3^2}(\grave{u}_1 - \grave{u}_2)^2} = \frac{1}{3}\sqrt{(\grave{u}_1 - \grave{u}_2)^2} = \frac{1}{3}||\grave{z}_1 - \grave{z}_2||,$$

$$||\Im\grave{z}_1 - \Im\grave{z}_2|| < \gamma||\grave{z}_1 - \grave{z}_2||.$$

This shows

$$||\Im\grave{z}_1 - \Im\grave{z}_2|| < \alpha'||\grave{z}_1 - \Im\grave{z}_1|| + \beta'||\grave{z}_2 - \Im\grave{z}_2|| + \gamma'||\grave{z}_1 - \grave{z}_2|| + \delta(||\grave{z}_1 - \Im\grave{z}_2|| + ||\grave{z}_2 - \Im\grave{z}_1||) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d(\mathcal{E}_1, \mathcal{E}_2),$$

for all $\grave{z}_1, \grave{z}_2 \in \mathcal{E}_1$ and $\Im(\mathcal{E}_1) \subseteq \mathcal{E}_2$, $\Im(\mathcal{E}_2) \subseteq \mathcal{E}_1$. Starting with point $\grave{u}_0 = (0,1) \in \mathcal{E}_1$, we construct a sequence as

$\grave{u}_{n+1}$	$\grave{u}_0$	$\grave{u}_1$	$\grave{u}_2$	$\grave{u}_3$	$\grave{u}_4$	$\grave{u}_5$	$\grave{u}_6$	...
$\Im\grave{x}_n$	$(0,1)$	$(0,0.3333)$	$(0,0.1111)$	$(0,0.0366)$	$(0,0.0122)$	$(0,0.0040)$	$(0,0.0013)$	...

We found that $\{\grave{u}_{2n}\}$ has a subsequence ((0,1) (0,0.1111), (0,0.0122), (0,0.0013), (0,0.0001),(0,0.0000)...), which converges to $(0,0)$. All the conditions of Theorem 2 are satisfied, and $\Im$ has a best proximity point $(0,0)$.

4. Applications

4.1. *Solving split feasibility problems*

The split feasibility problem (SFP), which is mathematically formulated as:

$$\text{find a point } \grave{u} \in \mathcal{C} \text{ such that } \mathcal{E}\grave{u} \in \mathcal{D}, \tag{4.1}$$

where $\mathcal{C}$ and $\mathcal{D}$ are non-empty convex and closed subsets of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and $\mathcal{E} : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. This problem was first proposed by Censor and Elfving [5] in Euclidean spaces. Assume that the SFP (4.1) has at least one solution and $\mathcal{S}$ its solution set, then (see [1]) $\grave{u} \in \mathcal{C}$ is a solution of (4.1) if and only if it is a solution of the fixed-point problem (FPP),

$$\grave{u} = \mathcal{P}_{\mathcal{C}}(I - \omega\mathcal{E}^*(I - \mathcal{P}_{\mathcal{D}})\mathcal{E})\grave{u},$$

where $\mathcal{P}_{\mathcal{C}}$ and $\mathcal{P}_{\mathcal{D}}$ are the near by point projections onto $\mathcal{C}$ and $\mathcal{D}$, respectively, $\omega > 0$, and $\mathcal{E}^*$ is the adjoint operator of $\mathcal{E}$. It has been shown in [4] if κ is the spectral radius of $\mathcal{E}^*\mathcal{E}$ and $\omega \in (0, \frac{2}{\kappa})$, then the operator

$$\Im = \mathcal{P}_{\mathcal{C}}(I - \omega\mathcal{E}^*(I - \mathcal{P}_{\mathcal{D}})\mathcal{E})$$

is nonexpansive and averaged and the so-called $\mathcal{CQ}$ algorithm,

$$\grave{u}_{n+1} = \mathcal{P}_{\mathcal{C}}(I - \omega\mathcal{E}^*(I - \mathcal{P}_{\mathcal{D}})\mathcal{E})\grave{u}_n,\ n \geq 0,$$

converges weakly to a solution of the SFP. Now we generlaize this using assumptions on two norm $||.||$ and $||.||_1$, defined on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $||.||_1 \leq ||.||$ and proved the convergence of the iterative algorithm to the solution of a SFP.

Theorem 3. *Assume that $\mathcal{C}$ is convex and closed subset of a $\mathcal{H}_1$ endowed with two norm $||.||$ and $||.||_1$ such that $||.||_1 \leq ||.||$ and $\mathcal{P}_{\mathcal{C}}(I-\omega\mathcal{E}^*(I-\mathcal{P}_{\mathcal{D}})\mathcal{E})$ satisfying (3.1), $\omega \in (0, \frac{2}{\kappa})$. If SFP (4.1) has at least a solution and $\mathcal{C}$ is complete with respect $||.||_1$, then*

$$\grave{u}_{n+1} = \mathcal{P}_{\mathcal{C}}(I-\omega\mathcal{E}^*(I-\mathcal{P}_{\mathcal{D}})\mathcal{E})(\grave{u}_n),$$

converges to the unique solution $\grave{u}$ of the SFP.

Proof. Given $\mathcal{C}$ is convex and closed, we can take $\mathcal{C} = \chi$ and

$$\mathfrak{I} = \mathcal{P}_{\mathcal{C}}(I-\omega\mathcal{E}^*(I-\mathcal{P}_{\mathcal{D}})\mathcal{E})$$

and using Corollary 5, we get the result. □

4.2. *Solving variational inequality problems*

Assume that $\mathcal{H}$ is a real Hilbert space and $\mathcal{C} \subset \mathcal{H}$ be a convex and closed. A function $\mathcal{S}'$ defined on Hilbert space $\mathcal{H}$ is said to be monotone if

$$\left\langle \mathcal{S}'\grave{u} - \mathcal{S}'\grave{v}, \grave{u} - \grave{v} \right\rangle \geq 0, \text{ for all } \grave{u}, \grave{v} \in \mathcal{H}.$$

The variational inequality problem (VIP) with respect to $\mathcal{S}'$ and $\mathcal{C}$, symbolized by $VIP(\mathcal{S}', \mathcal{C})$, is to discover $\grave{u}^* \in \mathcal{C}$ such that

$$\left\langle \mathcal{S}'\grave{u}^*, \grave{u} - \grave{u}^* \right\rangle \geq 0, \text{ for all } \grave{u} \in \mathcal{H}.$$

It is popular if $\omega > 0$, then $\grave{u}^* \in \mathcal{C}$ is a solution of $VIP(\mathcal{S}', \mathcal{C})$ if and only if $\grave{u}^*$ is a solution of the FPP

$$\grave{u} = \mathcal{P}_{\mathcal{C}}(I-\omega\mathcal{S}')\grave{u},$$

where $\mathcal{P}_{\mathcal{C}}$ is the near by point projection onto $\mathcal{C}$. It was confirmed by others results (see [4]), if $\mathcal{P}_{\mathcal{C}}(I-\omega\mathcal{S}')$ and $(I-\omega\mathcal{S}')$ are averaged nonexpansive mappings, then, under few more hypotheses, the iterative algorithm defined by,

$$\grave{u}_{n+1} = \mathcal{P}_{\mathcal{C}}(I-\omega\mathcal{S}')\grave{u}_n, n \in \mathbb{N},$$

converges weakly to a solution of $VIP(\mathcal{S}', \mathcal{C})$, if such solutions exist. Now we generalize this using assumptions on two norms $||.||$ and $||.||_1$, defined on the Hilbert space $\mathcal{H}$ such that $||.||_1 \leq ||.||$ and proved the convergence of iterative algorithm to the solution of a VIP.

Theorem 4. *Assume that $\mathcal{C}$ is convex and closed subset of a $\mathcal{H}$ endowed with two norms $||.||$ and $||.||_1$ such that $||.||_1 \leq ||.||$ and $\mathcal{P}_{\mathcal{C}}(I-\omega\mathcal{S}')$ satisfying (3.1) on $\mathcal{C}$, $\omega > 0$. If C is complete with respect $||.||_1$, then*

$$\grave{u}_{n+1} = \mathcal{P}_{\mathcal{C}}(I-\omega\mathcal{S}')\grave{u}_n,$$

converges to the unique solution $\grave{u}$ of the $VIP(S', C)$.

Proof. Given $\mathcal{C}$ is convex and closed, we can take $\mathcal{C} = \chi$ and $\mathfrak{I} = \mathcal{P}_{\mathcal{C}}(I-\omega\mathcal{S}')$ and using Corollary 5, we get result. □

5. Conclusion

We find some novel best proximity point results in binormed linear spaces using Hardy-Roger type contraction mappings. One can obtain interesting results by choosing different values of $\alpha', \beta', \gamma', \delta'$. Many known results in the literature are generalized and extended by our findings.

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