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ON A SOLVABLE SYSTEM OF DIFFERENCE EQUATIONS OF SIXTH-ORDER

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Abstract. In this paper, we study the following two-dimesional system of difference equations

$$x_n = \frac{x_{n-4}y_{n-5}x_{n-6}}{y_{n-1}x_{n-2}\left(a + by_{n-3}x_{n-4}y_{n-5}x_{n-6}\right)},\ y_n = \frac{y_{n-4}x_{n-5}y_{n-6}}{x_{n-1}y_{n-2}\left(c + dx_{n-3}y_{n-4}x_{n-5}y_{n-6}\right)},\ n \in \mathbb{N}_0,$$

where the parameters a,b,c,d and the initial values x_{-i}, y_{-i}, $i \in \{1,2,3,4,5,6\}$, are real numbers. We show that some subclasses of nonlinear two-dimensional system of difference equations are solvable in closed form. We also describe the forbidden set of solutions of the system of difference equations. Some numerical examples are given to demonstrate the theoretical results.

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1. Introduction and preliminaries

Difference equations and systems of difference equations usually state the natural models of discrete processes, containing non-linear difference equations such as rational difference equations, Riccati difference equations, exponential difference equations. They play a key role in numerous applications in applied sciences such as Genetics, Biomathematics, Population Dynamics, Bioengineering, Biology and other sciences. So, there has been a lot of interest in difference equations which can be solved in explicit form or closed form (see, [1–4, 13, 14, 17, 19, 20, 23, 24, 36, 40–42, 44]) as well as in difference equations systems (see, e.g. [6–9, 11, 16, 21, 25–28, 30–35, 37–39, 45, 46, 48–50]). For example, the difference equations

$$x_{n+1} = \frac{x_{n-3}x_{n-4}}{x_n\left(\pm 1 \pm x_{n-1}x_{n-2}x_{n-3}x_{n-4}\right)}, \qquad n \in \mathbb{N}_0, \tag{1.1}$$

where the initial conditions are arbitrary real numbers, were studied in [12]. In addition, they investigated the behavior of the solutions of equations in (1.1).

The authors of [43] studied the periodicity and the solutions of the following systems of difference equations

$$x_{n+1} = \frac{y_n x_{n-1}}{\pm x_{n-1} \pm y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{\pm y_{n-1} \pm x_n}, \quad n \in \mathbb{N}_0,$$

where the initial conditions are non-zero real numbers.

The existence, uniqueness and attractivity of prime period two solutions of the following difference equation

$$x_{n+1} = a + b x_{n-1} e^{-x_n}, \qquad n \in \mathbb{N}_0,$$

where a, b are positive constants and the initial values x_{-1}, x_0 are positive numbers, was studied in [15].

Recently, Yazlik and Gungor solved the following non-linear difference equation

$$x_n = \frac{x_{n-4}x_{n-5}x_{n-6}}{x_{n-1}x_{n-2}\left(a + b x_{n-3}x_{n-4}x_{n-5}x_{n-6}\right)}, \qquad n \in \mathbb{N}_0, \tag{1.2}$$

in closed form. Also, the asymptotic behavior of well-defined solution of equation (1.2) was obtained in [47].

In [22], Ibrahim gave the solutions of the rational difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}\left(a + b x_n x_{n-2}\right)}, \qquad n \in \mathbb{N}_0, \tag{1.3}$$

where initial values x_0, x_{-1}, x_{-2} are non-negative real numbers with $b x_0 x_{-2} \neq -a$ and $x_{-1} \neq 0$. In addition, he investigated some properties for difference equation (1.3) such as the local stability and the boundedness for the solutions.

Alzubaidi et al. [5] presented the solutions of the following recursive sequences

$$x_{n+1} = \frac{x_{n-2}x_{n-3}}{x_n\left(\pm 1 \pm x_{n-2}x_{n-3}\right)}, \qquad n \in \mathbb{N}_0, \tag{1.4}$$

where the initial conditions x_{-3}, x_{-2}, x_{-1} and x_0 are arbitrary real numbers. Also, they studied some dynamic behavior of equations in (1.4).

El-Dessoky et al. dealt with the existence of solutions and the periodicity character of the following systems of rational difference equations

$$x_{n+1} = \frac{x_n y_{n-3}}{y_{n-2}\left(\pm 1 \pm x_n y_{n-3}\right)}, \quad y_{n+1} = \frac{y_n x_{n-3}}{x_{n-2}\left(\pm 1 \pm y_n x_{n-3}\right)}, \quad n \in \mathbb{N}_0,$$

with initial conditions are non-zero real numbers in [10].

In [29], Kara et al. showed that the following system of difference equations

$$x_n = \frac{x_{n-2}y_{n-3}}{y_{n-1}\left(a_n + b_n x_{n-2}y_{n-3}\right)}, \quad y_n = \frac{y_{n-2}x_{n-3}}{x_{n-1}\left(\alpha_n + \beta_n y_{n-2}x_{n-3}\right)}, \quad n \in \mathbb{N}_0, \tag{1.5}$$

where the sequences $\forall n \in \mathbb{N}_0$, (a_n), (b_n), (α_n), (β_n) and the initial values x_{-j}, y_{-j}, $j \in \{1,2,3\}$ are non-zero real numbers, could be solved in the closed form. Further, they investigated the asymptotic behavior and periodicity of solutions of system (1.5) for the case when all the sequences (a_n), (b_n), (α_n), (β_n) are constant.

Also, Halim et al. [18] gave a representation formula for the general solutions to the following two-dimensional system of difference equations

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{y_n\left(a + b y_{n-1}x_{n-2}\right)}, \quad y_{n+1} = \frac{x_{n-1}y_{n-2}}{x_n\left(a + b x_{n-1}y_{n-2}\right)}, \quad n \in \mathbb{N}_0,$$

where parameters a, b and initial values x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0 are real numbers. In addition, they gave some theoretical explanations related to the representation.

Motivated by the above papers we will expand the difference equation (1.2) to following system of difference equations

$$x_n = \frac{x_{n-4}y_{n-5}x_{n-6}}{y_{n-1}x_{n-2}\left(a+by_{n-3}x_{n-4}y_{n-5}x_{n-6}\right)}, \qquad (1.6)$$
$$y_n = \frac{y_{n-4}x_{n-5}y_{n-6}}{x_{n-1}y_{n-2}\left(c+dx_{n-3}y_{n-4}x_{n-5}y_{n-6}\right)},$$

for $n \in \mathbb{N}_0$, where the parameters a, b and the initial values x_{-i}, $i \in \{1,2,3,4,5,6\}$, are real numbers. Our aim here is to show that system (1.6) is solvable in closed form by using the method of transformation. The forbidden set of initial values for solutions of system (1.6) is also described. Some numerical examples are given to demonstrate the theoretical results.

Definition 1. A solution $(x_n, y_n)_{n\geq -6}$ of system (1.6) is called *eventually periodic* with period p if there exist $n_0 \geq -6$ such that $x_{n+p} = x_n$ and $y_{n+p} = y_n$ for all $n \geq n_0$. If $n_0 = -6$, then the solution $(x_n, y_n)_{n\geq -6}$ of system (1.6) is said to be periodic with period p.

2. Solutions of the system (1.6) in closed form

In this section is studied solvability of system (1.6) and our main results are proved, also some applicateions are given. We will deal only with well-defined solutions to system of difference equations. Hence, we assume that

$$x_n \neq 0, \quad y_n \neq 0, \quad n \geq -6,$$

and

$$a + by_{n-3}x_{n-4}y_{n-5}x_{n-6} \neq 0, \quad c + dx_{n-3}y_{n-4}x_{n-5}y_{n-6} \neq 0, \quad n \in \mathbb{N}_0.$$

It is clear that if $x_{-j} = 0$ or $y_{-j} = 0$, for some $j \in \{1,2\}$, then x_0 or y_0 is not defined. Similarly, if $x_{-3} = 0$ or $y_{-3} = 0$, then $x_1 = 0$ or $y_1 = 0$ and y_2 or x_2 is not defined, respectively, while if $x_{-l} = 0$ or $y_{-l} = 0$, for some $l \in \{4,5,6\}$, then $x_0 = 0$ or $y_0 = 0$ so that x_1 or y_1 is not defined. On the other hand, we suppose that $x_{n_0} = 0$ for some $n_0 \in \mathbb{N}_0$ and $x_n \neq 0$, for every $n < n_0$, then from system (1.6), it follows that $x_{n_0-4} = 0$ or $y_{n_0-5} = 0$, which is impossible. Thus, the set

$$\left\{\vec{\mathbb{S}} : x_{-i} = 0 \text{ or } y_{-i} = 0,\ i \in \{1,2,3,4,5,6\}\right\}$$

is a subset of forbidden set of solutions of the initial values for system (1.6). Hence, for well-defined solutions of system (1.6), $(x_n, y_n)_{n\geq -6}$, we have that

$$x_n y_n \neq 0, \quad n \geq -6, \qquad (2.1)$$

if and only if $x_{-j}y_{-j} \neq 0$, $j \in \{1,2,3,4,5,6\}$. After this, we assume that $(x_n, y_n)_{n\geq -6}$ is a solution of system (1.6) with holding the condition (2.1). Now, we will investigate the solutions in 10 different cases depending on whether the parameters are zero or nonzero.

Case 1: Let $a = c = 0$, $bd \neq 0$. In this case, system (1.6) is equivalent to the system

$$x_n = \frac{1}{by_{n-1}x_{n-2}y_{n-3}}, \quad y_n = \frac{1}{dx_{n-1}y_{n-2}x_{n-3}}, \quad n \in \mathbb{N}_0. \tag{2.2}$$

Multiplying the first equation in system (2.2) by y_{n-1}, for all $n \in \mathbb{N}_0$ and the second by x_{n-1}, for all $n \in \mathbb{N}_0$, it follows that

$$x_ny_{n-1} = \frac{1}{bx_{n-2}y_{n-3}}, \quad y_nx_{n-1} = \frac{1}{dy_{n-2}x_{n-3}}, \quad n \in \mathbb{N}_0.$$

Now, we may use the change of variables

$$k_n = x_ny_{n-1}, \quad \widehat{k}_n = y_nx_{n-1}, \quad n \geq -2,$$

and transform (2.2) into the following equations

$$k_n = \frac{1}{bk_{n-2}} = k_{n-4}, \quad \widehat{k}_n = \frac{1}{d\widehat{k}_{n-2}} = \widehat{k}_{n-4}, \quad n \geq 2,$$

which means that $(k_n)_{n\geq -2}$ and $\left(\widehat{k}_n\right)_{n\geq -2}$ are four-periodic, that is,

$$k_{4n+i} = k_i, \quad \widehat{k}_{4n+i} = \widehat{k}_i,$$

where $n \in \mathbb{N}_0$, $i \in \{-2,-1,0,1\}$, from which along with the substitutions $k_n = x_ny_{n-1}$, $\widehat{k}_n = y_nx_{n-1}$, it follows that

$$x_{4n+i} = \frac{d}{b}x_{4(n-1)+i}, \quad y_{4n+i} = \frac{b}{d}y_{4(n-1)+i},$$

for every $n \in \mathbb{N}_0$, $i \in \{-2,-1,0,1\}$, and we get

$$x_{4n+i} = \left(\frac{d}{b}\right)^n x_i, \quad y_{4n+i} = \left(\frac{b}{d}\right)^n y_i, \tag{2.3}$$

for every $n \in \mathbb{N}_0$, $i \in \{-2,-1,0,1\}$.

Case 2: Let $b = d = 0$, $ac \neq 0$. In this case, system (1.6) is written as in the form

$$x_n = \frac{x_{n-4}y_{n-5}x_{n-6}}{ay_{n-1}x_{n-2}}, \quad y_n = \frac{y_{n-4}x_{n-5}y_{n-6}}{cx_{n-1}y_{n-2}}, \quad n \in \mathbb{N}_0. \tag{2.4}$$

Multiplying the first equation in system (2.4) by $y_{n-1}x_{n-2}$, for all $n \in \mathbb{N}_0$ and the second by $x_{n-1}y_{n-2}$, for all $n \in \mathbb{N}_0$, it follows that

$$x_ny_{n-1}x_{n-2} = \frac{x_{n-4}y_{n-5}x_{n-6}}{a}, \quad y_nx_{n-1}y_{n-2} = \frac{y_{n-4}x_{n-5}y_{n-6}}{c}, \quad n \in \mathbb{N}_0. \tag{2.5}$$

By change the change of variables

$$w_n = x_n y_{n-1} x_{n-2}, \quad \widehat{w}_n = y_n x_{n-1} y_{n-2}, \quad n \geq -4,$$

system (2.5) becomes

$$w_n = \frac{1}{a} w_{n-4}, \quad \widehat{w}_n = \frac{1}{c} \widehat{w}_{n-4}, \quad n \in \mathbb{N}_0. \tag{2.6}$$

From (2.6), we see that the sequences $(w_{4m+i})_{m\geq -1}$ and $(\widehat{w}_{4m+i})_{m\geq -1}$, for $i \in \{0,1,2,3\}$, are the solutions of the homogeneous linear first-order difference equation with constant coefficients, respectively,

$$s_m = \frac{1}{a} s_{m-1}, \quad \widehat{s}_n = \frac{1}{c} \widehat{s}_{n-1}, \quad m \in \mathbb{N}_0.$$

From which it follows that

$$s_m = \left(\frac{1}{a}\right)^{m+1} s_{-1}, \quad \widehat{s}_n = \left(\frac{1}{c}\right)^{m+1} \widehat{s}_{-1}, \quad m \in \mathbb{N}_0,$$

and consequently we have

$$w_{4m+i} = \left(\frac{1}{a}\right)^{m+1} w_{i-4}, \quad \widehat{w}_{4m+i} = \left(\frac{1}{c}\right)^{m+1} \widehat{w}_{i-4}, \quad m \in \mathbb{N}_0, \tag{2.7}$$

for $i \in \{0,1,2,3\}$. From (2.6), we easily get

$$x_n = \frac{w_n \widehat{w}_{n-3}}{\widehat{w}_{n-1} w_{n-4}} x_{n-6}, \quad y_n = \frac{\widehat{w}_n w_{n-3}}{w_{n-1} \widehat{w}_{n-4}} y_{n-6}, \quad n \in \mathbb{N}_0,$$

from which along with the solutions in (2.7), we obtain the general solutions of system (2.4)

$$x_{12m+4r+s} = x_{4r+s-12} \prod_{j=0}^{m} \left(\frac{w_{4(3j+r)+s} \widehat{w}_{4(3j+r)+s-3}}{\widehat{w}_{4(3j+r)+s-1} w_{4(3j+r-1)+s}} \frac{w_{4(3j+r-1)+s-2} \widehat{w}_{4(3j+r-2)+s-1}}{\widehat{w}_{4(3j+r-1)+s-3} w_{4(3j+r-2)+s-2}} \right),$$

and

$$y_{12m+4r+s} = y_{4r+s-12} \prod_{j=0}^{m} \left(\frac{\widehat{w}_{4(3j+r)+s} w_{4(3j+r)+s-3}}{w_{4(3j+r)+s-1} \widehat{w}_{4(3j+r-1)+s}}, \frac{\widehat{w}_{4(3j+r-1)+s-2} w_{4(3j+r-2)+s-1}}{w_{4(3j+r-1)+s-3} \widehat{w}_{4(3j+r-2)+s-2}} \right),$$

for every $m \in \mathbb{N}_0$, $r \in \{1,2,3\}$ and $s \in \{2,3,4,5\}$, from which along with (2.7) it follows that

$$x_{12m+4r+s} = x_{4r+s-12} \left(\frac{c}{a^2}\right)^{m+1}, \quad y_{12m+4r+s} = y_{4r+s-12} \left(\frac{a}{c^2}\right)^{m+1}, \tag{2.8}$$

for every $m \in \mathbb{N}_0$, $r \in \{1,2,3\}$ and $s \in \{2,3,4,5\}$.

Case 3: Let $a = d = 0$, $bc \neq 0$. In this case, system (1.6) is expressed as

$$x_n = \frac{1}{b y_{n-1} x_{n-2} y_{n-3}}, \quad y_n = \frac{y_{n-4} x_{n-5} y_{n-6}}{c x_{n-1} y_{n-2}}, \quad n \in \mathbb{N}_0. \tag{2.9}$$

Multiplying the first equation in system (2.9) by $y_{n-1}x_{n-2}y_{n-3}$, for all $n \in \mathbb{N}_0$ and the second by $x_{n-1}y_{n-2}x_{n-3}$, for all $n \in \mathbb{N}_0$, it follows that

$$x_n y_{n-1} x_{n-2} y_{n-3} = \frac{1}{b}, \quad y_n x_{n-1} y_{n-2} x_{n-3} = \frac{x_{n-3} y_{n-4} x_{n-5} y_{n-6}}{c}, \quad n \in \mathbb{N}_0. \tag{2.10}$$

By using the change of variables

$$r_n = x_n y_{n-1} x_{n-2} y_{n-3}, \quad \widehat{r}_n = y_n x_{n-1} y_{n-2} x_{n-3}, \quad n \in \mathbb{N}_0, \tag{2.11}$$

system (2.10) becomes

$$\begin{cases} r_n = \frac{1}{b}, & n \in \mathbb{N}_0, \\ \widehat{r}_n = \frac{r_{n-3}}{c} = \frac{1}{bc}, & n \geq 3. \end{cases} \tag{2.12}$$

From (2.11), we easily obtain

$$x_n = \frac{r_n}{y_{n-1} x_{n-2} y_{n-3}} = \frac{r_n}{\widehat{r}_{n-1}} x_{n-4}, \quad y_n = \frac{\widehat{r}_n}{x_{n-1} y_{n-2} x_{n-3}} = \frac{\widehat{r}_n}{r_{n-1}} y_{n-4}, \quad n \geq 1,$$

from which along with (2.12), we get the general solutions of system (2.9)

$$x_{4n+i} = c^n x_i, \quad y_{4n+i} = \left(\frac{1}{c}\right)^n y_i, \tag{2.13}$$

for every $n \in \mathbb{N}_0$ and $i \in \{0,1,2,3\}$.

Case 4: Let $b = c = 0$, $ad \neq 0$. In this case, we obtain the system

$$x_n = \frac{x_{n-4} y_{n-5} x_{n-6}}{a y_{n-1} x_{n-2}}, \quad y_n = \frac{1}{d x_{n-1} y_{n-2} x_{n-3}}, \quad n \in \mathbb{N}_0. \tag{2.14}$$

By interchanging variables x_n, y_n and d instead of b and a instead of c, system (2.9) is transformed into (2.14). So, by interchanging x_{-j} and y_{-j}, for $j \in \{1,2,3,4,5,6\}$, the formula in (2.13) is transformed into the formula

$$x_{4n+i} = \left(\frac{1}{a}\right)^n x_i, \quad y_{4n+i} = a^n y_i, \tag{2.15}$$

for every $n \in \mathbb{N}_0$ and $i \in \{0,1,2,3\}$.

Case 5: Let $a = 0$, $bcd \neq 0$. In this case, system (1.6) becomes

$$x_n = \frac{1}{b y_{n-1} x_{n-2} y_{n-3}}, \quad y_n = \frac{y_{n-4} x_{n-5} y_{n-6}}{x_{n-1} y_{n-2} \left(c + d x_{n-3} y_{n-4} x_{n-5} y_{n-6}\right)}, \quad n \in \mathbb{N}_0. \tag{2.16}$$

Multiplying the first equation in system (2.16) by $y_{n-1}x_{n-2}y_{n-3}$, for all $n \in \mathbb{N}_0$ and the second by $x_{n-1}y_{n-2}x_{n-3}$, for all $n \in \mathbb{N}_0$, it follows that

$$x_n y_{n-1} x_{n-2} y_{n-3} = \frac{1}{b}, \quad n \in \mathbb{N}_0,$$

$$y_n x_{n-1} y_{n-2} x_{n-3} = \frac{x_{n-3} y_{n-4} x_{n-5} y_{n-6}}{c + d x_{n-3} y_{n-4} x_{n-5} y_{n-6}}, \quad n \in \mathbb{N}_0. \tag{2.17}$$

By using the first equation of (2.17) in the second one, we get that

$$x_n y_{n-1} x_{n-2} y_{n-3} = \frac{1}{b}, \quad n \in \mathbb{N}_0,$$

$$y_n x_{n-1} y_{n-2} x_{n-3} = \frac{1}{bc+d}, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$x_{4n+i} = \left(\frac{bc+d}{b}\right)^n x_i, \quad y_{4n+i_1} = \left(\frac{b}{bc+d}\right)^n y_{i_1}, \tag{2.18}$$

where $n \in \mathbb{N}$, $i \in \{0,1,2,3\}$ and $i_1 \in \{-1,0,1,2\}$.

Case 6: Let $b = 0$, $acd \neq 0$. In this case, system (1.6) is expressed as

$$x_n = \frac{x_{n-4} y_{n-5} x_{n-6}}{a y_{n-1} x_{n-2}}, \quad y_n = \frac{y_{n-4} x_{n-5} y_{n-6}}{x_{n-1} y_{n-2} \left(c + d x_{n-3} y_{n-4} x_{n-5} y_{n-6}\right)}, \quad n \in \mathbb{N}_0. \tag{2.19}$$

Multiplying the first equation in system (2.19) by $y_{n-1} x_{n-2} y_{n-3}$, for all $n \in \mathbb{N}_0$ and the second by $x_{n-1} y_{n-2} x_{n-3}$, for all $n \in \mathbb{N}_0$, it follows that

$$x_n y_{n-1} x_{n-2} y_{n-3} = \frac{1}{a} y_{n-3} x_{n-4} y_{n-5} x_{n-6}, \quad n \in \mathbb{N}_0,$$

$$y_n x_{n-1} y_{n-2} x_{n-3} = \frac{x_{n-3} y_{n-4} x_{n-5} y_{n-6}}{c + d x_{n-3} y_{n-4} x_{n-5} y_{n-6}}, \quad n \in \mathbb{N}_0. \tag{2.20}$$

By using the first equation of (2.20) in the second one, we get that

$$y_n x_{n-1} y_{n-2} x_{n-3} = \frac{y_{n-6} x_{n-7} y_{n-8} x_{n-9}}{ac + d y_{n-6} x_{n-7} y_{n-8} x_{n-9}}, \quad n \geq 3. \tag{2.21}$$

By employing the substitution $y_n x_{n-1} y_{n-2} x_{n-3} = \frac{1}{s_n}$, $n \geq -3$, to (2.21), we obtain the linear six-order equation

$$s_n = ac s_{n-6} + d, \qquad n \geq 3, \tag{2.22}$$

from which it easily follows that

$$s_{6n+i_2} = \frac{d + (ac)^n \left((1-ac) s_{i_2} - d\right)}{1-ac}, \quad n \in \mathbb{N}_0, \tag{2.23}$$

if $ac \neq 1$, and

$$s_{6n+i_2} = s_{i_2} + dn, \qquad n \in \mathbb{N}_0, \tag{2.24}$$

if $ac = 1$, where $i_2 \in \{-3,-2,-1,0,1,2\}$. The equalities in (2.23) and (2.24) are formulas for the general solution of (2.22). From the substitution $y_n x_{n-1} y_{n-2} x_{n-3} = \frac{1}{s_n}$, for $n \geq -3$, and $x_n y_{n-1} x_{n-2} y_{n-3} = \frac{1}{a s_{n-3}}$, for $n \in \mathbb{N}_0$, we get

$$x_n = \frac{s_{n-1} s_{n-5} s_{n-9}}{a^3 s_{n-3} s_{n-7} s_{n-11}} x_{n-12}, \quad n \geq 8,$$

$$y_n=\frac{a^3 s_{n-12}}{s_n}y_{n-12},\quad n\geq 9,$$

from which it follows that

$$x_{12m+6r_1+i_1}=x_{6r_1+i_1-12}\frac{1}{a^{3m+3}}\prod_{j=0}^{m}\frac{s_{12j+6r_1+i_1-1}s_{12j+6r_1+i_1-5}s_{12j+6r_1+i_1-9}}{s_{12j+6r_1+i_1-3}s_{12j+6r_1+i_1-7}s_{12j+6r_1+i_1-11}},$$

$$\begin{aligned}y_{12m+6r_2+i_2}&=y_{6r_2+i_2-12}a^{3m+3}\prod_{j=0}^{m}\frac{s_{6(2j+r_2-2)+i_2}}{s_{6(2j+r_2)+i_2}}\\&=y_{6r_2+i_2-12}a^{3m+3}\frac{s_{6(r_2-2)+i_2}}{s_{6(2m+r_2)+i_2}},\end{aligned}$$

where

$m\in\mathbb{N}_0,\quad r_1\in\{1,2\},\quad r_2\in\{2,3\},\quad i_1\in\{2,3,4,5,6,7\}$ and $i_2\in\{-3,-2,-1,0,1,2\}$.

From this, along with (2.23) and (2.24), we get

$$\begin{aligned}x_{12m+6r_1+i_1}&=x_{6r_1+i_1-12}\frac{1}{a^{3m+3}}\\&\times\prod_{j=0}^{m}\frac{d+(ac)^{\left(2j+r_1+1+\lfloor\frac{i_1-4}{6}\rfloor\right)}\left((1-ac)s_{i_1-7-6\lfloor\frac{i_1-4}{6}\rfloor}-d\right)}{d+(ac)^{\left(2j+r_1+1+\lfloor\frac{i_1-6}{6}\rfloor\right)}\left((1-ac)s_{i_1-9-6\lfloor\frac{i_1-6}{6}\rfloor}-d\right)}\\&\times\frac{d+(ac)^{\left(2j+r_1+\lfloor\frac{i_1-2}{6}\rfloor\right)}\left((1-ac)s_{i_1-5-6\lfloor\frac{i_1-2}{6}\rfloor}-d\right)}{d+(ac)^{\left(2j+r_1+\lfloor\frac{i_1-4}{6}\rfloor\right)}\left((1-ac)s_{i_1-7-6\lfloor\frac{i_1-4}{6}\rfloor}-d\right)}\\&\times\frac{d+(ac)^{\left(2j+r_1+\lfloor\frac{i_1-6}{6}\rfloor\right)}\left((1-ac)s_{i_1-9-6\lfloor\frac{i_1-6}{6}\rfloor}-d\right)}{d+(ac)^{\left(2j+r_1-1+\lfloor\frac{i_1-2}{6}\rfloor\right)}\left((1-ac)s_{i_1-5-6\lfloor\frac{i_1-2}{6}\rfloor}-d\right)},\end{aligned}\tag{2.25}$$

$$\begin{aligned}y_{12m+6r_2+i_2}&=y_{6r_2+i_2-12}a^{3m+3}\prod_{j=0}^{m}\frac{d+(ac)^{(2j+r_2-2)}((1-ac)s_{i_2}-d)}{d+(ac)^{(2j+r_2)}((1-ac)s_{i_2}-d)}\\&=y_{6r_2+i_2-12}a^{3m+3}\frac{d+(ac)^{(r_2-2)}((1-ac)s_{i_2}-d)}{d+(ac)^{(2m+r_2)}((1-ac)s_{i_2}-d)},\end{aligned}\tag{2.26}$$

if $ac\neq 1$, and

$$x_{12m+6r_1+i_1}=x_{6r_1+i_1-12}\frac{1}{a^{3m+3}}\tag{2.27}$$

$$\times\prod_{j=0}^{m}\left(\frac{s_{i_1-7-6\lfloor\frac{i_1-4}{6}\rfloor}+d\left(2j+r_1+1+\lfloor\frac{i_1-4}{6}\rfloor\right)}{s_{i_1-9-6\lfloor\frac{i_1-6}{6}\rfloor}+d\left(2j+r_1+1+\lfloor\frac{i_1-6}{6}\rfloor\right)}\right)$$
$$\times\left(\frac{s_{i_1-5-6\lfloor\frac{i_1-2}{6}\rfloor}+d\left(2j+r_1+\lfloor\frac{i_1-2}{6}\rfloor\right)}{s_{i_1-7-6\lfloor\frac{i_1-4}{6}\rfloor}+d\left(2j+r_1+\lfloor\frac{i_1-4}{6}\rfloor\right)}\right)$$
$$\times\left(\frac{s_{i_1-9-6\lfloor\frac{i_1-6}{6}\rfloor}+d\left(2j+r_1+\lfloor\frac{i_1-6}{6}\rfloor\right)}{s_{i_1-5-6\lfloor\frac{i_1-2}{6}\rfloor}+d\left(2j+r_1-1+\lfloor\frac{i_1-2}{6}\rfloor\right)}\right),$$

$$y_{12m+6r_2+i_2}=y_{6r_2+i_2-12}a^{3m+3}\frac{s_{i_2}+d\left(r_2-2\right)}{s_{i_2}+d\left(2m+r_2\right)},\tag{2.28}$$

if $ac=1$, where $m\in\mathbb{N}_0$, $r_1\in\{1,2\}$, $r_2\in\{2,3\}$, $i_1\in\{2,3,4,5,6,7\}$ and $i_2\in\{-3,-2,-1,0,1,2\}$.

Case 7: Let $c=0$, $abd\neq 0$. In this case, system (1.6) becomes

$$x_n=\frac{x_{n-4}y_{n-5}x_{n-6}}{y_{n-1}x_{n-2}\left(a+by_{n-3}x_{n-4}y_{n-5}x_{n-6}\right)},\quad y_n=\frac{1}{dx_{n-1}y_{n-2}x_{n-3}},\quad n\in\mathbb{N}_0.\tag{2.29}$$

By interchanging variables x_n, y_n and d instead of b, a instead of c and b instead of d, system (2.16) is transformed into (2.29). So, by interchanging x_{-j} and y_{-j}, for $j\in\{1,2,3,4,5,6\}$, the formula in (2.18) is transformed into the formula

$$x_{4n+i_1}=\left(\frac{d}{ad+b}\right)^n x_{i_1},\quad y_{4n+i}=\left(\frac{ad+b}{d}\right)^n y_i,\tag{2.30}$$

where $n\in\mathbb{N}$, $i\in\{0,1,2,3\}$ and $i_1\in\{-1,0,1,2\}$.

Case 8: Let $d=0$, $abc\neq 0$. In this case, system (1.6) is equivalent to the system

$$x_n=\frac{x_{n-4}y_{n-5}x_{n-6}}{y_{n-1}x_{n-2}\left(a+by_{n-3}x_{n-4}y_{n-5}x_{n-6}\right)},\quad y_n=\frac{y_{n-4}x_{n-5}y_{n-6}}{cx_{n-1}y_{n-2}},\quad n\in\mathbb{N}_0.\tag{2.31}$$

Similarly, by interchanging variables x_n, y_n and b instead of d, c instead of a and a instead of c, system (2.19) is transformed into (2.31). So, by interchanging x_{-j} and y_{-j}, for $j\in\{1,2,3,4,5,6\}$, the formula in (2.25)-(2.28) is transformed into the formula

$$x_{12m+6r_2+i_2}=x_{6r_2+i_2-12}c^{3m+3}\prod_{j=0}^{m}\frac{b+(ac)^{(2j+r_2-2)}\left(\left(1-ac\right)s_{i_2}-b\right)}{b+(ac)^{(2j+r_2)}\left(\left(1-ac\right)s_{i_2}-b\right)}$$
$$=x_{6r_2+i_2-12}c^{3m+3}\frac{b+(ac)^{(r_2-2)}\left(\left(1-ac\right)s_{i_2}-b\right)}{b+(ac)^{(2m+r_2)}\left(\left(1-ac\right)s_{i_2}-b\right)},\tag{2.32}$$

$$y_{12m+6r_1+i_1}=y_{6r_1+i_1-12}\frac{1}{c^{3m+3}}\tag{2.33}$$

$$\times\prod_{j=0}^{m}\frac{b+(ac)^{\left(2j+r_1+1+\lfloor\frac{i_1-4}{6}\rfloor\right)}\left((1-ac)s_{i_1-7-6\lfloor\frac{i_1-4}{6}\rfloor}-b\right)}{b+(ac)^{\left(2j+r_1+1+\lfloor\frac{i_1-6}{6}\rfloor\right)}\left((1-ac)s_{i_1-9-6\lfloor\frac{i_1-6}{6}\rfloor}-b\right)}$$
$$\times\frac{b+(ac)^{\left(2j+r_1+\lfloor\frac{i_1-2}{6}\rfloor\right)}\left((1-ac)s_{i_1-5-6\lfloor\frac{i_1-2}{6}\rfloor}-b\right)}{b+(ac)^{\left(2j+r_1+\lfloor\frac{i_1-4}{6}\rfloor\right)}\left((1-ac)s_{i_1-7-6\lfloor\frac{i_1-4}{6}\rfloor}-b\right)}$$
$$\times\frac{b+(ac)^{\left(2j+r_1+\lfloor\frac{i_1-6}{6}\rfloor\right)}\left((1-ac)s_{i_1-9-6\lfloor\frac{i_1-6}{6}\rfloor}-b\right)}{b+(ac)^{\left(2j+r_1-1+\lfloor\frac{i_1-2}{6}\rfloor\right)}\left((1-ac)s_{i_1-5-6\lfloor\frac{i_1-2}{6}\rfloor}-b\right)},$$

if $ac\neq 1$, and

$$x_{12m+6r_2+i_2}=x_{6r_2+i_2-12}c^{3m+3}\frac{s_{i_2}+b(r_2-2)}{s_{i_2}+b(2m+r_2)},\tag{2.34}$$

$$\begin{aligned}y_{12m+6r_1+i_1}=&y_{6r_1+i_1-12}\frac{1}{c^{3m+3}}\prod_{j=0}^{m}\left(\frac{s_{i_1-7-6\lfloor\frac{i_1-4}{6}\rfloor}+b\left(2j+r_1+1+\lfloor\frac{i_1-4}{6}\rfloor\right)}{s_{i_1-9-6\lfloor\frac{i_1-6}{6}\rfloor}+b\left(2j+r_1+1+\lfloor\frac{i_1-6}{6}\rfloor\right)}\right)\\&\times\left(\frac{s_{i_1-5-6\lfloor\frac{i_1-2}{6}\rfloor}+b\left(2j+r_1+\lfloor\frac{i_1-2}{6}\rfloor\right)}{s_{i_1-7-6\lfloor\frac{i_1-4}{6}\rfloor}+b\left(2j+r_1+\lfloor\frac{i_1-4}{6}\rfloor\right)}\right)\\&\times\left(\frac{s_{i_1-9-6\lfloor\frac{i_1-6}{6}\rfloor}+b\left(2j+r_1+\lfloor\frac{i_1-6}{6}\rfloor\right)}{s_{i_1-5-6\lfloor\frac{i_1-2}{6}\rfloor}+b\left(2j+r_1-1+\lfloor\frac{i_1-2}{6}\rfloor\right)}\right),\end{aligned}\tag{2.35}$$

if $ac=1$, where $m\in\mathbb{N}_0$, $r_1\in\{1,2\}$, $r_2\in\{2,3\}$, $i_1\in\{2,3,4,5,6,7\}$ and $i_2\in\{-3,-2,-1,0,1,2\}$.

Case 9: Let $abcd\neq 0$. Employing the change of variables

$$u_n=\frac{1}{x_ny_{n-1}x_{n-2}y_{n-3}},\quad v_n=\frac{1}{y_nx_{n-1}y_{n-2}x_{n-3}},\quad n\geq -3,\tag{2.36}$$

system (1.6) is transformed into the following system of linear difference equations

$$u_n=av_{n-3}+b,\quad v_n=cu_{n-3}+d,\quad n\in\mathbb{N}_0,\tag{2.37}$$

from which it follows that

$$u_n=acu_{n-6}+b+ad,\tag{2.38}$$
$$v_n=acv_{n-6}+d+bc,\quad n\geq 3,\tag{2.39}$$

which are nonhomogeneous linear sixth-order difference equations with constant coefficients. If we apply the decomposition of indexes $n \to 6m + j$, for some $m \in \mathbb{N}_0$ and $j \in \{3,4,5,6,7,8\}$, to (2.38) and (2.39), then they become

$$u_{6m+j} = acu_{6(m-1)+j} + b + ad, \tag{2.40}$$

$$v_{6m+j} = acv_{6(m-1)+j} + d + bc, \tag{2.41}$$

which are first-order 6-equations. Let $u_{6m+j} = u_m^{(j)} = z_m$ and $v_{6m+j} = v_m^{(j)} = \widehat{z}_m$, for $m \geq -1$ and $j \in \{3,4,5,6,7,8\}$. Then Eqs. (2.40) and (2.41) can be written in the form

$$z_m = acz_{m-1} + b + ad, \tag{2.42}$$

$$\widehat{z}_m = ac\widehat{z}_{m-1} + d + bc, \quad m \in \mathbb{N}_0, \tag{2.43}$$

which are nonhomogeneous linear first-order difference equations with constant coefficients. Contrary to the usual, here, Eqs. (2.42) and (2.43) can be solved by using transformation $z_m = w_m + C$ and $\widehat{z}_m = \widehat{w}_m + \widehat{C}$, for chosen suitable values of C and $\widehat{C}$, to reduce them to homogeneous linear first-order difference equations with constant coefficients. That is, by employing these transformations to Eqs. (2.42) and (2.43), they become

$$w_m = acw_{m-1} + C(ac-1) + b + ad,$$

$$\widehat{w}_m = ac\widehat{w}_{m-1} + \widehat{C}(ac-1) + d + bc, \quad m \in \mathbb{N}_0,$$

which for $C = \frac{b+ad}{1-ac}$ and $\widehat{C} = \frac{d+bc}{1-ac}$, if $ac \neq 1$, reduce to the next equations

$$w_m = acw_{m-1},$$

$$\widehat{w}_m = ac\widehat{w}_{m-1}, \quad m \in \mathbb{N}_0,$$

whose general solutions are

$$w_m = w_{-1}(ac)^{m+1},$$

$$\widehat{w}_m = \widehat{w}_{-1}(ac)^{m+1}, \quad m \geq -1.$$

From this and by using the relation $z_m = w_m + \frac{b+ad}{1-ac}$ and $\widehat{z}_m = \widehat{w}_m + \frac{d+bc}{1-ac}$, we obtain

$$z_m = \left(z_{-1} + \frac{b+ad}{ac-1}\right)(ac)^{m+1} + \frac{b+ad}{1-ac}, \tag{2.44}$$

$$\widehat{z}_m = \left(\widehat{z}_{-1} + \frac{d+bc}{ac-1}\right)(ac)^{m+1} + \frac{d+bc}{1-ac}, \quad m \geq -1, \tag{2.45}$$

when $ac \neq 1$. If $ac = 1$, then Eqs. (2.42) and (2.43) becomes

$$z_m = z_{m-1} + b + ad,$$

$$\widehat{z}_m = \widehat{z}_{m-1} + d + bc, \quad m \geq -1,$$

from which they immediately follow that

$$z_m = z_{-1} + (m+1)(b+ad), \tag{2.46}$$

$$\widehat{z}_m = \widehat{z}_{-1} + (m+1)(d+bc), \quad m \geq -1. \tag{2.47}$$

From (2.44), (2.45), (2.46) and (2.47) and the substitutions $u_{6m+j} = u_m^{(j)} = z_m$ and $v_{6m+j} = v_m^{(j)} = \widehat{z}_m$, we can write the solutions of (2.40) and (2.41) as in the following

$$u_{6m+j} = \left(u_{j-6} + \frac{b+ad}{ac-1}\right)(ac)^{m+1} + \frac{b+ad}{1-ac},$$

$$v_{6m+j} = \left(v_{j-6} + \frac{d+bc}{ac-1}\right)(ac)^{m+1} + \frac{d+bc}{1-ac},$$

$m \geq -1$, $j \in \{3,4,5,6,7,8\}$, when $ac \neq 1$, or

$$u_{6m+j} = u_{j-6} + (m+1)(b+ad),$$

$$v_{6m+j} = v_{j-6} + (m+1)(d+bc).$$

$m \geq -1$, $j \in \{3,4,5,6,7,8\}$, when $ac = 1$. Now note that from (2.36) we have that

$$x_m = \frac{1}{u_m y_{m-1} x_{m-2} y_{m-3}} = \frac{v_{m-1}}{u_m} x_{m-4}$$
$$= \frac{v_{m-1} v_{m-5}}{u_m u_{m-4}} x_{m-8} = \frac{v_{m-1} v_{m-5} v_{m-9}}{u_m u_{m-4} u_{m-8}} x_{m-12},$$

$$y_m = \frac{1}{v_m x_{m-1} y_{m-2} x_{m-3}} = \frac{u_{m-1}}{v_m} y_{m-4}$$
$$= \frac{u_{m-1} u_{m-5}}{v_m v_{m-4}} y_{m-8} = \frac{u_{m-1} u_{m-5} u_{m-9}}{v_m v_{m-4} v_{m-8}} y_{m-12},$$

for $m \geq 6$, from which it follows that

$$x_{12m+6r+i} = x_{6r+i-12} \prod_{s=0}^{m} \frac{v_{6\left(2s+r+\lfloor \frac{i-4}{6} \rfloor\right)+i-1-6\lfloor \frac{i-4}{6} \rfloor}}{u_{6\left(2s+r+\lfloor \frac{i-3}{6} \rfloor\right)+i-6\lfloor \frac{i-3}{6} \rfloor}} \frac{v_{6\left(2s+r+\lfloor \frac{i-8}{6} \rfloor\right)+i-5-6\lfloor \frac{i-8}{6} \rfloor}}{u_{6\left(2s+r+\lfloor \frac{i-7}{6} \rfloor\right)+i-4-6\lfloor \frac{i-7}{6} \rfloor}}$$
$$\times \frac{v_{6\left(2s+r+\lfloor \frac{i-12}{6} \rfloor\right)+i-9-6\lfloor \frac{i-12}{6} \rfloor}}{u_{6\left(2s+r+\lfloor \frac{i-11}{6} \rfloor\right)+i-8-6\lfloor \frac{i-11}{6} \rfloor}}$$

and

$$y_{12m+6r+i} = y_{6r+i-12} \prod_{s=0}^{m} \frac{u_{6\left(2s+r+\lfloor \frac{i-4}{6} \rfloor\right)+i-1-6\lfloor \frac{i-4}{6} \rfloor}}{v_{6\left(2s+r+\lfloor \frac{i-3}{6} \rfloor\right)+i-6\lfloor \frac{i-3}{6} \rfloor}} \frac{u_{6\left(2s+r+\lfloor \frac{i-8}{6} \rfloor\right)+i-5-6\lfloor \frac{i-8}{6} \rfloor}}{v_{6\left(2s+r+\lfloor \frac{i-7}{6} \rfloor\right)+i-4-6\lfloor \frac{i-7}{6} \rfloor}}$$
$$\times \frac{u_{6\left(2s+r+\lfloor \frac{i-12}{6} \rfloor\right)+i-9-6\lfloor \frac{i-12}{6} \rfloor}}{v_{6\left(2s+r+\lfloor \frac{i-11}{6} \rfloor\right)+i-8-6\lfloor \frac{i-11}{6} \rfloor}},$$

for every $m \in \mathbb{N}_0$, $r \in \{1,2\}$ and $i \in \{0,1,2,3,4,5\}$.

Case $ac \neq 1$**:** In this case, we obtain that

$$x_{12m+6r+i} = x_{6r+i-12} \times \prod_{s=0}^{m} \frac{d+bc+\Big((1-ac)\,v_{-7+i-6\lfloor\frac{i-4}{6}\rfloor}-(d+bc)\Big)(ac)^{2s+r+1+\lfloor\frac{i-4}{6}\rfloor}}{b+ad+\Big((1-ac)\,u_{-6+i-6\lfloor\frac{i-3}{6}\rfloor}-(b+ad)\Big)(ac)^{2s+r+1+\lfloor\frac{i-3}{6}\rfloor}} \times \frac{d+bc+\Big((1-ac)\,v_{-11+i-6\lfloor\frac{i-8}{6}\rfloor}-(d+bc)\Big)(ac)^{2s+r+1+\lfloor\frac{i-8}{6}\rfloor}}{b+ad+\Big((1-ac)\,u_{-10+i-6\lfloor\frac{i-7}{6}\rfloor}-(b+ad)\Big)(ac)^{2s+r+1+\lfloor\frac{i-7}{6}\rfloor}} \times \frac{d+bc+\Big((1-ac)\,v_{-15+i-6\lfloor\frac{i-12}{6}\rfloor}-(d+bc)\Big)(ac)^{2s+r+1+\lfloor\frac{i-12}{6}\rfloor}}{b+ad+\Big((1-ac)\,u_{-14+i-6\lfloor\frac{i-11}{6}\rfloor}-(b+ad)\Big)(ac)^{2s+r+1+\lfloor\frac{i-11}{6}\rfloor}} \tag{2.48}$$

and

$$y_{12m+6r+i} = y_{6r+i-12} \times \prod_{s=0}^{m} \frac{b+ad+\Big((1-ac)\,u_{-7+i-6\lfloor\frac{i-4}{6}\rfloor}-(b+ad)\Big)(ac)^{2s+r+1+\lfloor\frac{i-4}{6}\rfloor}}{d+bc+\Big((1-ac)\,v_{-6+i-6\lfloor\frac{i-3}{6}\rfloor}-(d+bc)\Big)(ac)^{2s+r+1+\lfloor\frac{i-3}{6}\rfloor}} \times \frac{b+ad+\Big((1-ac)\,u_{-11+i-6\lfloor\frac{i-8}{6}\rfloor}-(b+ad)\Big)(ac)^{2s+r+1+\lfloor\frac{i-8}{6}\rfloor}}{d+bc+\Big((1-ac)\,v_{-10+i-6\lfloor\frac{i-7}{6}\rfloor}-(d+bc)\Big)(ac)^{2s+r+1+\lfloor\frac{i-7}{6}\rfloor}} \times \frac{b+ad+\Big((1-ac)\,u_{-15+i-6\lfloor\frac{i-12}{6}\rfloor}-(b+ad)\Big)(ac)^{2s+r+1+\lfloor\frac{i-12}{6}\rfloor}}{d+bc+\Big((1-ac)\,v_{-14+i-6\lfloor\frac{i-11}{6}\rfloor}-(d+bc)\Big)(ac)^{2s+r+1+\lfloor\frac{i-11}{6}\rfloor}}, \tag{2.49}$$

for every $m \in \mathbb{N}_0$, $r \in \{1,2\}$ and $i \in \{0,1,2,3,4,5\}$.

Case $ac = 1$**:** In this case, we get that

$$x_{12m+6r+i} = x_{6r+i-12} \prod_{s=0}^{m} \frac{v_{-7+i-6\lfloor\frac{i-4}{6}\rfloor}+\big(2s+r+1+\lfloor\frac{i-4}{6}\rfloor\big)(d+bc)}{u_{-6+i-6\lfloor\frac{i-3}{6}\rfloor}+\big(2s+r+1+\lfloor\frac{i-3}{6}\rfloor\big)(b+ad)} \times \frac{v_{-11+i-6\lfloor\frac{i-8}{6}\rfloor}+\big(2s+r+1+\lfloor\frac{i-8}{6}\rfloor\big)(d+bc)}{u_{-10+i-6\lfloor\frac{i-7}{6}\rfloor}+\big(2s+r+1+\lfloor\frac{i-7}{6}\rfloor\big)(b+ad)} \times \frac{v_{-15+i-6\lfloor\frac{i-12}{6}\rfloor}+\big(2s+r+1+\lfloor\frac{i-12}{6}\rfloor\big)(d+bc)}{u_{-14+i-6\lfloor\frac{i-11}{6}\rfloor}+\big(2s+r+1+\lfloor\frac{i-11}{6}\rfloor\big)(b+ad)} \tag{2.50}$$

and

$$y_{12m+6r+i} = y_{6r+i-12} \prod_{s=0}^{m} \frac{u_{-7+i-6\lfloor\frac{i-4}{6}\rfloor} + \left(2s+r+1+\lfloor\frac{i-4}{6}\rfloor\right)(b+ad)}{v_{-6+i-6\lfloor\frac{i-3}{6}\rfloor} + \left(2s+r+1+\lfloor\frac{i-3}{6}\rfloor\right)(d+bc)} \times \frac{u_{-11+i-6\lfloor\frac{i-8}{6}\rfloor} + \left(2s+r+1+\lfloor\frac{i-8}{6}\rfloor\right)(b+ad)}{v_{-10+i-6\lfloor\frac{i-7}{6}\rfloor} + \left(2s+r+1+\lfloor\frac{i-7}{6}\rfloor\right)(d+bc)} \times \frac{u_{-15+i-6\lfloor\frac{i-12}{6}\rfloor} + \left(2s+r+1+\lfloor\frac{i-12}{6}\rfloor\right)(b+ad)}{v_{-14+i-6\lfloor\frac{i-11}{6}\rfloor} + \left(2s+r+1+\lfloor\frac{i-11}{6}\rfloor\right)(d+bc)}, \tag{2.51}$$

for every $m \in \mathbb{N}_0$, $r \in \{1,2\}$ and $i \in \{0,1,2,3,4,5\}$.

From the above considerations, we see that the following result holds.

Theorem 1. *Assume that a,b,c,d and the initial values x_{-i}, y_{-i}, $i \in \{1,2,3,4,5,6\}$, are real numbers. Then, the next statements hold.*

a) *If $a = c = 0$ and $bd \neq 0$, then the general solutions of system (1.6) is given by formulas in (2.3).*
b) *If $b = d = 0$ and $ac \neq 0$, then the general solutions of system (1.6) is given by formulas in (2.8).*
c) *If $a = d = 0$ and $bc \neq 0$, then the general solutions of system (1.6) is given by formulas in (2.13).*
d) *If $b = c = 0$ and $ad \neq 0$, then the general solutions of system (1.6) is given by formulas in (2.15).*
e) *If $a = 0$ and $bcd \neq 0$, then the general solutions of system (1.6) is given by formulas in (2.18).*
f) *If $b = 0$, $acd \neq 0$ and $ac \neq 1$, then the general solutions of system (1.6) is given by formulas in (2.25)-(2.26).*
g) *If $b = 0$, $acd \neq 0$ and $ac = 1$, then the general solutions of system (1.6) is given by formulas in (2.27)-(2.28).*
h) *If $c = 0$ and $abd \neq 0$, then the general solutions of system (1.6) is given by formulas in (2.30).*
i) *If $d = 0$, $abc \neq 0$ and $ac \neq 1$, then the general solutions of system (1.6) is given by formulas in (2.32)-(2.33).*
j) *If $d = 0$, $abc \neq 0$ and $ac = 1$, then the general solutions of system (1.6) is given by formulas in (2.34)-(2.35).*
k) *If $abcd \neq 0$ and $ac \neq 1$, then the general solutions of system (1.6) is given by formulas in (2.48) and (2.49).*
l) *If $abcd \neq 0$ and $ac = 1$, then the general solutions of system (1.6) is given by formulas in (2.50) and (2.51).*

By the following theorem, we characterize the forbidden set of the initial values for system (1.6).

Theorem 2. *The forbidden set of the initial values for system (1.6) is the union of two sets*

$$\left\{\vec{\mathbb{S}} : x_{-i} = 0 \text{ or } y_{-i} = 0,\ i \in \{1,2,3,4,5,6\}\right\}$$

and

$$\bigcup_{m\in\mathbb{N}_0}\bigcup_{j=3}^{8}\left\{\vec{\mathbb{S}} : \frac{1}{x_{j-3}y_{j-4}x_{j-5}y_{j-6}} = (f\circ g)^{-m}\left(-\frac{d}{c}\right),\ or\right.$$
$$\left.\frac{1}{y_{j-3}x_{j-4}y_{j-5}x_{j-6}} = (g\circ f)^{-m}\left(-\frac{b}{a}\right)\right\},$$

where $\vec{\mathbb{S}} = (x_{-6},x_{-5},x_{-4},x_{-3},x_{-2},x_{-1},y_{-6},y_{-5},y_{-4},y_{-3},y_{-2},y_{-1})$.

Proof. We have already obtained that the set

$$\left\{\vec{\mathbb{S}} : x_{-i} = 0 \text{ or } y_{-i} = 0,\ i \in \{1,2,3,4,5,6\}\right\}$$

belongs to the forbidden set of the initial values for system (1.6). If $x_{-i} \neq 0 \neq y_{-i}$, $i \in \{1,2,3,4,5,6\}$, (i.e., $x_n y_n \neq 0,\ n \geq -6$), then such a solution $(x_n,y_n)_{n\geq -6}$ is not defined if and only if

$$a + b y_{n-3}x_{n-4}y_{n-5}x_{n-6} = 0, \quad c + d x_{n-3}y_{n-4}x_{n-5}y_{n-6} = 0,$$

for some $n \in \mathbb{N}_0$, which correspond to the statements $y_{n-3}x_{n-4}y_{n-5}x_{n-6} = -\frac{a}{b}$ and $x_{n-3}y_{n-4}x_{n-5}y_{n-6} = -\frac{c}{d}$, for $n \in \mathbb{N}_0$, respectively. Hence, from (2.36), we get

$$u_{n-3} = -\frac{d}{c} \quad \text{and} \quad v_{n-3} = -\frac{b}{a}, \tag{2.52}$$

for $n \in \mathbb{N}_0$. Now, we consider system (2.37) and the functions

$$f(t) = at + b, \quad g(t) = ct + d$$

which correspond to the equations of (2.37). Thus, we can describe the solutions of (2.38) and (2.39) as follows:

$$u_{6(m-1)+j} = (f\circ g)^m (u_{j-6}),$$
$$v_{6(m-1)+j} = (g\circ f)^m (v_{j-6}),$$

for $m \in \mathbb{N}_0$ and $j \in \{3,4,5,6,7,8\}$. By using (2.52), we get

$$u_{j-6} = (f\circ g)^{-m}\left(-\frac{d}{c}\right), \tag{2.53}$$

$$v_{j-6} = (g\circ f)^{-m}\left(-\frac{b}{a}\right), \tag{2.54}$$

for $m \geq -1$ and $j \in \{3,4,5,6,7,8\}$, where $f^{-1}(t) = \frac{t-b}{a}$ and $g^{-1}(t) = \frac{t-d}{c}$. This means that if one of the conditions in (2.53) and (2.54) holds, then $6(m-1)$-th or

$6m$-th iteration in (1.6) can not be calculated. Consequently, desired result follows from (2.36).

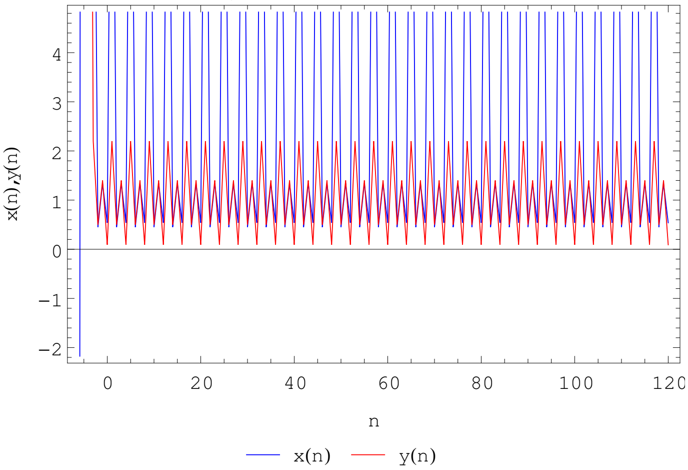
□

3. NUMERICAL EXAMPLES

To support our theoretical results, we present numerical examples for the solutions of system (1.6) regard to the different values of a, b, c and d.

Example 1. Consider the system (1.6) with the initial values $x_{-6} = -15.71$, $x_{-5} = 87.76$, $x_{-4} = 48.97$, $x_{-3} = 12.23$, $x_{-2} = 0.45$, $x_{-1} = 1.34$, $y_{-6} = 5.2$, $y_{-5} = 8.6$, $y_{-4} = 22.7$, $y_{-3} = 2.2$, $y_{-2} = 0.5$, $y_{-1} = 1.4$ and the parameters, $a = 0$, $b = 1.34$, $c = 0$, $d = 1.34$ the solutions are represented as in the Figure 1.

FIGURE 1. Plots of x_n, y_n in case $a = 0$, $b = 1.34$, $c = 0$, $d = 1.34$



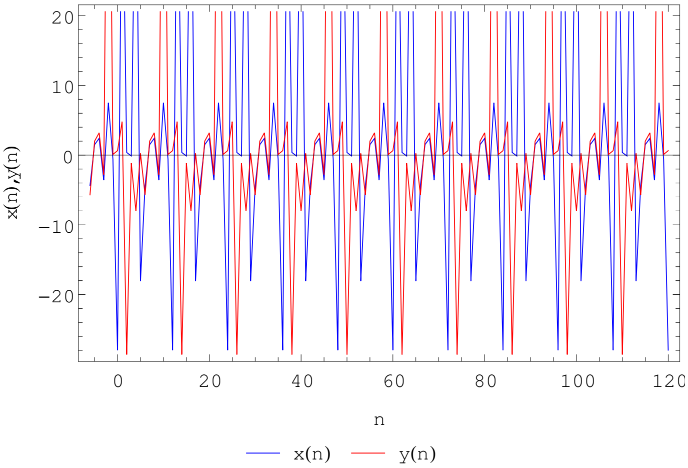
In this case, equations in (2.3) are satisfied. Hence, the solutions of system (1.6) have periodic solutions with period four.

Example 2. Consider the system (1.6) with the initial values $x_{-6} = -4.37$, $x_{-5} = 1.5$, $x_{-4} = 2.4$, $x_{-3} = -3.6$, $x_{-2} = 7.5$, $x_{-1} = -0.6$, $y_{-6} = -5.7$, $y_{-5} = 2$, $y_{-4} = 3.2$, $y_{-3} = -2.8$, $y_{-2} = 71.5$, $y_{-1} = 0.1$ and the parameters, $a = 1$, $b = 0$, $c = 1$, $d = 0$ the solutions are represented as in the Figure 2.

In this case, equations in (2.8) are satisfied. Hence, the solutions of system (1.6) have periodic solutions with period twelve.

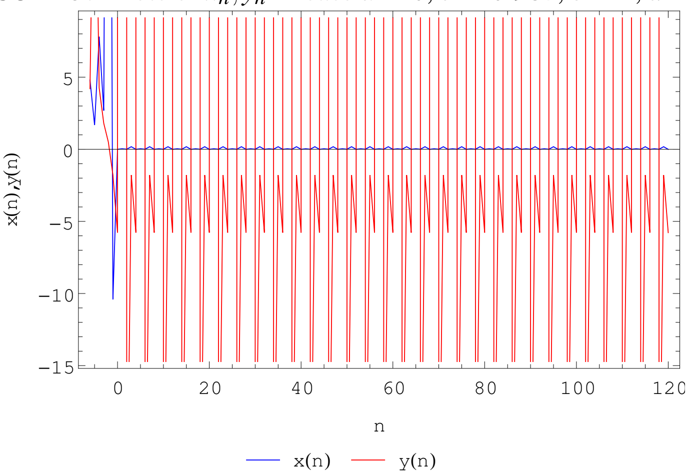
Example 3. Consider the system (1.6) with the initial values $x_{-6} = 4.8$, $x_{-5} = 1.68$, $x_{-4} = 7.8$, $x_{-3} = 2.68$, $x_{-2} = 70.8$, $x_{-1} = -10.4$, $y_{-6} = 4.2$, $y_{-5} = 25.6$, $y_{-4} = 4.26$,

FIGURE 2. Plots of x_n, y_n in case $a = 1$, $b = 0$, $c = 1$, $d = 0$



$y_{-3} = 1.8$, $y_{-2} = 0.5$, $y_{-1} = -1.79$ and the parameters, $a = 0$, $b = 0.987$, $c = 1$, $d = 0$ the solutions are represented as in the Figure 3.

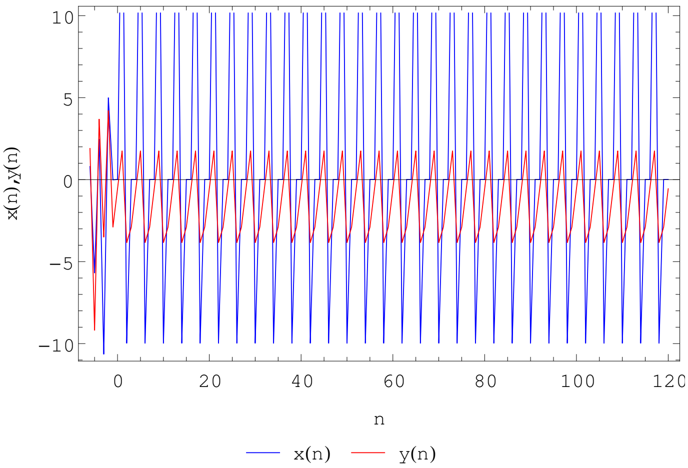
FIGURE 3. Plots of x_n, y_n in case $a = 0$, $b = 0.987$, $c = 1$, $d = 0$



In this case, equations in (2.13) are satisfied. Hence, the solutions of system (1.6) have periodic solutions with period four.

Example 4. Consider the system (1.6) with the initial values $x_{-6} = 0.8$, $x_{-5} = -5.7$, $x_{-4} = 2.47$, $x_{-3} = -10.65$, $x_{-2} = 5$, $x_{-1} = -0.01$, $y_{-6} = 1.9$, $y_{-5} = -9.2$, $y_{-4} = 3.7$, $y_{-3} = -3.51$, $y_{-2} = 4.22$, $y_{-1} = -2.9$ and the parameters, $a = 0$, $b = 2$, $c = 3$, $d = -4$ the solutions are represented as in the Figure 4.

FIGURE 4. Plots of x_n, y_n in case $a = 0$, $b = 2$, $c = 3$, $d = -4$



In this case, equations in (2.18) are satisfied. Hence, the solutions of system (1.6) have periodic solutions with period four.

4. CONCLUSION

In this study, we obtain solutions of the following system of difference equations

$$x_n = \frac{x_{n-4}y_{n-5}x_{n-6}}{y_{n-1}x_{n-2}\left(a + by_{n-3}x_{n-4}y_{n-5}x_{n-6}\right)}, \quad y_n = \frac{y_{n-4}x_{n-5}y_{n-6}}{x_{n-1}y_{n-2}\left(c + dx_{n-3}y_{n-4}x_{n-5}y_{n-6}\right)},$$

for $n \in \mathbb{N}_0$, where the parameters a, b and the initial values x_{-i}, $i \in \{1,2,3,4,5,6\}$, are real numbers. In addition, we show that some solvable subclasses of the class of nonlinear two-dimensional system of difference equations are solvable in closed form. We also describe the forbidden set of solutions of the system of difference equations. Some numerical examples are given to demonstrate the theoretical results.

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