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ON QUANTUM HERMITE-JENSEN-MERCER INEQUALITIES

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Abstract. A. M. Mercer prove a new version of well-known Jensen inequality which is called Jensen-Mercer inequality [16]. By using Jensen-Mercer inequality, Kian and Moslehian establish a new variant of Hermite-Hadamard inequality which is called Hermite-Jensen-Mercer inequality [15]. In this paper, we extend the Hermite-Jensen-Mercer inequality for quantum integrals by utilizing Jensen-Mercer inequality. Then we investigate the connections between our results and those in earlier works. Moreover we give some examples to illustrate the main results in this paper. This is the first paper focusing on Hermite-Jensen-Mercer inequalities for quantum integrals.

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1. INTRODUCTION

In quantum mathematical research, it is an unlimited analysis of calculus and is known as q - calculus. We get original mathematical formulas for q-calculus as q reaches 1^-. The beginning of the analysis of q-calculus was initiated by Euler (1707-1783). The results mentioned above lead to a thorough investigation into the q-calculus of the twentieth century. The concept of q-calculus is used in many areas in mathematics and physics such as numerical theory, orthogonal polynomials, integration, basic hypergeometric functions, mechanics, and quantum and relativity theory. For more information about q -calculus, one can refer to [5, 11, 13, 14] and references cited therein.

On the other hand, the celebrated discrete Jensen inequality states that: let $0 < \varkappa_1 \leq \varkappa_2 \leq \ldots \leq \varkappa_n$ and let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ nonnegative weights such that $\Sigma_{i=1}^{n}\mu = 1$. If ψ is a convex function on the interval $[\kappa_1, \kappa_2]$, then

$$\psi\left(\sum_{i=1}^{n}\mu_i\varkappa_i\right) \leq \sum_{i=1}^{n}\mu_i\psi(\varkappa_i),$$

where for all $\varkappa_i \in [\kappa_1, \kappa_2]$ and $\mu_i \in [0,1]$, $(i = \overline{1,n})$ [10]. The Hermite–Hadamard inequality states that, if a mapping $\psi\colon I \subseteq R \to R$ is a convex on I with $\kappa_1, \kappa_2 \in I$ and

$\kappa_1 < \kappa_2$, then

$$\psi\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\psi(\varkappa)d\varkappa \leq \frac{\psi(\kappa_1)+\psi(\kappa_2)}{2}.$$

The double inequality holds in the reversed direction if ψ is concave [9].

Theorem 1 ([16, Theoem 1.2])**.** *If ψ is convex function on $I=[\kappa_1,\kappa_2]$, then*

$$\psi\left(\kappa_1+\kappa_2-\sum_{i=1}^{n}\mu_i\varkappa_i\right) \leq \psi(\kappa_1)+\psi(\kappa_2)-\sum_{i=1}^{n}\mu_i\psi(\varkappa_i), \tag{1.1}$$

for each $\varkappa_i \in [\kappa_1,\kappa_2]$ and $\mu_i \in [0,1]$, $(i=\overline{1,n})$ with $\sum_{i=1}^{n}\mu = 1$.

Inequality (1.1) is known as the Jensen-Mercer inequality. By using inequality (1.1), Kian and Moslehian prove the Hermite-Jensen-Mercer inequality in [15]. Recently several papers have devoted to generalization of Hermite-Jensen-Mercer inequality. For more recent and related results connected with Jensen-Mercer inequality and Hermite-Jensen–Mercer inequality, see [1, 6–8, 19].

In this study, we extend the Hermite-Jensen-Mercer for quantum integrals. We also present some examples to illustrate our results.

2. Preliminaries of q-calculus and some inequalities

In this section we present some required definitions and related inequalities about q-calculus. Also, here and further we use the following notation(see [14]):

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+q^2+\ldots+q^{n-1}, \quad q\in(0,1).$$

In [13], Jackson gave the q-Jackson integral from 0 to κ_2 for $0<q<1$ as follows:

$$\int_0^{\kappa_2}\psi(\varkappa)\ d_q\varkappa = (1-q)\kappa_2\sum_{n=0}^{\infty}q^n\psi(\kappa_2 q^n)$$

provided the sum converge absolutely.

Jackson in [13] gave the q-Jackson integral in a generic interval $[\kappa_1,\kappa_2]$ as:

$$\int_{\kappa_1}^{\kappa_2}\psi(\varkappa)\ d_q\varkappa = \int_0^{\kappa_2}\psi(\varkappa)\ d_q\varkappa - \int_0^{\kappa_1}\psi(\varkappa)\ d_q\varkappa\ .$$

Definition 1 ([21, Definition 3.3])**.** Let $\psi\colon [\kappa_1,\kappa_2]\to\mathbb{R}$ be a continuous function. Then, the q_{κ_1}-definite integral on $[\kappa_1,\kappa_2]$ is defined as

$$\int_{\kappa_1}^{\kappa_2}\psi(\varkappa)\ {}_{\kappa_1}d_q\varkappa = (1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty}q^n\psi(q^n\kappa_2+(1-q^n)\kappa_1)$$

$$= (\kappa_2 - \kappa_1) \int_0^1 \psi((1-t)\kappa_1 + t\kappa_2) \ d_q t \ .$$

Many of the most well-known inequalities in the classical analysis such as Hölder's inequality, Hermite-Hadamard's inequality and Ostrowski's inequality, Cauchy-Bunyakovsky-Schwarz, Gruss, Gruss-Cebysev and other integral inequalities have been proven and applied for q-calculus . For more q-calculus results please refer [2–5, 12, 17, 18, 20–22].

In [4], Alp et al. proved the following q_{κ_1}-Hermite-Hadamard inequalities for convex functions in the setting of quantum calculus:

Theorem 2 ([4, Theoem 6])**.** *Let $\psi\colon [\kappa_1,\kappa_2] \to \mathbb{R}$ be a convex differentiable function on $[\kappa_1,\kappa_2]$ and $0<q<1$. Then q-Hermite-Hadamard inequalities are as follows:*

$$\psi\left(\frac{q\kappa_1+\kappa_2}{1+q}\right) \le \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \psi(\varkappa) \ {}_{\kappa_1}d_q\varkappa \ \le \frac{q\psi(\kappa_1)+\psi(\kappa_2)}{1+q}. \tag{2.1}$$

In [18] and [4], authors established some bounds for left and right hand sides of the inequality (2.1).

On the other hand, Bermudo et al. gave the following new definition and related Hermite-Hadamard type inequalities:

Definition 2 ([5, Definition 6])**.** Let $\psi\colon [\kappa_1,\kappa_2] \to \mathbb{R}$ be a continuous function. Then, the q^{κ_2}-definite integral on $[\kappa_1,\kappa_2]$ is defined as

$$\int_{\kappa_1}^{\kappa_2} \psi(\varkappa) \ {}^{\kappa_2}d_q\varkappa \ = (1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty} q^n\psi(q^n\kappa_1+(1-q^n)\kappa_2)$$

$$= (\kappa_2-\kappa_1)\int_0^1 \psi(t\kappa_1+(1-t)\kappa_2) \ d_q t \ .$$

Theorem 3 ([5, Theorem 11])**.** *Let $\psi\colon [\kappa_1,\kappa_2] \to \mathbb{R}$ be a convex function on $[\kappa_1,\kappa_2]$ and $0<q<1$. Then, q-Hermite-Hadamard inequalities are as follows:*

$$\psi\left(\frac{\kappa_1+q\kappa_2}{1+q}\right) \le \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \psi(\varkappa) \ {}^{\kappa_2}d_q\varkappa \ \le \frac{\psi(\kappa_1)+q\psi(\kappa_2)}{1+q}.$$

From Theorem 2 and Theorem 3, one can get the following inequalities:

Corollary 1 ([5, Corollary 14])**.** *For any convex function $\psi\colon [\kappa_1,\kappa_2] \to \mathbb{R}$ and $0<q<1$, we have*

$$\psi\left(\frac{q\kappa_1+\kappa_2}{1+q}\right)+\psi\left(\frac{\kappa_1+q\kappa_2}{1+q}\right) \tag{2.2}$$

$$\leq \frac{1}{\kappa_2-\kappa_1}\left\{\int_{\kappa_1}^{\kappa_2}\psi(\varkappa)\ {}_{\kappa_1}d_q\varkappa\ +\int_{\kappa_1}^{\kappa_2}\psi(\varkappa)\ {}^{\kappa_2}d_q\varkappa\right\}$$
$$\leq \psi(\kappa_1)+\psi(\kappa_2)$$

and

$$\psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\leq \frac{1}{2(\kappa_2-\kappa_1)}\left\{\int_{\kappa_1}^{\kappa_2}\psi(\varkappa)\ {}_{\kappa_1}d_q\varkappa\ +\int_{\kappa_1}^{\kappa_2}\psi(\varkappa)\ {}^{\kappa_2}d_q\varkappa\right\} \tag{2.3}$$
$$\leq \frac{\psi(\kappa_1)+\psi(\kappa_2)}{2}.$$

In the next section, we'll get integral inequalities that are our main results.

3. Quantum Hermite-Jensen-Mercer inequalities

In this section, we prove some Quantum Hermite-Jensen-Mercer type inequalities for quantum integrals.

Theorem 4. *Let $\psi\colon\ [\kappa_1,\kappa_2]\to\mathbb{R}$ be a convex function on $[\kappa_1,\kappa_2]$ and $0<q<1$. Then we have the following quantum Jensen-Mercer inequalities*

$$\psi\left(\kappa_1+\kappa_2-\frac{\varkappa+\gamma}{2}\right) \tag{3.1}$$
$$\leq \psi(\kappa_1)+\psi(\kappa_2)-\frac{1}{2}\left[\frac{1}{\gamma-\varkappa}\int_{\varkappa}^{\gamma}\psi(\upsilon)\ {}^{\gamma}d_q\upsilon+\frac{1}{\gamma-\varkappa}\int_{\varkappa}^{\gamma}\psi(\upsilon)\ {}_{\varkappa}d_q\upsilon\right]$$
$$\leq \psi(\kappa_1)+\psi(\kappa_2)-\frac{1}{2}\left[\psi\left(\frac{\varkappa+q\gamma}{[2]_q}\right)+\psi\left(\frac{q\varkappa+\gamma}{[2]_q}\right)\right]$$
$$\leq \psi(\kappa_1)+\psi(\kappa_2)-\psi\left(\frac{\varkappa+\gamma}{2}\right)$$

for all $\varkappa,\gamma\in[\kappa_1,\kappa_2]$ with $\varkappa<\gamma$.

Proof. It follows from the Jensen-Mercer inequality,

$$\psi\left(\kappa_1+\kappa_2-\frac{\upsilon+\nu}{2}\right)\leq \psi(\kappa_1)+\psi(\kappa_2)-\frac{1}{2}\left[\psi(\upsilon)+\psi(\nu)\right]$$

for $\upsilon,\nu\in[\kappa_1,\kappa_2]$. By choosing $\upsilon=t\varkappa+(1-t)\gamma$ and $\nu=t\gamma+(1-t)\varkappa$ for $t\in[0,1]$, we obtain

$$\psi\left(\kappa_1+\kappa_2-\frac{\upsilon+\nu}{2}\right)\leq \psi(\kappa_1)+\psi(\kappa_2) \tag{3.2}$$
$$-\frac{1}{2}\left[\psi(t\varkappa+(1-t)\gamma)+\psi(t\gamma+(1-t)\varkappa)\right].$$

By taking q-integral of both sides of (3.2), we have

$$\psi\left(\kappa_1+\kappa_2-\frac{\upsilon+\nu}{2}\right)\leq\psi(\kappa_1)+\psi(\kappa_2)-\frac{1}{2}\left[\int_0^1\psi(t\varkappa+(1-t)\gamma)\,d_qt+\int_0^1\psi(t\gamma+(1-t)\varkappa)\,d_qt\right]. \tag{3.3}$$

From Definition 1 and Theorem 2, we get

$$\int_0^1\psi(t\gamma+(1-t)\varkappa)\,d_qt=\frac{1}{\gamma-\varkappa}\int_\varkappa^\gamma\psi(\upsilon)\ {}_\varkappa d_q\upsilon\geq\psi\left(\frac{q\varkappa+\gamma}{[2]_q}\right) \tag{3.4}$$

and similarly by Definition 2 and Theorem 3, we have

$$\int_0^1\psi(t\varkappa+(1-t)\gamma)\,d_qt=\frac{1}{\gamma-\varkappa}\int_\varkappa^\gamma\psi(\upsilon)\ {}^\gamma d_q\upsilon\geq\psi\left(\frac{\varkappa+q\gamma}{[2]_q}\right). \tag{3.5}$$

By substituting the inequalities (3.4) and (3.5) in (3.3), we obtain

$$\begin{aligned}&\psi\left(\kappa_1+\kappa_2-\frac{\varkappa+\gamma}{2}\right)\\&\leq\psi(\kappa_1)+\psi(\kappa_2)-\frac{1}{2}\left[\frac{1}{\gamma-\varkappa}\int_\varkappa^\gamma\psi(\upsilon)\ {}^\gamma d_q\upsilon+\frac{1}{\gamma-\varkappa}\int_\varkappa^\gamma\psi(\upsilon)\ {}_\varkappa d_q\upsilon\right]\\&\leq\psi(\kappa_1)+\psi(\kappa_2)-\frac{1}{2}\left[\psi\left(\frac{\varkappa+q\gamma}{[2]_q}\right)+\psi\left(\frac{q\varkappa+\gamma}{[2]_q}\right)\right]\end{aligned}$$

which gives the proof of first and second inequalities in (3.1).

Since ψ is a convex function we have

$$\psi\left(\frac{\varkappa+\gamma}{2}\right)=\psi\left(\frac{1}{2}\frac{\varkappa+q\gamma}{[2]_q}+\frac{1}{2}\frac{q\varkappa+\gamma}{[2]_q}\right)\leq\frac{1}{2}\left[\psi\left(\frac{\varkappa+q\gamma}{[2]_q}\right)+\psi\left(\frac{q\varkappa+\gamma}{[2]_q}\right)\right].$$

This proves the last inequality in (3.1). □

Remark 1. If we take the limit $q\to1^-$ in Theorem 4, then inequalities (3.1) reduce to inequalities (2.1) of Theorem 2.1 in [15].

Corollary 2. *If we choose $\varkappa=\kappa_1$ and $\gamma=\kappa_2$ in Theorem 4, we have*

$$\psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\leq\psi(\kappa_1)+\psi(\kappa_2)-\frac{1}{2(\kappa_2-\kappa_1)}\left[\int_{\kappa_1}^{\kappa_2}\psi(\upsilon)\ {}^{\kappa_2}d_q\upsilon+\int_{\kappa_1}^{\kappa_2}\psi(\upsilon)\ {}_{\kappa_1}d_q\upsilon\right]$$

$$\leq \psi(\kappa_1)+\psi(\kappa_2)-\frac{1}{2}\left[\psi\left(\frac{\kappa_1+q\kappa_2}{[2]_q}\right)+\psi\left(\frac{q\kappa_1+\kappa_2}{[2]_q}\right)\right]$$
$$\leq \psi(\kappa_1)+\psi(\kappa_2)-\psi\left(\frac{\kappa_1+\kappa_2}{2}\right).$$

This inequalities give the first inequalities in (2.2)and (2.3).

Theorem 5. *Let $\psi\colon [\kappa_1,\kappa_2]\to\mathbb{R}$ be a convex function on $[\kappa_1,\kappa_2]$ and $0<q<1$. Then we have the following quantum Jensen-Mercer inequalities*

$$\psi\left(\kappa_1+\kappa_2-\frac{\varkappa+q\gamma}{[2]_q}\right)\leq \frac{1}{\gamma-\varkappa}\int_{\kappa_1+\kappa_2-\gamma}^{\kappa_1+\kappa_2-\varkappa}\psi(\upsilon)\ {}_{(\kappa_1+\kappa_2-\gamma)}d_q\upsilon \tag{3.6}$$
$$\leq \psi(\kappa_1)+\psi(\kappa_2)-\frac{\psi(\varkappa)+q\psi(\gamma)}{[2]_q}$$

for all $\varkappa,\gamma\in[\kappa_1,\kappa_2]$ with $\varkappa<\gamma$, where $0<q<1$.

Proof. From Definition 1 and Theorem 2, we get

$$\int_0^1 \psi(\kappa_1+\kappa_2-(t\varkappa+(1-t)\gamma))\,d_qt$$
$$=\int_0^1 \psi(t(\kappa_1+\kappa_2-\varkappa)+(1-t)(\kappa_1+\kappa_2-\gamma))\,d_qt$$
$$=\frac{1}{\gamma-\varkappa}\int_{\kappa_1+\kappa_2-\gamma}^{\kappa_1+\kappa_2-\varkappa}\psi(\upsilon)\ {}_{(\kappa_1+\kappa_2-\gamma)}d_q\upsilon$$
$$\geq \psi\left(\frac{q(\kappa_1+\kappa_2-\gamma)+\kappa_1+\kappa_2-\varkappa}{[2]_q}\right)$$
$$=\psi\left(\kappa_1+\kappa_2-\frac{\varkappa+q\gamma}{[2]_q}\right)$$

which gives prof of first inequality in (3.6). By Jensen-Mercer inequality, we obtain

$$\frac{1}{\gamma-\varkappa}\int_{\kappa_1+\kappa_2-\gamma}^{\kappa_1+\kappa_2-\varkappa}\psi(\upsilon)\ {}_{(\kappa_1+\kappa_2-\gamma)}d_q\upsilon=\int_0^1 \psi(\kappa_1+\kappa_2-(t\varkappa+(1-t)\gamma))\,d_qt$$
$$\leq \int_0^1 \psi(\kappa_1)+\psi(\kappa_2)-[t\psi(\varkappa)+(1-t)\psi(\gamma)]\,d_qt$$

$$= \psi(\kappa_1) + \psi(\kappa_2) - \left[\frac{\psi(\varkappa) + q\psi(\gamma)}{[2]_q} \right].$$

This completes the proof. $\square$

Remark 2. If we take the limit $q \to 1^-$ in Theorem 5, then inequalities (3.6) reduce to inequalities (2.2) of Theorem 2.1 in [15]..

Remark 3. If we choose $\varkappa = \kappa_1$ and $\gamma = \kappa_2$ in Theorem 5, then Theorem 5 reduces to Theorem 2.

Theorem 6. *Let $\psi\colon [\kappa_1, \kappa_2] \to \mathbb{R}$ be a convex function on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then we have the following quantum Jensen-Mercer inequalities*

$$\psi\left(\kappa_1 + \kappa_2 - \frac{q\varkappa + \gamma}{[2]_q}\right) \le \frac{1}{\gamma - \varkappa} \int_{\kappa_1+\kappa_2-\gamma}^{\kappa_1+\kappa_2-\varkappa} \psi(\upsilon)\ {}^{(\kappa_1+\kappa_2-\varkappa)}d_q\upsilon \tag{3.7}$$
$$\le \psi(\kappa_1) + \psi(\kappa_2) - \frac{q\psi(\varkappa) + \psi(\gamma)}{[2]_q}$$

for all $\varkappa, \gamma \in [\kappa_1, \kappa_2]$ with $\varkappa < \gamma$, where $0 < q < 1$.

Proof. From Definition 2 and Theorem 3, we get

$$\begin{aligned}
&\int_0^1 \psi(\kappa_1 + \kappa_2 - (t\gamma + (1-t)\varkappa))\, d_qt \\
&= \int_0^1 \psi(t(\kappa_1 + \kappa_2 - \gamma) + (1-t)(\kappa_1 + \kappa_2 - \varkappa))\, d_qt \\
&= \frac{1}{\gamma - \varkappa} \int_{\kappa_1+\kappa_2-\gamma}^{\kappa_1+\kappa_2-\varkappa} \psi(\upsilon)\ {}^{(\kappa_1+\kappa_2-\varkappa)}d_q\upsilon \\
&\ge \psi\left(\frac{q(\kappa_1 + \kappa_2 - \varkappa) + \kappa_1 + \kappa_2 - \gamma}{[2]_q}\right) \\
&= \psi\left(\kappa_1 + \kappa_2 - \frac{\gamma + q\varkappa}{[2]_q}\right)
\end{aligned}$$

which proves the first inequality in (3.7). By Jensen-Mercer Inequality, we have

$$\int_0^1 \psi(\kappa_1 + \kappa_2 - (t\gamma + (1-t)\varkappa))\, d_qt \le \int_0^1 \psi(\kappa_1) + \psi(\kappa_2) - [t\psi(\gamma) + (1-t)\psi(\varkappa)]\, d_qt$$

$$= \psi(\kappa_1) + \psi(\kappa_2) - \left[\frac{\psi(\gamma) + q\psi(\varkappa)}{[2]_q} \right]$$

which completes the proof. □

Remark 4. If we take the limit $q \to 1^-$ in Theorem 6, then inequalities (3.7) reduce to inequalities (2.2) of Theorem 2.1 in [15].

Remark 5. If we choose $\varkappa = \kappa_1$ and $\gamma = \kappa_2$ in Theorem 6, then Theorem 6 reduces to Theorem 3.

4. Examples

We provide some examples of our main theorems in this section.

Example 1. We define a convex function $\psi\colon [\kappa_1, \kappa_2] = [-1, 2] \to \mathbb{R}$ by $\psi(x) = x^2$ for $\varkappa = 0$, $\gamma = 1$ and $q = \frac{1}{2}$. Then we have

$$\psi\left(\kappa_1 + \kappa_2 - \frac{\varkappa + \gamma}{2}\right) = \psi\left(\frac{1}{2}\right) = \frac{1}{4}, \tag{4.1}$$

and

$$\psi(\kappa_1) + \psi(\kappa_2) = 5,$$

$$\frac{1}{\gamma - \varkappa} \int_{\varkappa}^{\gamma} \psi(\upsilon)\ {}^{\gamma}d_q\upsilon = \int_{0}^{1} \psi(\upsilon)\ {}^{1}d_{\frac{1}{2}}\upsilon = \frac{5}{21}$$

$$\frac{1}{\gamma - \varkappa} \int_{\varkappa}^{\gamma} \psi(\upsilon)\ {}_{\varkappa}d_q\upsilon = \int_{0}^{1} \psi(\upsilon)\ {}_{0}d_{\frac{1}{2}}\upsilon = \frac{4}{7}.$$

Thus,

$$\psi(\kappa_1) + \psi(\kappa_2) - \frac{1}{2}\left[\frac{1}{\gamma - \varkappa} \int_{\varkappa}^{\gamma} \psi(\upsilon)\ {}^{\gamma}d_q\upsilon + \frac{1}{\gamma - \varkappa} \int_{\varkappa}^{\gamma} \psi(\upsilon)\ {}_{\varkappa}d_q\upsilon \right] = \frac{193}{42}. \tag{4.2}$$

Now, we observe that

$$\psi\left(\frac{\varkappa + q\gamma}{[2]_q}\right) = \psi\left(\frac{1}{3}\right) = \frac{1}{9} \qquad \psi\left(\frac{q\varkappa + \gamma}{[2]_q}\right) = \psi\left(\frac{2}{3}\right) = \frac{4}{9}.$$

Thus,

$$\psi(\kappa_1) + \psi(\kappa_2) - \frac{1}{2}\left[\psi\left(\frac{\varkappa + q\gamma}{[2]_q}\right) + \psi\left(\frac{q\varkappa + \gamma}{[2]_q}\right) \right] = \frac{85}{18}. \tag{4.3}$$

Again, we observe that

$$\psi(\kappa_1) + \psi(\kappa_2) - \psi\left(\frac{\varkappa + \gamma}{2}\right) = \frac{19}{4} \tag{4.4}$$

Finally, we conclude from the equalities (4.1–4.4) that the inequality (3.1) is valid and

$$\frac{1}{4}<\frac{193}{42}<\frac{85}{18}<\frac{19}{4}.$$

Example 2. We also consider the convex function $\psi\colon[-1,2]\to\mathbb{R}$ by $\psi(x)=x^2$ with $\varkappa=0$, $\gamma=1$ and $q=\frac{1}{2}$. Then we get

$$\psi\left(\kappa_1+\kappa_2-\frac{\varkappa+q\gamma}{[2]_q}\right)=\frac{4}{9}, \tag{4.5}$$

$$\frac{1}{\gamma-\varkappa}\int\limits_{\kappa_1+\kappa_2-\gamma}^{\kappa_1+\kappa_2-\varkappa}\psi(\upsilon)\ {}_{(\kappa_1+\kappa_2-\gamma)}d_q\upsilon=\int\limits_0^1\psi(\upsilon)\ {}_0d_{\frac{1}{2}}\upsilon=\frac{4}{7} \tag{4.6}$$

and

$$\psi(\kappa_1)+\psi(\kappa_2)-\frac{\psi(\varkappa)+q\psi(\gamma)}{[2]_q}=\frac{14}{3}. \tag{4.7}$$

By the equalities (4.5–4.7), the inequality (3.6) is valid and

$$\frac{4}{9}<\frac{4}{7}<\frac{14}{3}.$$

Example 3. Let's consider again the convex function $\psi\colon[-1,2]\to\mathbb{R}$, $\psi(x)=x^2$ with $\varkappa=0$, $\gamma=1$ and $q=\frac{1}{2}$. Then we have

$$\psi\left(\kappa_1+\kappa_2-\frac{\varkappa+q\gamma}{[2]_q}\right)=\frac{1}{9}, \tag{4.8}$$

$$\frac{1}{\gamma-\varkappa}\int\limits_{\kappa_1+\kappa_2-\gamma}^{\kappa_1+\kappa_2-\varkappa}\psi(\upsilon)\ {}^{(\kappa_1+\kappa_2-\varkappa)}d_q\upsilon=\int\limits_0^1\psi(\upsilon)\ {}^1d_{\frac{1}{2}}\upsilon=\frac{5}{21} \tag{4.9}$$

and

$$\psi(\kappa_1)+\psi(\kappa_2)-\frac{\psi(\varkappa)+q\psi(\gamma)}{[2]_q}=\frac{13}{3}. \tag{4.10}$$

From the equalities (4.8–4.10), the inequality (3.7) is valid and

$$\frac{1}{9}<\frac{5}{21}<\frac{13}{3}.$$

5. Conclusion

In this research paper, we generalize Hermite-Jensen-Mercer inequality for quantum integrals. This is the first study obtaining Hermite-Jensen-Mercer inequalities for quantum integrals. In the future works, authors can try to generalize our results by utilizing different kinds of convex function classes.

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