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ON THE DIOPHANTINE EQUATION $x^2 + 7^\alpha \cdot 11^\beta = y^n$

GOKHAN SOYDAN

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Abstract. In this paper, we give all the solutions of the Diophantine equation $x^2 + 7^\alpha \cdot 11^\beta = y^n$, for the nonnegative integers $\alpha,\ \beta,\ x,\ y,\ n \geq 3$, where x and y coprime, except when $\alpha.x$ is odd and β is even.

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1. INTRODUCTION

The Diophantine equation

$$x^2 + C = y^n, \qquad n \geq 3 \tag{1.1}$$

in positive integers $x,\ y,\ n$ for given a C has a rich history. In 1850, Lebesgue [25] proved that the above equation has no solutions when $C = 1$. The equation of the title is a special case of the Diophantine equation $ay^2 + by + c = dx^n$, where $a \neq 0, b,\ c$ and $d \neq 0$ are integers with $b^2 - 4ac \neq 0$, which has at most finitely many integer solutions $x,\ y,\ n \geq 3$ (see [23]). In 1993, J.H.E. Cohn [17] solved the Diophantine equation (1.1) for several values of the parameter C in the range $1 \leq C \leq 100$. The solution for the cases $C = 74, 86$ was completed by Mignotte and de Weger [31]. That had not been covered by Cohn (indeed, Cohn solved these two equations of type (1.1) except for $p = 5$, in which case difficulties occur as the class numbers of the corresponding imaginary quadratic fields are divisible by 5). In [12], Bugeaud, Mignotte and Siksek improved modular methods to solve completely (1.1) when $n \geq 3$, for C in the range $[1, 100]$. So they covered the remaining cases.

Different types of the Diophantine equation (1.1) were studied also by various mathematicians. For effectively computable upper bounds for the exponent n, we refer to [8] and [22]. However, these estimates are based on Baker's theory of lower bounds for linear forms in logarithms of algebraic numbers, so they are quite impractical. In [37], Tengely gave a method to solve the equation $x^2 + a^2 = y^n$ and applied it to $3 \leq a \leq 501$, so it includes $x^2 + 7^2 = y^n$ and $x^2 + 11^2 = y^n$. In [4], the equation $x^2 + C = 2y^n$, where C is a fixed positive integer, under the similar restrictions

$n \geq 3$ and $\gcd(x, y) = 1$ was studied. Recently, Luca, Tengely and Togbé studied the Diophantine equation $x^2 + C = 4y^n$ for nonnegative integers $x, y, n \geq 3$ with x and y coprime for various shapes of the positive integer C in [28].

In recent years, a different form of the above equation has been considered, namely where C is a power of a fixed prime. In [6], the equation $x^2 + 2^k = y^n$ was studied under some conditions by Arif and Muriefah. A conjecture of Cohn (see [16]) was verified. It says that $x^2 + 2^k = y^n$ has no solutions with x odd and even $k > 2$ by Le [24]. In [7], Abu Muriefah and Arif, gave all the solutions of $x^2 + 3^k = y^n$ with k odd and, Luca [27], gave all the solutions with k even. Again the same equation was independently solved in 2008 by Liqun in [35] for both odd and even m. All solutions of $x^2 + 5^k = y^n$ are given with k odd in [3] and with k even in [2]. Liqun solves the same equation again in 2009, in [36]. Recently, Bérczes and Pink [9], gave all the solutions of the Diophantine equation (1.1) when $C = p^k$ and k is even, where p is any prime in the interval $[2, 100]$.

The last variant of the Diophantine equation (1.1) where C is a product of at least two prime powers were studied in some recent papers. In 2002, Luca gave complete solution of $x^2 + 2^a.3^b = y^n$ in [30]. Since then, in 2006, all the solutions of the Diophantine equation $x^2 + 2^a.5^b = y^n$ were found by Luca and Togbé in [30]. In 2008, the equations $x^2 + 5^a.13^b = y^n$ and $x^2 + 2^a 5^b.13^c = y^n$ were solved in [5] and [21]. Recently, in [14] and [13], complete solutions of the equations $x^2 + 2^a.11^b = y^n$ and $x^2 + 2^a.3^b.11^c = y^n$ were found. In [20], the complete solution (n, a, b, x, y) of the equation $x^2 + 5^a.11^b = y^n$ when $\gcd(x, y) = 1$, except for the case when xab is odd, is given. In [34], Pink gave all the *non-exceptional solutions* (in the terminology of that paper) with $C = 2^a.3^b.5^c.7^d$. Note that finding all the *exceptional solutions* of this equation seems to be a very difficult task. A more exhaustive survey on this type of problems is [32].

Here, we study the Diophantine equation

$$x^2 + 7^\alpha \cdot 11^\beta = y^n, \qquad gcd(x, y) = 1 \qquad \text{and} \qquad n \geq 3. \tag{1.2}$$

There are three papers concerned with partial solutions for equation (1.2). The known results include the following theorem:

Theorem 1. (i) *If α is even and $\beta = 0$, then the only integer solutions of the Diophantine equation*

$$x^2 + 7^{2k} = y^n$$

are

$$n = 3 \quad (x, y, k) = (524 \cdot 7^{3\lambda}, 65 \cdot 7^{2\lambda}, 1 + 3\lambda),$$

$$n = 4 \quad (x, y, k) = (24 \cdot 7^{2\lambda}, 5 \cdot 7^{\lambda}, 1 + 2\lambda) \text{ where } \lambda \geq 0 \text{ is any integer.}$$

(ii) If $\alpha=1$ and $\beta=0$, then the only integer solutions (x,y,n) to the generalized Ramanujan–Nagell equation

$$x^2+7=y^n$$

are

$$(1,2,3),(181,32,3),(3,2,4),(5,2,5),(181,8,5),(11,2,7),(181,2,15).$$

(iii) If $\alpha=0$, then the only integer solutions of the Diophantine equation

$$x^2+11^\beta=y^n$$

are

$$(x,y,\beta,n)\ =(2,5,2,3),(4,3,1,3),(58,15,1,3),(9324,443,3,3)$$

Proof. See [29], [12] and [14]. □

Our main result is the following.

Theorem 2. *The only solutions of the Diophantine equation (1.2) are*

$$\begin{aligned} n=3:\quad & (x,y,\alpha,\beta)\in\{(57,16,1,2),(797,86,1,2),(4229,284,3,4),\\ & (3093,478,7,2)(4,3,0,1),(58,15,0,1),(2,5,0,2),(9324,443,0,3),\\ & (1,2,1,0),(181,32,1,0),(524,65,2,0),(13,8,3,0)\};\\ n=4:\quad & (x,y,\alpha,\beta)\in\{(2,3,1,1),(57,8,1,2),(8343,92,5,2),(3,2,1,0),\\ & (24,5,2,0)\};\\ n=6:\quad & (x,y,\alpha,\beta)=(57,4,1,2);\\ n=9:\quad & (x,y,\alpha,\beta)=(13,2,3,0);\\ n=12:\ & (x,y,\alpha,\beta)=(57,2,1,2); \end{aligned}$$

When $n\geq 5, n\neq 6,9,12$, equation (1.2) has no solutions (x,y,α,β) with at least one of α,x even or with β is odd.

Remark 1. For $n\geq 5, n\neq 6,9,12$ the above theorem lefts out the solutions (α,β,x,y) when $\alpha.x$ is odd and β is even. These are exactly the exceptional solutions of the equation (1.2) in the terminology of [34]; see also the remark 2 at the end of this paper.

One can deduce from the Theorem 1 and Theorem 2 the following corollary.

Corollary 1. *The only integer solutions of the Diophantine equation (1.2) are*

$$\begin{aligned} n=3:\quad & (x,y,\alpha,\beta)\in\{(57,16,1,2),(797,86,1,2),(4229,284,3,4),\\ & (3093,478,7,2),(4,3,0,1),(58,15,0,1),(2,5,0,2),\\ & (9324,443,0,3),\ (1,2,1,0),(181,32,1,0),(524,65,2,0),(13,8,3,0)\};\\ n=4:\quad & (x,y,\alpha,\beta)\in\{(2,3,1,1),(57,8,1,2),(8343,92,5,2),(3,2,1,0), \end{aligned}$$

$$\begin{aligned}
&\quad\quad (24,5,2,0)\};\\
n=5:&\quad (x,y,\alpha,\beta)=(5,2,1,0),(181,8,1,0);\\
n=6:&\quad (x,y,\alpha,\beta)=(57,4,1,2);\\
n=7:&\quad (x,y,\alpha,\beta)=(11,2,1,0);\\
n=9:&\quad (x,y,\alpha,\beta)=(13,2,3,0);\\
n=12:&\quad (x,y,\alpha,\beta)=(57,2,1,2);\\
n=15:&\quad (x,y,\alpha,\beta)=(181,2,1,0).
\end{aligned}$$

2. THE PROOF OF THEOREM 2

We distinguish the cases $n = 3,6,9,12,\ n = 4$ and $n > 4$, devoting a subsection to the treatment of each case. We first treat the cases $n = 3$ and $n = 4$. This is achieved in Section 2.1 and Section 2.2, respectively. For the case $n = 3$, we transform equation (1.2) into several elliptic equations in Weierstrass form for which we need to determine all their $\{7,11\}-$integral points. In Section 2.2, we use the same method as in Section 2.1 to determine the solutions of (1.2) for $n = 4$. In the last section, we assume that $n > 4$ is prime and study the equation (1.2) under this assumption. Here we use the method of primitive divisors for Lucas sequences. All the computations are done with MAGMA [11] and with Cremona's program mwrank.

2.1. *The Cases* $n = 3,6,9$ *and* 12

Lemma 1. *When* $n = 3$, *then only solutions to equation* (1.2) *are*

$$\begin{aligned}
&(57,16,1,2),(797,86,1,2),(4229,284,3,4),(3093,478,7,2),\\
&(4,3,0,1),(58,15,0,1),(2,5,0,2),(9324,443,0,3),\\
&(1,2,1,0),(181,32,1,0),(524,65,2,0),(13,8,3,0);
\end{aligned}\tag{2.1}$$

when $n = 6$, *then only solution to equation* (1.2) *is* $(57,4,1,2)$; *when* $n = 9$, *then only solution to equation* (1.2) *is* $(13,2,3,0)$; *when* $n = 12$, *then only solution to equation* (1.2) *is* $(57,2,1,2)$.

Proof. Suppose $n = 3$. Writing $\alpha = 6k+\alpha_1,\ \beta = 6l+\beta_1$ in (1.2) with $\alpha_1,\beta_1 \in \{0,1,2,3,4,5\}$, we get that

$$\left(\frac{x}{7^{3k}11^{3l}},\frac{y}{7^{2k}11^{2l}}\right)$$

is an $S-$Integral point (X,Y) on the elliptic curve

$$X^2 = Y^3 - 7^{\alpha_1}\cdot 11^{\beta_1},\tag{2.2}$$

where $S = \{7,11\}$ with the numerator of Y being coprime to 77, in view of the restriction $\gcd(x,y) = 1$. Now we need to determine all the $\{7,11\}$-integral points on the above 36 elliptic curves. At this stage we note that in [33] Pethő, Zimmer, Gebel

and Herrmann developed a practical method for computing all $S-$Integral points on Weierstrass elliptic curve and their method has been implemented in MAGMA [11] as a routine under the name `SIntegralPoints`.The subroutine `SIntegralPoints` of MAGMA worked
without problems for all (α_1,β_1) except for $(\alpha_1,\beta_1)=(5,5)$. MAGMA determined the appropriate Mordell-Weil groups except this case and we deal with this exceptional case separately. By computations done for equation (2.2) when $n=3$, we obtain the following solutions for the $\{7,11\}-$integral points on the curves:

$$\begin{aligned}&(1,0,0,0),(3,4,0,1),(15,58,0,1),(5,2,0,2),(11,0,0,3),(443,9324,0,3),\\&(2,1,1,0),(32,181,1,0),\ (478/49,3093/3431,2),(11,22,1,2),(16,57,1,2),\\&(1899062/117649,2338713355/40353607,1,2),(22,99,1,2),(86,797,1,2),\\&(88,825,1,2),(638,16115,1,2),(657547,533200074,1,2),(242,3751,1,4),\\&(65,524,2,0),(7,0,3,0),(8,13,3,0),(14,49,3,0),(28,147,3,0),\\&(154,1911,3,0),(77,0,3,3),(242,3025,3,4),(284,4229,3,4),\\&(1435907/49,1720637666/343,3,4).\end{aligned}$$

We use the above points on the elliptic curves to find the corresponding solutions for equation (2.2). Identifying the coprime positive integers x and y from the above list, one obtains the solutions listed in (2.2) (note that not all of them lead to coprime values for x and y).

We give the details in case $(\alpha_1,\beta_1)=(5,5)$ of equation (2.2). Observe that if Y is even, then X is odd and $X^2+\ 7^5 11^5\equiv 0 \pmod 8$, and hence $X^2\equiv 3 \pmod 8$, which is a contradiction. Therefore Y is always odd. We consider solutions such that X and Y are coprime.

Write $\mathbb{K}=\mathbb{Q}(i\sqrt{77})$. In this field, the primes $2,7,11$ (all primes dividing the discriminant $d_{\mathbb{K}}=4d$) ramify so there are prime ideals P_2,P_7,P_{11} such that $2\mathcal{O}_{\mathbb{K}}=P_2^2$, $7\mathcal{O}_{\mathbb{K}}=P_7^2$, $11\mathcal{O}_{\mathbb{K}}=P_{11}^2$ respectively. Now, we show that the ideals $(X+7^2 11^2\sqrt{77}i)\mathcal{O}_{\mathbb{K}}$ and $(X-7^2 11^2\sqrt{77}i)\mathcal{O}_{\mathbb{K}}$ are coprime in the ring of integers $\mathcal{O}_{\mathbb{K}}$. To show this, let us assume that the ideals $(X+7^2 11^2\sqrt{77}i)\mathcal{O}_{\mathbb{K}}$ and $(X-7^2 11^2\sqrt{77}i)\mathcal{O}_{\mathbb{K}}$ are not coprime. So, these ideals have a gcd that divides $2.7^2.11^2\sqrt{77}i$. Hence there is an ideal $P_2^a P_7^b P_{11}^c$ with $a\leq 2$, and $b,c\leq 5$. If $b>0$ then $7\mid X$. Hence $7\mid Y$, hence $7^3\mid X^2$, hence $7^2\mid X$, hence $7^4| \ Y^3$, hence $7^2\mid Y$, hence $7^5\mid X^2$, hence $7^3|\ X$. So, we have a contradiction as $7^6\mid X^2-Y^3$. Thus $b=0$. Similarly we can prove that $c=0$.

Now let $(X+7^2 11^2\sqrt{77}i)\mathcal{O}_{\mathbb{K}}=P_2^a\wp^3$ for some ideal $\wp$ not divisible by P_2, and $(X-7^2 11^2\sqrt{77}i)\mathcal{O}_{\mathbb{K}}=P_2^a\wp'^3$ (for its conjugate ideal). If we take norms, then we get that $y^3=2^a[N_{\mathbb{K}}(\wp)]^3$, where $N_{\mathbb{K}}(\wp)$ is odd. It follows that $a=0$ (as it could be at most 2). So, we showed that the ideals $(X+7^2 11^2\sqrt{77}i)\mathcal{O}_{\mathbb{K}}$ and $(X-$

$7^2 11^2\sqrt{77}i)\mathcal{O}_{\mathbb{K}}$ are coprime. Equation (2.2) now implies that

$$(X+7^2 11^2\sqrt{77}i)\mathcal{O}_{\mathbb{K}}=\wp^3 \text{ and } (X-7^2 11^2\sqrt{77}i)\mathcal{O}_{\mathbb{K}}=\wp'^3$$

for the ideals $\wp$ and $\wp'$. Let $h(\mathbb{K})$ be the class number of the field $\mathbb{K}$, then $\delta^{h(\mathbb{K})}$ is principal for any ideal δ. Note that, $h(\mathbb{K})=8$ and so $(3,h(\mathbb{K}))=1$. Thus since $\wp^3$ and $\wp'^3$ are principal, $\wp$ and $\wp'$ are also principal. Moreover, since the units of $\mathbb{Q}(i\sqrt{77})$ are 1 and -1, which are both cubes, we conclude that

$$(X+7^2 11^2\sqrt{77}i)=(u+\sqrt{77}iv)^3 \tag{2.3}$$

$$(X-7^2 11^2\sqrt{77}i)=(u-\sqrt{77}iv)^3 \tag{2.4}$$

for some integers u and v. After subtracting the conjugate equation we obtain

$$7^2\cdot 11^2=v(3u^2v-77v^2). \tag{2.5}$$

Since u and v are coprime, we have the following possibilities in equation (2.5)

$$v=\pm 1;\ v=\pm 7^2;\ v=\pm 11^2;\ v=\pm 7^2 11^2$$

All cases lead to the conclusion that no solution is obtained.

For $n=6$, equation

$$x^2+7^\alpha\cdot 11^\beta=y^6$$

becomes equation

$$x^2+7^\alpha\cdot 11^\beta=(y^2)^3.$$

Again, here we look in the list of solutions of equation (2.1) and observe that the only solution whose y is a perfect square is $(57,16,1,2)$.Therefore the only solution to equation (1.2) is $(57,4,1,2)$. In the same way, one can see that the value of y above which is a perfect square is $y=4$ for the solution $(57,4,1,2)$, therefore the only solution with $n=12$ is $(57,2,1,2)$.

For $n=9$, equation

$$x^2+7^\alpha\cdot 11^\beta=y^9$$

becomes equation

$$x^2+7^\alpha\cdot 11^\beta=(y^3)^3.$$

Again here, we look in the list of solutions of (2.1) and observe that only solution whose y is a perfect cube is $(13,8,3,0)$.Therefore the only solution to equation (1.2) is $(13,2,3,0)$.This completes the proof of lemma. □

If (x,y,α,β,n) is a solution of the Diophantine equation (1.2) and d is any proper divisor of n, then $(x,y^d,\alpha,\beta,n/d)$ is also a solution of the same equation. Since $n>3$ and we have already dealt with case $n=3$, it follows that it suffices to look at the solutions n for which $p\mid n$ for some odd prime p. In this case, we may certainly replace n by p, and thus assume for the rest of the paper that $n\in\{4,p\}$.

2.2. *The Case* $n=4$

Lemma 2. *The only solutions with* $n=4$ *of the Diophantine equation* (1.2) *are given by*

$$(x,y,\alpha,\beta)=(2,3,1,1),(57,8,1,2),(8343,92,5,2),(3,2,1,0),(24,5,2,0)$$

Proof. Suppose that $n=4$. Rewrite equation (1.2) as

$$7^\alpha\cdot 11^\beta=(y^2+x)(y^2-x). \tag{2.6}$$

From equation (2.6), we have that

$$y^2+x=7^{a_1}.11^{b_1}$$
$$y^2-x=7^{a_2}.11^{b_2}$$

where $a_1,a_2,b_1,b_2\geq 0$. Then we get that

$$2y^2=7^{a_1}.11^{b_1}+7^{a_2}.11^{b_2}$$

from the sum of two equations. We multiply the above equation by 2 and we can write the equation

$$Z^2=2.(7^{a_1}.11^{b_1}+7^{a_2}.11^{b_2}) \tag{2.7}$$

as

$$2U+2V=Z^2 \tag{2.8}$$

where $Z=2y,\ U=7^{a_1}.11^{b_1}$ and $V=7^{a_2}.11^{b_2}$.

Let $p_1,p_2,...,p_s$ $(s\geq 1)$ be fixed distinct primes. The set of $S-$Units is defined as $S=\left\{\pm p_1^{x_1}p_2^{x_2}...p_s^{x_s}\,|\,x_i\in\mathbb{Z},\text{ for } i=1...k\right\}$. Let $a,b\in\mathbb{Q}-\{0\}$ be fixed. In [19], B.M.M. de Weger dealt with the solutions of the Diophantine equation $ax+by=z^2$, in $a,b\in S,\ z\in\mathbb{Q}$. He showed that this equation has essentially only finitely many solutions. Moreover, he indicated how to find all the solutions of this equation for any given set of parameters $a,b,\ p_1,...,p_s$. The tools are the theory of p-adic linear forms in logarithms, and a computational p-adic diophantine approximation method. He actually performed all the necessary computations for solving (2.8) completely for $p_1,...,p_s=2,3,5,7$ and $a=b=1$, and reported on this elsewhere (see [18], Chapter 7). Then we can find all the solutions of the Diophantine equation (2.7). But this requires a lot of additional manual effort. To solve the equation $x^2+7^\alpha\cdot 11^\beta=y^4$ instead of this method, we prefer using MAGMA (see [11]).

Writing in (1.2) $\alpha=4k+\alpha_1,\ \beta=4l+\beta_1$ with $\alpha_1,\beta_1\in\{0,1,2,3\}$ we get that

$$\left(\frac{x}{7^{2k}11^{2l}},\frac{y}{7^{2k}11^{2l}}\right)$$

is an $S-$Integral point (X,Y) on the hyperelliptic curve

$$X^2=Y^4-7^{\alpha_1}\cdot 11^{\beta_1}, \tag{2.9}$$

where $S = \{7, 11\}$ with the numerator of Y being prime to 77, in view of the restriction $\gcd(x, y) = 1$. We use the subroutine `SIntegralLjunggrenPoints` of MAGMA to determine the $\{7, 11\}$-integral points on the above hyperelliptic curves and we only find the following solutions

$$(X, Y, \alpha_1, \beta_1) = \{(1,0,0,0), (2,3,1,0), (3,2,1,1), (8,57,1,2), \\ (92/7, 8343/49, 1, 2), (5, 24, 2, 0)\}$$

With the conditions on x and y and the definition of X, Y, one can obtain the solutions listed in the statement of the lemma. □

2.3. *The Case $n > 4$ and Prime*

Lemma 3. *The Diophantine equation (1.2) has no solutions with $n > 4$ prime except possibly for α and x are odd and β even.*

Proof. Since in section 2 we have finished the study of equation $x^2 + 7^\alpha \cdot 11^\beta = y^n$ with $n = 3$, we can assume that n is a prime > 4. One can write the Diophantine equation (1.2) as $x^2 + dz^2 = y^n$, where

$$d \in \{1,\ 7,\ 11,\ 77\}, \quad z = 7^{\alpha_1} \cdot 11^{\beta_1} \tag{2.10}$$

the relation of α_1 and β_1 with α and β, respectively, is clear. If x is odd, then by z also being odd we have that y is even, so $y^n \equiv 0 \pmod 8$. As $x^2 = z^2 \equiv 1 \pmod 8$ we have $1 + d \equiv 0 \pmod 8$, so $d = 7$, implying $\alpha \equiv 1 \pmod 2$ and $\beta \equiv 0 \pmod 2$. This case is excluded in the lemma. Hence we have that x is even, and y is odd. We study in the field $\mathbb{K} = \mathbb{Q}(i\sqrt{d})$. As $\gcd(x, z) = 1$ standard argument tells us now that in $\mathbb{K}$ we have

$$(x + i\sqrt{d}z)(x - i\sqrt{d}z) = y^n, \tag{2.11}$$

where the ideals generated by $x + iz\sqrt{d}$ and $x - iz\sqrt{d}$ are coprime in $\mathbb{K}$. Hence, we obtain the ideal equation

$$\langle x + i\sqrt{d}z \rangle = \theta^n \tag{2.12}$$

Then, since the ideal class number of $\mathbb{K}$ is 1 or 8, and n is odd, we conclude that the ideal θ is principal. The cardinality of the group of units of $\mathcal{O}_{\mathbb{K}}$ is 2 or 4, all coprime to n. Furthermore, $\{1, i\sqrt{d}\}$ is always an integral base for $\mathcal{O}_{\mathbb{K}}$ except for when $d = 7$, and $d = 11$, in which cases an integral basis for $\mathcal{O}_{\mathbb{K}}$ is $\{1, (1 + i\sqrt{d})/2\}$. Thus, we may assume that

$$x + i\sqrt{d}z = \varphi^n, \ \varphi = \frac{u + i\sqrt{d}v}{2} \tag{2.13}$$

the relation holds with some algebraic integer $\varphi \in \mathcal{O}_{\mathbb{K}}$. The algebraic integers in this number field are of the form $\varphi = \frac{u+i\sqrt{d}v}{2}$, where $u, v \in \mathbb{Z}$, with u, v both even, if $d = 1, 77$ and u, v both odd if $d = 7, 11$. Note that

$$\varphi - \overline{\varphi} = vi\sqrt{d}, \ \varphi + \overline{\varphi} = i\sqrt{d}v, \ \varphi\overline{\varphi} = \frac{u^2 + dv^2}{4}$$

We thus obtain

$$\frac{2\cdot 7^{\alpha_1}\cdot 11^{\beta_1}}{v}=\frac{2z}{v}=\frac{\varphi^n-\overline{\varphi}^n}{\varphi-\overline{\varphi}}\in\mathbb{Z}. \tag{2.14}$$

Let $(L_m)_{m\geq 0}$ be the sequence with general term $L_m=(\varphi^m-\overline{\varphi}^m)/(\varphi-\overline{\varphi})$ for all $m\geq 0$. This is called a *Lucas sequence*. Note that

$$L_0=0, L_1=1 \text{ and } L_m=uL_{m-1}-\frac{u^2+dv^2}{4}L_{m-2},\ m\geq 2. \tag{2.15}$$

Following the nowadays standard strategy based on the important paper [10], we distinguish two cases according as L_n has or has not primitive divisors.

Suppose first that L_n has a primitive divisor, say q. By definition, this means that the prime q divides L_n and q does not divide $(\mu-\overline{\mu})^2L_1...L_{n-1}$, hence

$$q\nmid(\varphi-\overline{\varphi})^2L_1...L_4=(dv^2).u.\frac{3u^2-dv^2}{4}.\frac{u^2-dv^2}{2}. \tag{2.16}$$

If $q=2$, then (2.16) implies that uv is odd, hence $d=11$ or 77. If $d=11$, then third factor in the right hand-most side of (2.16) is even, a contradiction. If $d=77$, then, from (2.15) we see that $L_m\equiv L_{m-1} \pmod 2$, hence L_m is odd for every $m\geq 1$, implying that 2 cannot be a primitive divisor of L_n.

If $q=7$, then (2.16) implies that $d=1,11$ and 7 does not divide $uv(3u^2-dv^2)(u^2-dv^2)$. It follows easily then that $v^2\equiv -du^2 \pmod 7$, so that, by (2.15), $L_m\equiv uL_{m-1} \pmod 8$ for every $m\geq 2$. Therefore, $7\nmid L_n$, so that 7 can not be a prime divisor of L_n.

If $q=11$, then by (2.16), $d=1$ or 7. If $d=1$ then we write $u=2v_1, v=2v_1$ with $u_1,v_1\in\mathbb{Z}$, so that $\varphi=u_1+i\sqrt{d}v_1$ and (2.16) becomes $q\nmid u_1v_1(3u_1^2-dv_1^2)(u_1^2-dv_1^2)$. Moreover, $L_m=2u_1L_{m-1}-(u_1^2+dv_1^2)L_{m-2}$ for $m\geq 2$. Note that $\varphi\overline{\varphi}=u_1^2+dv_1^2\not\equiv 0 \pmod 8$; therefore, by corollary 2.2 of [10], there exists a positive integer m_{11} such that $11\mid L_{m_{11}}$ and $m_{11}\mid m$ for every m such that $11\mid L_m$. It follows then that $11\mid\gcd(L_n,L_{m_{11}})=L_{\gcd(n,m_{11})}$. Because of the minimality property of m_{11}, we conclude that $\gcd(n,m_{11})$, hence, since n is a prime, $m_{11}=n$. On the other hand, the Legendre symbol $\left(\frac{(\varphi-\overline{\varphi})^2}{11}\right)=-1$, hence by Theorem XII of [15] (or by theorem 2.2.4 (iv) of [26]), $11\mid L_{12}$. Therefore $m_{11}\mid 12$, i.e. $n\mid 12$, a contradiction, since n is a prime≥ 5. If $d=7$, then (2.16) implies $11\nmid u_1v_1(3u_1^2-dv_1^2)(u_1^2-dv_1^2)$. Moreover, $L_m=2u_1L_{m-1}-(u_1^2+dv_1^2)L_{m-2}$ for $m\geq 2$. Note that $\varphi\overline{\varphi}=u_1^2+dv_1^2\not\equiv 0 \pmod 8$; therefore, by corollary 2.2 of [10], there exists a positive integer m_{11} such that $11\mid L_{m_{11}}$ and $m_{11}\mid m$ for every m such that $11\mid L_m$.It follows then that $11\mid\gcd(L_n,L_{m_{11}})=L_{\gcd(n,m_{11})}$. Because of the minimality property of m_{11}, we conclude that $\gcd(n,m_{11})$, hence, since n is a prime, $m_{11}=n$. On the other hand, the Legendre symbol $\left(\frac{(\varphi-\overline{\varphi})^2}{11}\right)=1$, hence by Theorem XII of [15] (or by theorem 2.2.4 (iii) of [26]), $11 \mid L_{10}$. Therefore $m_{11} \mid 10$, i.e. $n \mid 10$. Since $n \geq 5$ is a prime, we get that $n = 5$.

We conclude that 11 is primitive divisor for $d = 7$.

In particular, u and v are integers. Since 11 is coprime to $-4dv^2 = -28v^2$, we get that $v = \pm 7^{\alpha_1}$. Since $y = u^2 + 7v^2$, we get that u is even.

In the case $v = \pm 7^{\alpha_1}$, equation (2.14) becomes

$$\pm 11^{\beta_1} = 5u^4 - 70u^2v^2 + 49v^4.$$

Since u is even, it follows that the right hand side of the last equation above is congruent to $1 \pmod 8$. So $\pm 11^{\beta_1} \equiv 1 \pmod 8$, showing that the sign on the left hand side is positive and β_1 is odd, or the sign on the left hand side is negative and β_1 is even.

Assume first that $\beta_1 = 2\beta_0 + 1$ be odd. We get

$$11V^2 = 5U^4 - 70U^2 + 49,$$

where $(U,V) = (u/v, 11^{\beta_0}/v^2)$ is a $\{7\}$-integral point on the above elliptic curve. We get that the only such points on the above curve are $(U,V) = (\pm 7, \pm 28)$. This does not lead to solutions of our original equation.

Assume now that $\beta_1 = 2\beta_0$ is even and we get that

$$V^2 = 5U^4 - 70U^2 + 49,$$

where $(U,V) = (u/v, 11^{\beta_0}/v^2)$ is a $\{7\}$-integral point on the above elliptic curve. With MAGMA, we get that the only such point on the above curve are $(U,V) = (0,7)$. This does not lead to solutions of our original equation.

We now recall that a particular instance of the Primitive Divisor Theorem for Lucas sequences implies that, if $n \geq 5$ is prime, then L_n always has a prime factor except for finitely many *exceptional triples* $(\varphi, \overline{\varphi}, n)$, and all of them appear in the Table 1 in [10] (see also [1]). These exceptional Lucas numbers are called *defective.*

Let us assume that we are dealing with a number L_n without primitive divisors. Then a quick look at Table 1 in [10] reveals that this is impossible. Indeed, all exceptional triples have $n = 5, 7$ or 13. The defective Lucas numbers whose roots are in $\mathbb{K} = \mathbb{Q}(i\sqrt{d})$ with $d = 7$ and $n = 5, 7$ or 13 appearing in the list (2.10) is $(\varphi,\ \overline{\varphi}) = ((1+i\sqrt{7})/2,\ (1-i\sqrt{7})/2)$ for which $L_7 = 7,\ L_{13} = -1$. Furthermore, with such a value for φ we get that $y = |\varphi|^2 = 2$. However, this is not convenient since for us x and y are coprime so y cannot be even. For $n = 5$ and $d = 11$,we get $L_5 = 1$ and $y = 3$ with $(\varphi,\ \overline{\varphi}) = ((1+i\sqrt{11})/2,\ (1-i\sqrt{11})/2)$. Therefore the equation is $x^2 + C = 3^5$, where $C = 7^{\alpha} \cdot 11^{\beta}$, with a even and b odd. Since $11^3 > 3^5$, we have $b = 1$, and next that $a = 0$. But it doesn't yield an integer value for x. The proof is completed. □

Remark 2. We mention here why the method applied for the proof of Lemma 3 does not apply when α and x are odd, β is even. In this case $d = 7$, the class number of $\mathbb{Q}(\sqrt{7}i)$ is 1. With $\omega=\frac{1+\sqrt{7}i}{2}$ a prime dividing 2, and ω' its conjugate, let us now write $(x+z\sqrt{7}i)=\omega^b\omega^c\xi$, where ξ is an integer in $\mathbb{Q}(\sqrt{7}i)$ of odd norm, not divisible by 7 and ξ' its conjugate. As both x and z are odd and they are coprime, we may take $c=1, b\geq 1$. Taking norms we get $y^n=2^{b+1}\xi\xi'$, and it easily follows that $\xi=c^n$ and $b+1=k.n$. Now we take $\varphi=2^{k-1}c$, $\wp=2\omega^{n-2}$, and then we have $x+z\sqrt{7}i=\wp\varphi^n$. A way to look at the rest of argument why this case is essentially different from the primitive divisors in Lucas sequences thing: From $x+z\sqrt{7}i=\wp\varphi^n$ and its conjugate it follows that

$$z=\frac{\wp\varphi^n-\overline{\wp}\,\overline{\varphi}^n}{2\sqrt{7}i}$$

If $\wp$ is in $\mathbb{Q}$ then the right hand side is the n-th term of a Lucas sequence. As z has a very nice prime factorization 7^p11^q then theory of primitive divisors will work. But in our case $\wp$ is not in $\mathbb{Q}$. Hence the right side, while it is the n-th term of a recurrence sequence, this is not a Lucas sequence, and does not have the nice divisibility properties of Lucas sequences. That's why the method of [10] fails in our case.

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Author's address

Gokhan Soydan
Isiklar Air Force High School, 16039 Bursa, TURKEY
E-mail address: gsoydan@uludag.edu.tr