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POSITIVE SOLUTIONS FOR INTEGRAL BOUNDARY VALUE PROBLEMS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we consider integral boundary value problems of nonlinear fractional differential equations. Existence results of positive solutions for the problem are obtained based on the Guo-Krasnoselskii theorem and the Five functional fixed point theorem. Simple examples follow the main results in successive sections.

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1. INTRODUCTION

In this study, we consider the following IBVP

$$\begin{cases} D^{\beta}(\varphi_p({}^cD^{\alpha}y(t))) + f(t,y(t)) = 0, & t \in [0,1], \\ y(0) = y''(0) = 0, \quad y(1) = k\int_0^1 y(s)ds, & \\ \varphi_p({}^cD^{\alpha}y(0)) = [\varphi_p({}^cD^{\alpha}y(0))]' = 0, & \end{cases} \tag{1.1}$$

where $2 < \alpha \leq 3,\ 1 < \beta \leq 2,\ 0 < k < 2,\ {}^cD^{\alpha}$ and D^{β} are the Caputo and Riemann-Liouville derivatives respectively, $\varphi_p(y) = |y|^{p-2}y$ such that $p > 1$, $\varphi_p^{-1} = \varphi_q$ with $1/p + 1/q = 1$ and $f(t,y) \in C([0,1] \times [0,+\infty), [0,+\infty))$.

A great deal of interest has emerged in the field of fractional differential equations. To be more specific, in the scientific discipline, precisely the area of mathematical modelling of processes in polymer rheology, physics, aerodynamics and chemistry among others, has fully embraced fractional differential equations as a vital tool in the description of hereditary properties for some materials and processes. Advances in fractional calculus theories has led to its applications in engineering, mechanics, chemistry, physics, among others, see [1, 3, 9, 10, 12–17, 20, 24].

An appreciable amount of authors have intensively concentrated on the existence and multiplicity of positive solution for boundary value problems of nonlinear fractional differential equations by applying certain fixed point theorems which include the Guo-Krasnosel'skii fixed point theorem, upper and lower solutions method, the Leggett-Williams fixed point theorem and the Schauder fixed-point theorem to mention but a few [5,6,8,26–28]. Some work has been covered involving integral boundary value problems, detailed approaches on integral boundary value problems can be seen in [4] and references entailed therein.

However, as far as we know, a limited number of work involving both a combination of Caputo-Riemann Liouville derivatives and integral boundary value conditions have been considered some of which include [7, 18, 19, 25]. Our work presents an in depth expansion and comprehensive approach inevitable in the field of fractional differential equations addressing the limitation outlined above.

This paper is organized in such a manner, Section 2 presents some necessary background material, lemmas and definitions. Section 3 deals with the existence of single and multiple positive solutions for the functional differential equation with fractional order (1.1) based on the Guo-Krasnoselskii theorem. Section 4 focuses on the existence of multiple positive solutions for the fractional differential equation by means of applying the Five functional fixed point theorem.

2. Basic definitions and preliminaries

In this section, we introduce some necessary definitions and lemmas.

Definition 1 ([4, Definition 2.2])**.** The integral

$$I^{\beta}g(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds,$$

where $\beta > 0$, is the fractional integral of order β for a function $g(t)$.

Definition 2. The gamma function is defined by the integral formula

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt.$$

The integral converges absolutely for $Re(z) > 0$. For any positive integer n, $\Gamma(n) = (n-1)!$.

Definition 3 ([4, Definition 2.3])**.** For a function $g(t)$ the expression

$$D_{0^+}^{\beta}g(t) = \frac{1}{\Gamma(n-\beta)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\beta-1}g(s)ds,$$

is called the Riemann-Liouville fractional derivative of order β, where $n = [\beta]+1$, and $[\beta]$ denotes the integer part of number β, provided that the right-hand side of the previous equation is point-wise defined on $(0,\infty)$.

Definition 4 ([21, Definition 2.2]). The α order Caputo fractional derivatives for a function $f(t)$ is defined as follows:

$$^{c}D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds, \qquad n-1<\alpha<n.$$

Definition 5 ([21, Definition 2.3]). Let $P\subseteq K$ be a nonempty, convex closed set and K a real Banach space. Then P is called a cone in K provided that

(1) $\lambda y\in P$, for all $y\in P$ and $\lambda\geq 0$;
(2) $y,-y\in P$ implies that $y=0$.

Definition 6 ([21, Definition 2.4]). Let P be a cone in real Banach space K. If the map
$\vartheta\colon P\to[0,\infty)$ is continuous and satisfies

$$\vartheta(tx+(1-t)y)\geq t\vartheta(x)+(1-t)\vartheta(y), \quad x,y\in P,\ t\in[0,1],$$

then ϑ is called a nonnegative continuous concave functional on P.

In a similar way, the map Υ is a nonnegative continuous convex function on a cone P of a real Banach space K provided that $\Upsilon\colon P\to[0,\infty)$ is continuous and

$$\Upsilon(tx+(1-t)y)\leq t\Upsilon(x)+(1-t)\Upsilon(y),$$

for all $x,y\in P$ and $t\in[0,1]$.

Lemma 1 ([2, Lemma 2.2]). *Assume that $g\in C(0,1)\cap L(0,1)$ with a fractional derivative of order $\beta>0$ that belongs to $C(0,1)\cap L(0,1)$. Then*

$$I^{\beta}D^{\beta}g(t)=g(t)+c_1t^{\beta-1}+c_2t^{\beta-2}+\cdots+c_Nt^{\beta-N},$$

for some $c_i\in\mathbb{R}$, $i=1,2,\cdots,N$, where N is the smallest integer greater than or equal to β.

Lemma 2 ([21, Lemma 2.1]). *Assume that $\alpha>0$ and $n=[\alpha]+1$. If the function $y\in L[0,1]\cap C[0,1]$, then there exists $c_i\in\mathbb{R}$, $i=1,2,\ldots,n$, such that*

$$I^{\alpha}(^{c}D^{\alpha}f(t))=f(t)-c_1-c_2t\cdots-c_nt^{n-1}.$$

Lemma 3. *The problem (1.1) has a unique solution as follows:*

$$y(t)=\int_0^1 G(t,s)\varphi_q(I^{\beta}f(s,y(s)))ds, \quad t\in[0,1]$$

where

$$G(t,s)=\begin{cases}\frac{2t(1-s)^{\alpha-1}(\alpha-k(1-s))-\alpha(2-k)(t-s)^{\alpha-1}}{(2-k)\Gamma(\alpha+1)}, & 0\leq s\leq t\leq 1,\\ \frac{2t(1-s)^{\alpha-1}(\alpha-k(1-s))}{(2-k)\Gamma(\alpha+1)}, & 0\leq t\leq s\leq 1,\end{cases} \tag{2.1}$$

is the Green's function.

Proof. Let $u(t) = \varphi_p({}^cD^\alpha y(t))$, we now show that IBVP (1.1) can be expressed as the following IBVP:

$$\begin{cases} D^\beta u(t) + f(t, y(t)) = 0, \\ u(0) = u'(0) = 0 \end{cases} \tag{2.2}$$

and

$$\begin{cases} {}^cD^\alpha y(t) = \varphi_q(u(t)), & t \in (0,1) \\ y(0) = y''(0) = 0, \quad y(1) = k\int_0^1 y(s)ds. \end{cases} \tag{2.3}$$

Using Lemma 1 and (2.2), we get

$$u(t) = -I^\beta f(t, y(t)) + c_1 t^{\beta-1} + c_2 t^{\beta-2},$$

since $u(0) = u'(0) = 0$, then $c_1 = c_2 = 0$ and we have

$$u(t) = -I^\beta f(t, y(t)) = \frac{-1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, y(s))ds.$$

Also, from (2.3) and [4], we get

$$\begin{aligned} y(t) &= \int_0^t \frac{2t(1-s)^{\alpha-1}(\alpha - k(1-s)) - \alpha(t-s)^{\alpha-1}(2-k)}{(2-k)\Gamma(\alpha+1)} \varphi_q(I^\beta f(s, y(s)))ds \\ &\quad + \int_t^1 \frac{2t(1-s)^{\alpha-1}(\alpha - k(1-s))}{(2-k)\Gamma(\alpha+1)} \varphi_q(I^\beta f(s, y(s)))ds \\ &= \int_0^1 G(t,s)\varphi_q(I^\beta f(s, y(s)))ds. \end{aligned}$$

This completes the proof. □

Lemma 4 ([21, Lemma 2.3])**.** *The function $G(t,s)$ defined in (2.1) satisfies the following properties:*

1. $0 < G(t,s) \leq \frac{2}{(2-k)\Gamma(\alpha)}$*, for $t,s \in (0,1)$ if and only if $0 < k < 2$.*
2. $tG(1,s) \leq G(t,s) \leq \frac{2\alpha}{k(\alpha-2)}G(1,s)$*, for all $t,s \in (0,1)$, $2 < \alpha < 3$ and $0 < k < 2$.*
3. $G(t,s)$ *and* $\frac{G(t,s)}{t}$ *are two continuous functions for all $t,s \in [0,1]$, $2 < \alpha < 3$ and $k \neq 2$.*

Lemma 5 ([11, Theorem 2.3])**.** *Let K be a Banach space and let $X \subset K$ be a cone in K. Assume that Ω_1 and Ω_2 are open subsets of K with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : X \cap (\overline{\Omega}_2 \setminus \Omega_1) \to X$ be completely continuous operator. In addition, suppose that either*

1. $\|Ty\| \leq \|y\|$*, for all $y \in X \cap \partial\Omega_1$ and $\|Ty\| \geq \|y\|$, for all $y \in X \cap \partial\Omega_2$ or*
2. $\|Ty\| \leq \|y\|$*, for all $y \in X \cap \partial\Omega_2$ and $\|Ty\| \geq \|y\|$, for all $y \in X \cap \partial\Omega_1$ holds. Then T has a fixed point in $X \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

Lemma 6 ([22, Lemma 2.6]). *Let K be a real Banach space, $P \subset K$ be a cone, $\Omega_r = \{y \in P\colon \|y\| \leq r\}$. Let the operator $T\colon P \cap \Omega_r \to P$ be completely continuous and satisfying $Tu \neq u$, $\forall u \in \partial\Omega_r$.*

Then

(1) *if $\|Tu\| \leq \|u\|$, $\forall u \in \partial\Omega_r$, then $i(T, \Omega_r, P) = 1$,*
(2) *if $\|Tu\| \geq \|u\|$, $\forall u \in \partial\Omega_r$, then $i(T, \Omega_r, P) = 0$.*

3. Main results

In this section we show the existence results for (1.1). We define the operator $T\colon C[0,1] \to C[0,1]$ as

$$Ty(t)\colon = \int_0^1 G(t,s)\varphi_q(I^\beta f(s,y(s)))ds, \tag{3.1}$$

with G defined in (2.1). Let $K = C[0,1]$ be a Banach space endowed with norm $\|\cdot\|$ and $P \subset K$ be the cone expressed as follows

$$P = \left\{ y \in K,\ \frac{y(t)}{t} \in K,\ y(t) \geq \frac{tk(\alpha-2)}{2\alpha}\|y\|, \text{ for all } t \in [0,1] \right\}. \tag{3.2}$$

We set

$$f_0^* = \lim_{y\to 0^+}\left\{\min_{t\in[0,1]} \frac{f(t,y)}{\varphi_p(y)}\right\} \text{ and } f_\infty^* = \lim_{y\to\infty}\left\{\max_{t\in[0,1]} \frac{f(t,y)}{\varphi_p(y)}\right\}.$$

Theorem 1. *Suppose that either of the two following conditions is satisfied:*

(1) *$f_0^* = \infty$ and $f_\infty^* = 0$.*
(2) *$f_0^* = 0$, $f_\infty^* = \infty$ and there exists $\nu > 0$ for which $f(t,\omega\iota) \geq \nu\iota^\delta f(t,\omega)$ for all $\iota \in (0,1]$.*

Then, for all $\alpha \in (2,3)$ and $k \in (0,2)$, the problem (1.1) has at least one solution that belongs to the cone P defined in (3.2)

Proof. Firstly, we prove that $T\colon P \to P$ is completely continuous. By the continuity and the non-negativeness of functions G and f on their domains of definition, we get that if $y \in P$ then $Ty \in K$ and $Ty(t) \geq 0$ for all $t \in [0,1]$.

Furthermore,

$$\frac{Ty(t)}{t} = \int_0^1 \frac{G(t,s)}{t}\varphi_q(I^\beta f(s,y(s))))ds,$$

that is, from condition (3) in Lemma 4, a continuous function for all $y \in K$. We can see that $T(P) \subset P$. We take $y \in P$, then for all $t \in [0,1]$ the following inequalities are satisfied

$$Ty(t) = \int_0^1 G(t,s)\varphi_q(I^\beta f(s,y(s)))ds$$

$$\geq \frac{tk(\alpha-2)}{2\alpha} \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s)\varphi_q(I^\beta f(s,y(s)))ds \right\} = \frac{tk(\alpha-2)}{2\alpha}\|Ty\|.$$

By the continuity of functions G and f, the operator $T: P \to P$ is continuous. Let $\Omega \subset P$ be bounded, that is, there exists a positive constant $\mathcal{M} > 0$ such that $\|y\| \leq \mathcal{M}$ for all $y \in \Omega$.

We define $\mathcal{L} = \max_{0\leq t\leq 1, 0\leq y\leq \mathcal{M}} |f(t,y)| + 1$.

Then, for all $y \in \Omega$, it is satisfied that

$$\begin{aligned}
|Ty(t)| &\leq \int_0^1 G(t,s)\varphi_q(I^\beta f(s,y(s)))ds \\
&\leq \frac{\mathcal{L}^{q-1}}{(\Gamma(\beta+1))^{q-1}} \int_0^1 G(t,s)ds, \quad \text{for all } t \in [0,1].
\end{aligned}$$

Therefore, the set $T(\Omega)$ is bounded in K. For each $y \in \Omega$, we get

$$\begin{aligned}
|(Ty)'(t)| = &\left| -\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi_q(I^\beta f(s,y(s)))ds \right. \\
&\left. + \int_0^1 \frac{2(1-s)^{\alpha-1}(\alpha-k+ks)}{(2-k)\alpha\Gamma(\alpha)}\varphi_q(I^\beta f(s,y(s)))ds \right| \\
\leq &\frac{\mathcal{L}^{q-1}}{(\Gamma(\beta+1))^{q-1}} \left[\frac{1}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha+1)} \right] := \mathcal{N}.
\end{aligned}$$

Consequently, for all $t_1, t_2 \in [0,1], t_1 < t_2$, we obtain

$$|(Ty)(t_2) - Ty)(t_1)| \leq \int_{t_1}^{t_2} |(Ty)'(s)|ds \leq \mathcal{N}(t_2 - t_1).$$

and the set $T(\Omega)$ is equicontinuous. By the Arzela-Ascoli Theorem we conclude that $\overline{T(\Omega)}$ is compact, that is, $T: P \to P$ is a completely continuous operator. We now consider the following cases:

Case 1: In part 1 of Theorem 1, $f_0^* = \infty$ and $f_\infty^* = 0$. Since $f_0^* = \infty$, then there exists a constant $\eta_1 > 0$ such that $f(t,y) \geq \varphi_p(\vartheta_1 y)$ for all $0 < y \leq \eta_1$, where $\vartheta_1 > 0$ satisfies

$$\vartheta_1 \frac{k(\alpha-2)}{2\alpha(\Gamma(\beta+1))^{q-1}} \int_0^1 G(t,s)s^{\beta(q-1)+1}ds \geq 1. \tag{3.3}$$

We assign $y \in P$, such that $\|y\| = \eta_1$, then from (3.3), we get

$$\begin{aligned}
\|Ty\| &= \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s)\varphi_q(I^\beta f(s,y(s)))ds \right\} \\
&\geq \|y\|\vartheta_1 \frac{k(\alpha-2)}{2\alpha(\Gamma(\beta+1))^{q-1}} \int_0^1 G(t,s)s^{\beta(q-1)+1}ds \geq \|y\|.
\end{aligned}$$

Since $f(t,\cdot)$ is a continuous function on $[0,\infty]$, we denote

$$\overline{f}(t,y) = \max_{u\in[0,y]} \{f(t,u)\}.$$

Trivially, $\overline{f}(t,\cdot)$ is non-decreasing on $[0,\infty)$. Also, since $f_\infty^* = 0$, it is easy to see from [4] that

$$\lim_{y\to\infty} \left\{ \max_{t\in[0,1]} \frac{\overline{f}(t,y)}{y} \right\} = 0.$$

We assign $\vartheta_2 > 0$ satisfying

$$\vartheta_2 \frac{2}{(2-k)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \le 1. \tag{3.4}$$

Then, there exists η_2, $\eta_1 > 0$ such that $\overline{f}(t,y) \le \varphi_p(\vartheta_2 y)$ for all $y \ge \eta_2$. We let $y \in P$ be such that $\|y\| = \eta_2$, therefore, from the definition of $\overline{f}$ and (3.4), we obtain the inequalities:

$$\begin{aligned}\|Ty\| &= \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s)\varphi_q(I^\beta f(s,y(s)))ds \right\} \\ &\le \|y\|\vartheta_2 \frac{2}{(2-k)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \le \|y\|.\end{aligned}$$

Therefore, from the Guo Krasnoselskii fixed point theorem, we conclude that IBVP (1.1) has at least one positive solution.

Case 1: We let $\vartheta_2 > 0$ be as given in (3.4), since $f_0 = 0$, there exists a constant $x_1 > 0$ such that $f(t,y) \le \varphi_p(\vartheta_2 y)$ for $0 \le y \le x_1$. We set $y \in P$, such that $\|y\| = x_1$. Then, we get

$$\begin{aligned}Ty(t) = \int_0^1 G(t,s)\varphi_q(I^\beta f(s,y(s)))ds &\le \vartheta_2 \int_0^1 G(t,s)\varphi_q(I^\beta f(s,y(s)))ds \\ &\le \vartheta_2\|y\| \frac{2}{(2-k)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \\ &\le \|y\|.\end{aligned}$$

We now consider $\vartheta_3 > 0$ which satisfies

$$\nu\vartheta_3 \frac{k(\alpha-2)}{2\alpha(\Gamma(\beta+1))^{q-1}} \max_{t\in[0,1]} \left\{ \int_0^1 s^{\beta(q-1)+\delta} G(t,s)ds \right\} \ge 1,$$

where $\nu > 0$ and $\delta > 0$ the constants given in Condition 2 of Theorem 1. $f_\infty^* = \infty$ implies that there exists constants $x_2 > x_1 > 0$ such that $f(t,y) \ge \varphi_p(\vartheta_3 y)$ for all $y \ge x_2$. We let $y \in P$ such that $\|y\| = \frac{2\alpha(\Gamma(\beta+1))^{q-1}}{k(\alpha-2)} x_2$. Therefore, for all $t > 0$, the following inequality holds:

$$\frac{y(t)}{t} \ge \frac{k(\alpha-2)}{2\alpha(\Gamma(\beta+1))^{q-1}} \|y\| = x_2.$$

Condition 2 of Theorem 1 gives us the following property:

$$\begin{aligned}\|Ty\| &= \max_{t\in[0,1]}\left\{\int_0^1 G(t,s)\varphi_q(I^{\beta}f(s,y(s)))ds\right\}\\ &\geq \nu \max_{t\in[0,1]}\left\{\int_0^1 s^{\delta}G(t,s)\varphi_q\left(I^{\beta}f\left(s,\frac{y(s)}{s}\right)\right)ds\right\}\\ &\geq \nu\vartheta_3\frac{k(\alpha-2)}{2\alpha(\Gamma(\beta+1))^{q-1}}\|y\|\max_{t\in[0,1]}\left\{\int_0^1 s^{\beta(q-1)+\delta}G(t,s)ds\right\}\geq\|y\|.\end{aligned}$$

Since $\frac{y(t)}{t}$ is a continuous on (0,1] and $\lim_{t\to0^+}\frac{y(t)}{t}$ exist, then the integrals in the previous inequalities are well defined. Thus by the Guo Krasnoselskii fixed point theorem, we conclude that IBVP (1.1) has at least one positive solution.

□

Remark 1. Condition 2 of Theorem 1 for problem (1.1) generalizes: $f\colon[0,\infty)\to(0,\infty)$ is non-decreasing and there exists $\delta\in(0,1)$ such that $f(\iota\omega)\geq\iota^{\delta}f(\omega)$ for all $\iota\in(0,1)$ and $\omega\in[0,\infty)$, which is condition (A3) imposed in [4]. This condition is less restrictive if δ is greater since $\iota\in(0,1)$. In addition, assumptions of monotonicity are not imposed on the function f.

Example 1. Consider the fractional differential equation:

$$\begin{cases}D^{\frac{3}{2}}(\varphi_2({}^cD^{\frac{5}{2}}y(t)))+1+\frac{1}{2}\sin(y)=0, & t\in[0,1],\\ y(0)=y''(0)=0,\quad y(1)=\frac{\pi}{3+\sin(1)}\int_0^1 y(s)ds, & \\ \varphi_2({}^cD^{\frac{5}{2}}y(0))=[\varphi_2({}^cD^{\frac{5}{2}}y(0))]'=0. & \end{cases}\tag{3.5}$$

Trivially, $f_0^*=\infty$ and $f_\infty^*=0$. Therefore, from Theorem 1, we conclude that problem (3.5) has at least one positive solution.

We now proceed to show that IBVP (1.1) has at least two positive solutions. We define

$$f^0=\lim_{y\to0^+}\sup_{t\in[0,1]}\frac{f(t,y)}{\varphi_p(l_1\|y\|)},\quad f_0=\lim_{y\to0^+}\inf_{t\in[0,1]}\frac{f(t,y)}{\varphi_p(l_2\|y\|)},$$

$$f^\infty=\lim_{y\to+\infty}\sup_{t\in[0,1]}\frac{f(t,y)}{\varphi_p(l_3\|y\|)},\quad f_\infty=\lim_{y\to+\infty}\inf_{t\in[0,1]}\frac{f(t,y)}{\varphi_p(l_4\|y\|)}.$$

Let

$$B=\frac{2}{(2-k)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}}\quad\text{and}\quad B_1=\frac{1}{4(\Gamma(\beta+1))^{q-1}}\int_{s\in I}s^{\beta(q-1)}G(1,s)ds,$$

where $I\in[\frac{1}{4},\frac{3}{4}]$.

Theorem 2. *Assume that $f\in C([0,1]\times[0,+\infty),[0,+\infty))$, and the following conditions hold;*

(N_1) $f_0 = f_\infty = +\infty$.
(N_2) *There exists a constant* $\rho_1 > 0$ *such that* $f(t,y) \le \varphi_p(l_5\|y\|)$ *for* $t \in [0,1]$, $y \in [0,\rho_1]$.

Then, IBVP (1.1) *has at least two positive solutions* y_1 *and* y_2 *such that*

$$0 < \|y_1\| < \rho_1 < \|y_2\|, \qquad \textit{for}$$
$$0 < \frac{1}{l_2 B_1} < 1 < \frac{1}{l_5 B} < +\infty \qquad \textit{and} \qquad 0 < \frac{1}{l_4 B_1} < 1 < \frac{1}{l_5 B} < +\infty. \tag{3.6}$$

Proof. Since

$$f_0 = \lim_{y \to 0^+} \inf_{t\in[0,1]} \frac{f(t,y)}{\varphi_p(l_2\|y\|)} = +\infty,$$

there is $\rho_0 \in (0,\rho_1)$ such that

$$f(t,y) \ge \varphi_p(l_2\|y\|) \text{ for } t \in [0,1],\ y \in [0,\rho_0].$$

Let

$$\Omega_{\rho_0} = \{y \in P\colon \|y\| \le \rho_0\}.$$

Then, for any $y \in \partial\Omega_{\rho_0}$, it follows from Lemma 4 that

$$\begin{aligned}(Ty)(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 I^\beta f(s,y(s))\right) ds\\ &\ge \min_{t\in I}\left\{\frac{l_2 t}{(\Gamma(\beta+1))^{q-1}}\int_0^1 s^{\beta(q-1)} G(1,s)ds\|y\|\right\}\\ &\ge l_2 \frac{1}{4(\Gamma(\beta+1))^{q-1}}\int_{s\in I} s^{\beta(q-1)} G(1,s)ds\|y\|.\end{aligned}$$

Therefore,

$$\|Ty\| \ge l_2 B_1 \|y\|.$$

Considering also (3.6), we get

$$\|Ty\| \ge \|y\|,\ \forall\, y \in \partial\Omega_{\rho_0}.$$

By Lemma 6, we get

$$i(T,\Omega_{\rho_0},P) = 0. \tag{3.7}$$

Also,

$$f_\infty = \lim_{y \to \infty} \inf_{t\in[0,1]} \frac{f(t,y)}{\varphi_p(l_4\|y\|)} = +\infty,$$

there is ρ_0^*, $\rho_0^* > \rho_1$, such that

$$f(t,y) \ge \varphi_p(l_4\|y\|) \text{ for } t \in [0,1],\ y \in [\rho_0^*, +\infty).$$

Let

$$\Omega_{\rho_0^*} = \{y \in P \colon \|y\| \le \rho_0^*\}.$$

Then, for any $y \in \partial\Omega_{\rho_0^*}$, it follows from Lemma 4 that

$$\begin{aligned}(Ty)(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 I^\beta f(s,y(s))\right)ds \\ &\ge \min_{t\in I}\left\{\frac{l_4 t}{(\Gamma(\beta+1))^{q-1}}\int_0^1 s^{\beta(q-1)}G(1,s)ds\|y\|\right\} \\ &\ge l_4\frac{1}{4(\Gamma(\beta+1))^{q-1}}\int_{s\in I} s^{\beta(q-1)}G(1,s)ds\|y\|.\end{aligned}$$

Therefore,

$$\|Ty\| \ge l_4 B_1\|y\|.$$

Considering also (3.6), we get

$$\|Ty\| \ge \|y\|,\ \forall\, y \in \partial\Omega_{\rho_0^*}.$$

By Lemma 6, we get

$$i(T,\Omega_{\rho_0^*},P) = 0.$$

Finally, let $\Omega_{\rho_1} = \{y \in P \colon \|y\| \le \rho_1\}$ for any $y \in \partial\Omega_{\rho_1}$, it follows from Lemma 4, 6 and (N_2) that

$$\begin{aligned}(Ty)(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 I^\beta f(s,y(s))\right)ds \\ &\le \max_{0\le t\le 1}\left\{\frac{l_5}{(\Gamma(\beta+1))^{q-1}}\int_0^1 s^{\beta(q-1)}G(t,s)ds\|y\|\right\} \\ &= l_5\frac{2}{(2-k)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}}\|y\|.\end{aligned}$$

Therefore,

$$\|Ty\| \le l_5 B\|y\|.$$

Considering also (3.6), we get

$$\|Ty\| \le \|y\|,\ \forall\, y \in \partial\Omega_{\rho_1}.$$

By Lemma 6, we get

$$i(T,\Omega_{\rho_1},P) = 1. \tag{3.8}$$

From (3.7)-(3.8) and $\rho_0 < \rho_1 < \rho_0^*$, we get

$$i(T,\Omega_{\rho_0^*}\setminus\overline{\Omega}_{\rho_1},P) = -1, \qquad i(T,\Omega_{\rho_1}\setminus\overline{\Omega}_{\rho_0},P) = 1.$$

Thus, T has a fixed point $y_1 \in \Omega_{\rho_1}\setminus\overline{\Omega}_{\rho_0}$ and a fixed point $y_2 \in \Omega_{\rho_0^*}\setminus\overline{\Omega}_{\rho_1}$. Trivially, $y_1,\ y_2$ are both positive solutions of IBVP (1.1) and $0 < \|y_1\| < \rho_1 < \|y_2\|$. □

Similarly, we get the following results;

Corollary 1. *Assume that* $f \in C([0,1]\times[0,+\infty),[0,+\infty))$ *and the following conditions hold:*

(N_1) $f^0 = f^\infty = 0$.

(N_2) *There exists a constant* $\rho_2 > 0$ *such that* $f(t,y) \geq \varphi_p(l_6\|y\|)$ *for* $t \in [0,1]$, $y \in [0,\rho_2]$. *Then IBVP* (1.1) *has at least two positive solutions* y_1 *and* y_2 *such that*

$$0 < \|y_1\| < \rho_2 < \|y_2\| \quad \textit{for}$$
$$0 < \frac{1}{l_6B_1} < 1 < \frac{1}{l_3B} < +\infty \qquad \textit{and} \qquad 0 < \frac{1}{l_6B_1} < 1 < \frac{1}{l_1B} < +\infty.$$

Example 2. Consider the following integral boundary value problem:

$$\begin{cases} D^{\frac{9}{5}}(\varphi_2({}^cD^{\frac{7}{3}}y(t))) + \frac{1}{3}(\|y\|^{\frac{1}{2}} + \frac{2}{3}\|y\|) = 0, & t \in [0,1],\\ y(0) = y''(0) = 0, \quad y(1) = \frac{3}{4}\int_0^1 y(s)ds,\\ \varphi_2({}^cD^{\frac{7}{3}}y(0)) = [\varphi_2({}^cD^{\frac{7}{3}}y(0))]' = 0, \end{cases} \tag{3.9}$$

where $\alpha = \frac{7}{3},\ \beta = \frac{9}{5},\ p = 2,\ k = \frac{3}{4}$. By computation we see that $B = 0.80159$ and $B_1 = 0.0022194$. Taking $\rho_1 = 9,\ l_5 = \frac{1}{3}$, we get

$$f(t,y) \leq \frac{1}{3}\left(3 + \frac{2}{3}.9\right) = 3 = \varphi_p(l_5\|y\|) = \varphi_2\left(\frac{1}{3}.9\right), \text{ for } t \in [0,1],\ y \in [0,\rho_1].$$

Therefore, condition (N_2) is satisfied. It can be easily seen that condition (N_1) holds. Also, let $l_2 = 130$ and $l_4 = 121$, we get

$$0 < \frac{1}{l_2B_1} < 1 < \frac{1}{l_5B} < +\infty \quad \text{and} \quad 0 < \frac{1}{l_4B_1} < 1 < \frac{1}{l_5B} < +\infty.$$

Hence, by Theorem 2, IBVP (3.9) has at least two solutions y_1 and y_2 such that $0 < \|y_1\| < 9 < \|y_2\|$ for the given values of $l_5,\ l_2$ and l_4.

4. Existence of multiple positive solutions

Sufficient conditions for the existence of at least three positive solutions for the p-Laplacian IBVP (1.1) are derived by means of the Five functional fixed point theorem in this section. Let γ,Υ and ω be nonnegative continuous convex functionals on P, ψ and ϑ be a nonnegative continuous concave functionals on P. Therefore, for positive real numbers h',a',b',d' and c', we denote the following convex sets

$$\begin{aligned} &P(\gamma,c') = \{y \in P\colon \gamma(y) < c'\},\\ &P(\gamma,\vartheta,a',c') = \{y \in P\colon a' \leq \vartheta(y); \gamma(y) \leq c'\},\\ &R(\gamma,\Upsilon,d',c') = \{y \in P\colon \Upsilon(y) \leq d'; \gamma(y) \leq c'\},\\ &P(\gamma,\omega,\vartheta,a',b',c') = \{y \in P\colon a' \leq \vartheta(y); \omega(y) \leq b'; \gamma(y) \leq c'\}, \end{aligned}$$

$$R(\gamma,\Upsilon,\psi,h',d',c') = \{y\in P\colon h'\le\psi(y);\Upsilon(y)\le d';\gamma(y)\le c'\}.$$

Theorem 3 ([23, Theorem 4.1]). *Suppose P be a cone in a real Banach space K. Let ϑ and ψ be nonnegative continuous concave functionals on P, γ,Υ and ω be a nonnegative continuous convex functionals on P, such that for some positive numbers c' and e', $\vartheta(y)\le\Upsilon(y)$ and $\|y\|\le e'\gamma(y)$, for all $y\in\overline{P(\gamma,c')}$. Suppose further that $T\colon\overline{P(\gamma,c')}\to\overline{P(\gamma,c')}$ is completely continuous and there exist constants h',d',a' and $b'\ge 0$ with $0<d'<a'$ such that each of the following is satisfied*

- (C_1) *$\{y\in P(\gamma,\omega,\vartheta,a',b',c')\colon \vartheta(y)>a\}\neq\varnothing$ and $\vartheta(Ty)>a'$ for $y\in P(\gamma,\omega,\vartheta,a',b',c')$,*
- (C_2) *$\{y\in R(\gamma,\Upsilon,\psi,h',d',c')\colon \Upsilon(y)>d'\}\neq\varnothing$ and $\Upsilon(Ty)>d'$ for $y\in R(\gamma,\Upsilon,\psi,h',d',c')$,,*
- (C_3) *$\vartheta(Ty)>a'$ provided $y\in P(\gamma,\vartheta,a',c')$ with $\omega(Ty)>b'$,*
- (C_4) *$\Upsilon(Ty)<d'$ provided $y\in R(\gamma,\Upsilon,\psi,h',d',c')$ with $\psi(Ty)<h'$.*

Then, T has at least three fixed points $y_1,y_2,y_3\in\overline{P(\gamma,c')}$ such that $\Upsilon(y_1)\le d', a'<\vartheta(y_2)$ and $d'<\Upsilon(y_3)$ with $\vartheta(y_3)<a$.

We denote the nonnegative continuous concave functionals ϑ,ψ and the nonnegative continuous convex functionals Υ,ω,γ on P by

$$\vartheta(y)=\min_{t\in I}y(t),\ \psi(y)=\min_{t\in I_1}y(t),$$
$$\gamma(y)=\max_{t\in[0,1]}y(t),\ \Upsilon(y)=\max_{t\in I_1}y(t),\ \omega(y)=\max_{t\in I}y(t),$$

where $I=[\frac14,\frac34]$ and $I_1=[\frac13,\frac23]$. For any $y\in P$,

$$\vartheta(y)=\min_{t\in I}y(t)\le\max_{t\in I_1}y(t)=\Upsilon(y) \tag{4.1}$$

and

$$\|y\|\le 4M\min_{t\in I}y(t)\le 4M\max_{t\in[0,1]}y(t)=4M\gamma(y), \tag{4.2}$$

where $M=\frac{2\alpha}{k(\alpha-2)}$.

Let

$$E=\left[\frac{1}{4(\Gamma(\beta+1))^{q-1}}\int_{s\in I}s^{\beta(q-1)}G(1,s)ds\right]^{-1},$$
$$Z=\left[\frac{M}{(\Gamma(\beta+1))^{q-1}}\int_0^1 s^{\beta(q-1)}G(1,s)ds\right]^{-1}.$$

Theorem 4. *Suppose there exist constants $0<a'<b'<4Mb'\le c'$ and assume that f satisfies the following conditions:*

- (H_1) *$f(t,y)<\varphi_p(a'Z)$ for $(t,y)\in[0,1]\times[\frac{1}{4M}a',a']$;*
- (H_2) *$f(t,y)>\varphi_p(b'E)$ for $(t,y)\in[\frac14,\frac34]\times[b',4Mb']$;*
- (H_3) *$f(t,y)<\varphi_p(c'Z)$ for $(t,y)\in[0,1]\times[0,c']$.*

Then the integral boundary value problem (1.1) has at least three positive solutions y_1,y_2 and y_3 such that $\Upsilon(y_1)<a', b'<\vartheta(y_2)$ and $a'<\Upsilon(y_3)$ with $\vartheta(y_3)<b'$.

Proof. We need to show that there exist three fixed points $y_1, y_2, y_3 \in P$ of T defined in (3.1).

Since T is completely continuous and from (4.1) - (4.2), for each $y \in P$, $\vartheta(y) \leq \Upsilon(y)$ and $\|y\| \leq 4M\gamma(y)$, we now show that $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$. Let $y \in \overline{P(\gamma, c')}$, then

$0 \leq y \leq c'$. We use (H_3) to get

$$\begin{aligned}\gamma(Ty) &= \max_{t\in[0,1]} \int_0^1 G(t,s)\varphi_q(I^\beta f(s,y(s)))ds \\ &\leq \frac{c'ZM}{(\Gamma(\beta+1))^{q-1}} \int_0^1 s^{\beta(q-1)} G(1,s)ds < c'.\end{aligned}$$

Thus, $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$. We show that conditions (C_1) and (C_2) of Theorem 3 are satisfied. Trivially,

$$\frac{b' + 4Mb'}{2} \in \{y \in P(\gamma, \omega, \vartheta, b', 4Mb', c') : \vartheta(y) > b\} \neq \varnothing$$

and

$$\frac{\frac{1}{4M}a' + a'}{2} \in \{y \in R(\gamma, \Upsilon, \psi, \frac{1}{4M}a', a', c') : \Upsilon(y) < a'\} \neq \varnothing.$$

We proceed by letting $y \in P(\gamma, \omega, \vartheta, b', 4Mb', c')$ or $y \in R(\gamma, \Upsilon, \psi, \frac{1}{4M}a', a', c')$. Then, $b' \leq y \leq 4Mb'$ and $\eta a' \leq y \leq a'$, where $\eta > 0$. We apply condition (H_2) to obtain

$$\begin{aligned}\vartheta(Ty) &= \min_{t\in I} \int_0^1 G(t,s)\varphi_q(I^\beta f(s,y(s)))ds \\ &> \frac{1}{4(\Gamma(\beta+1))^{q-1}} b'E \int_{s\in I} s^{\beta(q-1)} G(1,s)ds = b'.\end{aligned}$$

Intuitively, by condition (H_1), we get

$$\begin{aligned}\Upsilon(Ty) &= \max_{t\in I_1} \int_0^1 G(t,s)\varphi_q(I^\beta f(s,y(s)))ds \\ &< \frac{Ma'Z}{(\Gamma(\beta+1))^{q-1}} \int_0^1 s^{\beta(q-1)} G(1,s)ds = a'.\end{aligned}$$

To show that (C_3) is satisfied, let $y \in P(\gamma, \vartheta, b', c')$ with $\omega(Ty) > 4Mb'$. Then

$$\begin{aligned}\vartheta(Ty) &= \min_{t\in I} \int_0^1 G(t,s)\varphi_q(I^\beta f(s,y(s)))ds \\ &\geq \frac{1}{4M} \max_{t\in I} \int_0^1 G(t,s)\varphi_q(I^\beta f(s,y(s)))ds = \frac{1}{4M}\omega(Ty) > b'.\end{aligned}$$

Finally, we prove that (H_4) holds. Let $y \in R(\gamma, \Upsilon, a', c')$ with $\psi(Ty) < \frac{1}{4M}a'$. Then, we get

$$\Upsilon(Ty) = \max_{t \in I_1} \int_0^1 G(t,s)\varphi_q(I^\beta f(s,y(s)))ds$$
$$\leq 4M \min_{t \in I_1} \int_0^1 G(t,s)\varphi_q(I^\beta f(s,y(s)))ds = 4M\psi(Ty) < a'.$$

We have shown that all conditions of Theorem 3 are satisfied. Therefore, the IBVP (1.1) has at least three positive solutions y_1, y_2 and y_3 such that $\Upsilon(y_1) < a', b' < \vartheta(y_2)$ and $a' < \Upsilon(y_3)$ with $\vartheta(y_3) < b'$. This completes the proof. $\square$

Example 3. Consider the fractional differential equation:

$$\begin{cases} D^{\frac{3}{2}}(\varphi_2({}^cD^{\frac{5}{2}}y(t))) + f(t,y(t)) = 0, & t \in [0,1], \\ y(0) = y''(0) = 0, \quad y(1) = \frac{1}{2}\int_0^1 y(s)ds, \\ \varphi_2({}^cD^{\frac{5}{2}}y(0)) = [\varphi_2({}^cD^{\frac{5}{2}}y(0))]' = 0. \end{cases} \tag{4.3}$$

where

$$f(t,y) = \begin{cases} \frac{t}{20} + y^2 + 216y^3, & y \leq 1, \\ \frac{t}{20} + y + 216, & y > 1. \end{cases}$$

We set $a' = \frac{1}{10}$, $b' = 1$ and $c' = 100$. By computation, $4Mb' = 80$, $E = 216{,}02$ and $Z = 4{,}0755$. As a result, $f(t,y)$ satisfies

$$f(t,y) = \frac{t}{20} + y^2 + 216y^3 \leq 0{,}28 < \varphi_2\left(a'Z\right) \approx 0{,}41 \text{ for } (t,y) \in [0,1] \times \left[\frac{1}{400}, \frac{1}{10}\right],$$
$$f(t,y) = \frac{t}{20} + y + 216 \geq 217{,}01 > \varphi_2\left(b'E\right) \approx 216{,}02 \text{ for } (t,y) \in I \times [1,80],$$
$$f(t,y) = \frac{t}{20} + y + 216 \leq 316{,}05 < \varphi_2\left(c'Z\right) \approx 407{,}55 \text{ for } (t,y) \in [0,1] \times [0,100].$$

Since all conditions of Theorem 4 hold. Therefore, problem (4.3) has at least three positive solutions by Theorem 4.

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