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ON OPERATORS WHOSE CORE–EP INVERSE IS n-POTENT

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Abstract. The main contribution of this paper is to establish a number of equivalent conditions for the core–EP inverse of an operator, to be n-potent. We prove that the core–EP inverse of an operator is n-potent if and only if the Drazin inverse of the same operator is n-potent. Thus, we present new characterizations for n-potency of the Drazin inverse. Consequently, we get many characterizations for the core–EP inverse (and Drazin inverse) to be an idempotent. We observe that the core–EP inverse of an operator is idempotent if and only it is the orthogonal projector. Furthermore, we show that the n-potency of an operator implies n-potency of its core–EP inverse and develop the condition for the converse to hold. Applying these results, we obtain necessary and sufficient conditions for the n-potency and idempotency of the core inverse. Notice that the core inverse of an operator is n-potent (or idempotent) if and only if the given operator is n-potent (idempotent).

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1. INTRODUCTION

Let X and Y be arbitrary Hilbert spaces, and let $\mathcal{B}(X,Y)$ be the set of all bounded linear operators from X to Y. Especially, $\mathcal{B}(X) = \mathcal{B}(X,X)$. Denote by A^*, $R(A)$ and $N(A)$ the adjoint, range and null space of $A \in \mathcal{B}(X,Y)$, respectively.

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For $n \geq 2$, an operator $A \in \mathcal{B}(X)$ is called n-potent if $A^n = A$. In the case that $n = 2$, an operator A satisfying $A^2 = A$ is an idempotent (or projector). If an idempotent A satisfies $A = A^*$, we say that A is the orthogonal projector. Some interesting results about n-potent elements of rings can be found in [12].

It is well known that $B \in \mathcal{B}(Y,X)$ is the Moore–Penrose inverse of $A \in \mathcal{B}(X,Y)$ if

$$ABA = A, \qquad BAB = B, \qquad (AB)^* = AB, \qquad (BA)^* = BA.$$

The Moore–Penrose inverse of A is unique (if it exists) and denoted by $A^\dagger$ [13, 14]. Recall that $A^\dagger$ exists if and only if $R(A)$ is closed in Y. For $A \in \mathcal{B}(X,Y)$, we set $A\{1,3\} = \{B \in \mathcal{B}(Y,X) : ABA = A \text{ and } (AB)^* = AB\}$.

The Drazin inverse of $A \in \mathcal{B}(X)$ is an operator $B \in \mathcal{B}(X)$ for which

$$AB = BA, \quad BAB = B \quad \text{and} \quad A^{k+1}B = A^k,$$

where k is the index of A (denoted by $ind(A)$), i.e. the smallest non-negative integer k such that the previous three equations are satisfied. The Drazin inverse of A is unique (if it exists) and denoted by A^D [14]. In the case that $ind(A) = 1$, the Drazin inverse becomes the group inverse $A^\#$ of A. We use $\mathcal{B}(X)^D$ and $\mathcal{B}(X)^\#$ to denote the sets of all Drazin invertible and group invertible operators in $\mathcal{B}(X)$, respectively.

Firstly, the core–EP inverse was presented in [17] for a square matrix and then it was generalized in [15,16] for Hilbert space operators. If $A \in \mathcal{B}(X)^D$ and $k = ind(A)$, there exists the unique core–EP inverse $B \in \mathcal{B}(X)$ of A (denoted by $A^{\textcircled{\dagger}}$) satisfying [15]:

$$BAB = B \quad \text{and} \quad R(B) = R(B^*) = R(A^k).$$

When $ind(A) = 1$, the core–EP inverse of A reduces to the core inverse $A^{\textcircled{\#}}$ of A [1]. The core and core–EP inverses have attracted attentions of many authors [3, 4, 8, 10, 23]. In particular, various expressions of core-EP inverse were given in [5, 11, 20], iterative method for computing core-EP inverse was proved in [18]; limit representations for core-EP inverse in [21]; continuity of core-EP inverse was presented in [7]. Some generalizations of the core-EP inverse were considered for tensors in [19].

Inspired by some matrix equations occurring in physics and involving assumption that the matrices in them have an an idempotent Moore-Penrose inverse, the class of square matrices which have idempotent Moore-Penrose inverse was investigated in [2]. Beside of characterizations for an idempotent Moore-Penrose inverse, the authors studied the relation between idempotency of a given matrix and its Moore-Penrose inverse. These results were extended to elements of rings in [22].

The aim of this paper is to consider the n-potency of the core–EP inverse of an operator. Precisely, we prove many equivalent conditions for the core–EP inverse of a Drazin invertible operator A, to be n-potent. We observe that $A^{\textcircled{\dagger}}$ is n-potent if and only if A^D is n-potent. As a consequence, we obtain a set of characterizations for $A^{\textcircled{\dagger}}$ to be an idempotent. Remark that we present new characterizations for n-potency and idempotency of the Drazin inverse. Notice that $A^{\textcircled{\dagger}}$ is an idempotent if and only $A^{\textcircled{\dagger}}$ is the orthogonal projector. Also, we verify that the n-potency of A

implies n-potency of its core–EP inverse and consider a condition for the converse to be satisfied. Applying these results, we get necessary and sufficient conditions for the n-potency and idempotency of the core inverse. Remark that $A^{\#\!\!\!\!\bigcirc}$ is n-potent (or idempotent) if and only if A is n-potent (idempotent).

This is the content of this paper. In Section 2, we present a number of characterizations for the n-potency and idempotency of the core–EP inverse as well as the relation between the n-potency of a given operator and its core–EP inverse. Section 3 contains equivalent conditions for the n-potency and idempotency of the core inverse.

2. Main results

To develop a list of equivalent conditions for the core–EP inverse to be n-potent, we firstly present one auxiliary result related to the expressions for the power of the core–EP inverse.

Lemma 1. *If $A \in \mathcal{B}(X)^D$ and $k = ind(A)$, then*

$$(A^{\textcircled{D}})^n = (A^D)^n A^k (A^k)^\dagger = (A^D)^n A^k (A^k)^{(1,3)},$$

for any $n \geq 1$ and $(A^k)^{(1,3)} \in (A^k)\{1,3\}$.

Proof. It is well known, by [6, Theorem 2.3], that $A^{\textcircled{D}} = A^D A^k (A^k)^\dagger$. Assume that $(A^{\textcircled{D}})^n = (A^D)^n A^k (A^k)^\dagger$, for $n \geq 1$. Then

$$(A^{\textcircled{D}})^{n+1} = (A^{\textcircled{D}})^n A^{\textcircled{D}} = (A^D)^n (A^k (A^k)^\dagger A^k) A^D (A^k)^\dagger = (A^D)^{n+1} A^k (A^k)^\dagger.$$

The rest is clear. □

In the following theorem, for a Drazin invertible operator A, notice that $A^{\textcircled{D}}$ is n-potent if and only if A^D is n-potent.

Theorem 1. *If $A \in \mathcal{B}(X)^D$, $k = ind(A)$ and $n \geq 2$, then the following statements are equivalent:*

(i) $A^{\textcircled{D}}$ *is n-potent;*
(ii) $(A^{\textcircled{D}})^{n-1} A^k = A^k$;
(iii) $A^{k+n-1} = A^k$;
(iv) $A^D A^k = A^{k+n-2}$;
(v) $A^k (A^k)^{(1,3)} = (A^{\textcircled{D}})^{n-1}$, *for* $(A^k)^{(1,3)} \in (A^k)\{1,3\}$;
(v') $A^k (A^k)^\dagger = (A^{\textcircled{D}})^{n-1}$;
(vi) $A^D (A^{\textcircled{D}})^{n-1} = A^{\textcircled{D}}$;
(vii) $AA^{\textcircled{D}} = (A^{\textcircled{D}})^{n-1}$;
(viii) $(A^{\textcircled{D}})^{n-1}$ *is orthogonal projector;*
(ix) $(A^{\textcircled{D}})^{n-1}$ *is an idempotent;*
(x) $A^D A = (A^D)^{n-1}$;
(xi) A^D *is n-potent;*
(xii) $(A^k)^* = (A^k)^* (A^{\textcircled{D}})^{n-1}$;

(xiii) $(A^{\textcircled{D}})^m = (A^{\textcircled{D}})^{m+n-1}$, *for some/any* $m \geq 1$*;*
(xiv) $(A^D)^m = (A^D)^{m+n-1}$, *for some/any* $m \geq 1$.

Proof. (i) $\Rightarrow$ (ii): By the hypothesis $(A^{\textcircled{D}})^n = A^{\textcircled{D}}$, we obtain

$$A^k = A^{\textcircled{D}}A^{k+1} = (A^{\textcircled{D}})^n A^{k+1} = (A^{\textcircled{D}})^{n-1}A^k.$$

(ii) $\Rightarrow$ (iii): Using $(A^{\textcircled{D}})^{n-1}A^k = A^k$, we get

$$A^{k+n-1} = A^kA^{n-1} = (A^{\textcircled{D}})^{n-1}A^{k+n-1} = (A^{\textcircled{D}})^{n-2}A^{k+n-2} = \cdots = A^{\textcircled{D}}A^{k+1} = A^k.$$

(iii) $\Rightarrow$ (iv): Multiplying the equality $A^{k+n-1} = A^k$ by A^D from the left hand side, notice that $A^DA^k = A^DA^{k+n-1} = A^{k+n-2}$.

(iv) $\Rightarrow$ (v): The hypothesis $A^DA^k = A^{k+n-2}$ and Lemma 1 imply

$$A^k(A^k)^{(1,3)} = (A^D)^{n-2}A^{k+n-2}(A^k)^{(1,3)} = (A^D)^{n-1}A^k(A^k)^{(1,3)} = (A^{\textcircled{D}})^{n-1},$$

for $(A^k)^{(1,3)} \in (A^k)\{1,3\}$.

(v) $\Rightarrow$ (vi): If $A^k(A^k)^{(1,3)} = (A^{\textcircled{D}})^{n-1}$, for $(A^k)^{(1,3)} \in (A^k)\{1,3\}$, then $A^D(A^{\textcircled{D}})^{n-1} = A^DA^k(A^k)^{(1,3)} = A^{\textcircled{D}}$.

(vi) $\Rightarrow$ (vii): Applying $A^D(A^{\textcircled{D}})^{n-1} = A^{\textcircled{D}}$ and Lemma 1, we have

$$AA^{\textcircled{D}} = AA^D(A^{\textcircled{D}})^{n-1} = AA^D(A^D)^{n-1}A^k(A^k)^{(1,3)} = (A^D)^{n-1}A^k(A^k)^{(1,3)} = (A^{\textcircled{D}})^{n-1},$$

for $(A^k)^{(1,3)} \in (A^k)\{1,3\}$.

(vii) $\Rightarrow$ (i): Multiplying $AA^{\textcircled{D}} = (A^{\textcircled{D}})^{n-1}$ by $A^{\textcircled{D}}$ from the left hand side, we see that $A^{\textcircled{D}} = (A^{\textcircled{D}})^n$.

(v) $\Rightarrow$ (viii) $\Rightarrow$ (ix): These implications are evident.

(ix) $\Rightarrow$ (vii): Because $(A^{\textcircled{D}})^{n-1}$ is orthogonal projector, then

$$(A^{\textcircled{D}})^n = A^{\textcircled{D}}(A^{\textcircled{D}})^{n-1} = A^{\textcircled{D}}(A^{\textcircled{D}})^{2n-2} = (A^{\textcircled{D}})^{2n-1}.$$

Therefore,

$$AA^{\textcircled{D}} = A^n(A^{\textcircled{D}})^n = A^n(A^{\textcircled{D}})^{2n-1} = AA^{\textcircled{D}}(A^{\textcircled{D}})^{n-1} = (A^{\textcircled{D}})^{n-1}.$$

(iii) $\Rightarrow$ (x): Since $A^{k+n-1} = A^k$, then

$$A^DA = (A^D)^{k+n-1}A^{k+n-1} = (A^D)^{k+n-1}A^k = (A^D)^{n-1}.$$

(x) $\Rightarrow$ (xi): The condition $A^DA = (A^D)^{n-1}$ yields $A^D = (A^DA)A^D = (A^D)^n$.

(xi) $\Rightarrow$ (iii): From $A^D = (A^D)^n$, we get

$$A^{k+n-1} = A^{k+n}A^D = A^{k+n}(A^D)^n = A^{k+1}A^D = A^k.$$

(v') $\Rightarrow$ (xii): Multiplying $A^k(A^k)^\dagger = (A^{\textcircled{D}})^{n-1}$ by $(A^k)^*$ from the left hand side, we see that $(A^k)^* = (A^k)^*(A^{\textcircled{D}})^{n-1}$.

(xii) $\Rightarrow$ (v'): The assumption $(A^k)^* = (A^k)^*(A^{\textcircled{D}})^{n-1}$ and Lemma 1 imply

$$\begin{aligned} A^k(A^k)^\dagger &= ((A^k)^\dagger)^*(A^k)^* = ((A^k)^\dagger)^*(A^k)^*(A^{\textcircled{D}})^{n-1} \\ &= (A^k(A^k)^\dagger A^k)(A^D)^{n-1}(A^k)^\dagger = (A^D)^{n-1}A^k(A^k)^\dagger = (A^{\textcircled{D}})^{n-1}. \end{aligned}$$

(i) $\Rightarrow$ (xiii): It is clear.

(xiii) $\Rightarrow$ (vii): Assume that $(A^{\textcircled{D}})^m = (A^{\textcircled{D}})^{m+n-1}$, for $m \geq 1$. Then

$$AA^{\textcircled{D}} = A^m (A^{\textcircled{D}})^m = A^m (A^{\textcircled{D}})^{m+n-1} = AA^{\textcircled{D}} (A^{\textcircled{D}})^{n-1} = (A^{\textcircled{D}})^{n-1}.$$

(x) $\Rightarrow$ (xiv): This implication is obvious.

(xiv) $\Rightarrow$ (x): Applying $(A^D)^m = (A^D)^{m+n-1}$, for $m \geq 1$, we have

$$A^D = A^{m-1}(A^D)^m = A^{m-1}(A^D)^{m+n-1} = (A^D)^n.$$

□

By Theorem 1, we can obtain more characterizations for $A^{\textcircled{D}}$ to be n-potent operator. Recall that $B \in \mathcal{B}(Y,X)\backslash\{0\}$ is an outer inverse of $A \in \mathcal{B}(X,Y)$ if $BAB = B$ is satisfied.

Corollary 1. *If $A \in \mathcal{B}(X)^D$, $k = ind(A)$ and $n \geq 2$, then the following statements are equivalent:*

(i) *$A^{\textcircled{D}}$ is n-potent;*
(ii) *$(A^{\textcircled{D}})^n$ is an outer inverse of A;*
(iii) *$A^D (A^{\textcircled{D}})^{n-1}$ is an outer inverse of A;*
(iv) *$(A^D)^n$ is an outer inverse of A;*
(v) *$A^D A$ is an outer inverse of A^{n-1}.*

Proof. (i) $\Rightarrow$ (ii): Since $(A^{\textcircled{D}})^n = A^{\textcircled{D}}$, we conclude that

$$(A^{\textcircled{D}})^n = A^{\textcircled{D}} = A^{\textcircled{D}} A A^{\textcircled{D}} = (A^{\textcircled{D}})^n A (A^{\textcircled{D}})^n,$$

i.e. $(A^{\textcircled{D}})^n$ is an outer inverse of A.

(ii) $\Rightarrow$ (i): Notice that $(A^{\textcircled{D}})^n = (A^{\textcircled{D}})^n A (A^{\textcircled{D}})^n = (A^{\textcircled{D}})^{2n-1}$. By Theorem 1(xiii), for $m = n$, we deduce that $A^{\textcircled{D}}$ is n-potent.

(i) $\Rightarrow$ (iii): Using Theorem 1(vi), $A^D (A^{\textcircled{D}})^{n-1} = A^{\textcircled{D}}$ and so $A^D (A^{\textcircled{D}})^{n-1}$ is an outer inverse of A.

(iii) $\Rightarrow$ (i): Firstly, by Lemma 1, we can check that $AA^D (A^{\textcircled{D}})^m = (A^{\textcircled{D}})^m$, for $m \geq 1$. Now, from $A^D (A^{\textcircled{D}})^{n-1} = A^D (A^{\textcircled{D}})^{n-1} A A^D (A^{\textcircled{D}})^{n-1} = A^D (A^{\textcircled{D}})^{2n-2}$, we get $AA^D (A^{\textcircled{D}})^{n-1} = AA^D (A^{\textcircled{D}})^{2n-2}$, that is, $(A^{\textcircled{D}})^{n-1} = (A^{\textcircled{D}})^{2n-2}$. The rest is clear by Theorem 1(xiii).

(i) $\Rightarrow$ (iv): Applying Theorem 1(x), we see that $(A^D)^n = A^D$ is an outer inverse of A.

(iv) $\Rightarrow$ (i): Because $(A^D)^n = (A^D)^n A (A^D)^n = (A^D)^{2n-1}$, by Theorem 1(xiii), we conclude that $A^{\textcircled{D}}$ is n-potent.

(i) $\Rightarrow$ (v): According to Theorem 1(ix), $A^D A = (A^D)^{n-1} = (A^{n-1})^D$ is an outer inverse of A^{n-1}.

(v) $\Rightarrow$ (i): Suppose that $A^D A$ is an outer inverse of A^{n-1}. Then $A^D A = A^D A A^{n-1} A^D A = A^D A^n$ gives

$$(A^D)^{n-1} = (A^D A)(A^D)^{n-1} = A^D A^n (A^D)^{n-1} = (A^D)^n A^n = A^D A.$$

By Theorem 1(ix), $A^{\textcircled{D}}$ is n-potent. □

In the case that $n = 2$ in Theorem 1 and Corollary 1, we present necessary and sufficient conditions for $A^{\textcircled{D}}$ to be an idempotent. Remark that $A^{\textcircled{D}}$ is an idempotent if and only if $A^{\textcircled{D}}$ is the orthogonal projector.

Corollary 2. *If $A \in \mathcal{B}(X)^D$ and $k = ind(A)$, then the following statements are equivalent:*

(i) $A^{\textcircled{D}}$ *is an idempotent;*
(ii) $A^{\textcircled{D}}A^k = A^k$;
(iii) $A^{k+1} = A^k$;
(iv) $A^D A^k = A^k$;
(v) $A^k(A^k)^{(1,3)} = A^{\textcircled{D}}$, *for* $(A^k)^{(1,3)} \in (A^k)\{1,3\}$;
(v') $A^k(A^k)^\dagger = A^{\textcircled{D}}$;
(vi) $A^D A^{\textcircled{D}} = A^{\textcircled{D}}$;
(vii) $AA^{\textcircled{D}} = A^{\textcircled{D}}$;
(viii) $A^{\textcircled{D}}$ *is orthogonal projector;*
(ix) $A^D A = A^D$;
(x) A^D *is an idempotent;*
(xi) $(A^k)^* = (A^k)^* A^{\textcircled{D}}$;
(xii) $(A^{\textcircled{D}})^m = (A^{\textcircled{D}})^{m+1}$, *for some/any* $m \geq 1$;
(xiii) $(A^D)^m = (A^D)^{m+1}$, *for some/any* $m \geq 1$;
(xiv) $(A^{\textcircled{D}})^2$ *is an outer inverse of* A;
(xv) $A^D A^{\textcircled{D}}$ *is an outer inverse of* A;
(xvi) $(A^D)^2$ *is an outer inverse of* A;
(xvii) $A^D A$ *is an outer inverse of* A.

Now, we consider the relation between n-potency of A and $A^{\textcircled{D}}$. We firstly show that if A is n-potent, then $A^{\textcircled{D}}$ is n-potent.

Lemma 2. *Let $n \geq 2$. If $A \in \mathcal{B}(X)^D$ is n-potent, then $A^{\textcircled{D}}$ is n-potent.*

Proof. Using $A^n = A$, we obtain $A^{\textcircled{D}} = A(A^{\textcircled{D}})^2 = A^n(A^{\textcircled{D}})^{n+1} = A(A^{\textcircled{D}})^{n+1} = (A^{\textcircled{D}})^n$. □

In the following example, we remark that the converse of Lemma 2 does not hold in general.

Example 1. Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

on $X = \mathbb{C}^2$. Because

$$A^{\textcircled{\dagger}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad A^n = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad n \geq 2,$$

we deduce that $(A^{\textcircled{\dagger}})^n = A^{\textcircled{\dagger}}$ and $A^n \neq A$, for $n \geq 2$. Hence, $A^{\textcircled{\dagger}}$ is n-potent and A is not n-potent, for $n \geq 2$.

We establish an additional condition under which n-potency of $A^{\textcircled{\dagger}}$ implies n-potency of A. Also, we give more characterizations of n-potency of A which involve the core–EP inverse of A.

Theorem 2. *If $A \in \mathcal{B}(X)^D$ and $n \geq 2$, then the following statements are equivalent:*

(i) *A is n-potent;*
(ii) *$A^{\textcircled{\dagger}}$ is n-potent and $(I - A^{n-1})A(I - AA^{\textcircled{\dagger}}) = 0$;*
(iii) *$A^{\textcircled{\dagger}}A^n = A^{\textcircled{\dagger}}A$ and $A - AA^{\textcircled{\dagger}}A$ is n-potent;*
(iv) *$A^{\textcircled{\dagger}}A^n = A^{\textcircled{\dagger}}A$ and $(I - AA^{\textcircled{\dagger}})A(I - A^{n-1}) = 0$;*
(v) *$A^2A^{\textcircled{\dagger}}$ is n-potent and $(I - A^{n-1})A(I - AA^{\textcircled{\dagger}}) = 0$.*

Proof. (i) $\Rightarrow$ (ii): By Lemma 2, A is n-potent implies that $A^{\textcircled{\dagger}}$ is n-potent. The rest is evident by $A^n = A$.

(ii) $\Rightarrow$ (i): According to [15, Corollary 2.2], for $k = ind(A)$, the operators A and $A^{\textcircled{\dagger}}$ can be represented with respect to the orthogonal sum $X = R(A^k) \oplus N((A^k)^*)$ as:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \qquad \text{and} \qquad A^{\textcircled{\dagger}} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.1}$$

where $A_1 \in \mathcal{B}(R(A^k))$ is invertible and $A_3 \in \mathcal{B}[N((A^k)^*)]$ is nilpotent. We observe that

$$A^n = \begin{bmatrix} A_1^n & U \\ 0 & A_3^n \end{bmatrix} \qquad \text{and} \qquad I - AA^{\textcircled{\dagger}} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix},$$

where $U = \sum\limits_{j=0}^{n-1} A_1^{n-1-j} A_2 A_3^j$. Since $A^{\textcircled{\dagger}}$ is n-potent, from

$$\begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} = A^{\textcircled{\dagger}} = (A^{\textcircled{\dagger}})^n = \begin{bmatrix} A_1^{-n} & 0 \\ 0 & 0 \end{bmatrix},$$

notice that $A_1^{n-1} = I$. Further, $(I - A^{n-1})A(I - AA^{\textcircled{\dagger}}) = 0$ is equivalent to

$$\begin{bmatrix} 0 & A_2 \\ 0 & A_3 \end{bmatrix} = A(I - AA^{\textcircled{\dagger}}) = A^n(I - AA^{\textcircled{\dagger}}) = \begin{bmatrix} 0 & U \\ 0 & A_3^n \end{bmatrix},$$

which gives $A_2 = U$ and $A_3 = A_3^n$. Thus, $A = A^n$.

(i) $\Leftrightarrow$ (iii): Using the same representations of A and $A^{\textcircled{D}}$ as in (2.1), we have that A is n-potent if and only if $A_1^{n-1} = I$, $A_2 = U$ and $A_3 = A_3^n$. We can see that

$$\begin{bmatrix} A_1^{n-1} & A_1^{-1}U \\ 0 & 0 \end{bmatrix} = A^{\textcircled{D}}A^n = A^{\textcircled{D}}A = \begin{bmatrix} I & A_1^{-1}A_2 \\ 0 & 0 \end{bmatrix}$$

is equivalent to $A_1^{n-1} = I$ and $A_2 = U$. Since

$$A - AA^{\textcircled{D}}A = (I - AA^{\textcircled{D}})A = \begin{bmatrix} 0 & 0 \\ 0 & A_3 \end{bmatrix},$$

then $A - AA^{\textcircled{D}}A$ is n-potent if and only if A_3 is n-potent.

(i) $\Leftrightarrow$ (iv): This equivalence can be proved as (i) $\Leftrightarrow$ (iii) when we note that $(I - AA^{\textcircled{D}})A(I - A^{n-1}) = 0$ is equivalent to

$$\begin{bmatrix} 0 & 0 \\ 0 & A_3 \end{bmatrix} = (I - AA^{\textcircled{D}})A = (I - AA^{\textcircled{D}})A^n = \begin{bmatrix} 0 & 0 \\ 0 & A_3^n \end{bmatrix},$$

that is $A_3 = A_3^n$.

(i) $\Leftrightarrow$ (v): Applying (2.1), $A^2A^{\textcircled{D}} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$ is n-potent if and only if $A_1^n = A_1$, i.e. $A_1^{n-1} = I$. The rest follows as in the part (ii) $\Rightarrow$ (i) of this proof. $\square$

For $n = 2$ in Theorem 2, we get the next result.

Corollary 3. *If $A \in \mathcal{B}(X)^D$, then the following statements are equivalent:*

(i) *A is an idempotent;*
(ii) *$A^{\textcircled{D}}$ is an idempotent and $(I - A)A(I - AA^{\textcircled{D}}) = 0$;*
(iii) *$A^{\textcircled{D}}A^2 = A^{\textcircled{D}}A$ and $A - AA^{\textcircled{D}}A$ is an idempotent;*
(iv) *$A^{\textcircled{D}}A^2 = A^{\textcircled{D}}A$ and $(I - AA^{\textcircled{D}})A(I - A) = 0$;*
(v) *$A^2A^{\textcircled{D}}$ is an idempotent and $(I - A)A(I - AA^{\textcircled{D}}) = 0$.*

We also study equivalent conditions under which $A^{\textcircled{D}} = A^{k+1}$.

Theorem 3. *If $A \in \mathcal{B}(X)^D$ and $k = ind(A)$, then the following statements are equivalent:*

(i) *$A^{\textcircled{D}} = A^{k+1}$;*
(ii) *$A^k(A^k)^\dagger = A^{k+2}$;*
(ii') *$A^k(A^k)^{(1,3)} = A^{k+2}$, for $(A^k)^{(1,3)} \in (A^k)\{1,3\}$;*
(iii) *A^{k+2} is orthogonal projector;*
(iv) *$A^k = A^{2k+2}$ and $(A^{k+2})^* = A^{k+2}$;*
(v) *$A^DA^k = A^{2k+1}$ and $(A^{k+2})^* = A^{k+2}$.*

Proof. (i) $\Rightarrow$ (ii): The equalities $A^{\textcircled{D}} = A^{k+1}$ and $A^{\textcircled{D}} = A^DA^k(A^k)^\dagger$ yield

$$A^{k+2} = AA^{k+1} = AA^{\textcircled{D}} = (AA^DA^k)(A^k)^\dagger = A^k(A^k)^\dagger.$$

(ii) $\Rightarrow$ (iii): It is clear.

(iii) $\Rightarrow$ (iv): Because A^{k+2} is orthogonal projector, $(A^{k+2})^* = A^{k+2}$ and $A^{k+2} = A^{2k+4}$ which gives $A^k = (A^D)^2 A^{k+2} = (A^D)^2 A^{2k+4} = A^{2k+2}$.

(iv) $\Rightarrow$ (v): Multiplying $A^k = A^{2k+2}$ by A^D from the left hand side, we obtain $A^D A^k = A^{2k+1}$.

(v) $\Rightarrow$ (i): The assumptions $A^D A^k = A^{2k+1}$ and $(A^{k+2})^* = A^{k+2}$ imply

$$\begin{aligned} A^{\textcircled{D}} &= (A^D A^k)(A^k)^\dagger = A^{2k+1}(A^k)^\dagger = A^D A^{k+2} A^k (A^k)^\dagger = A^D (A^{k+2})^* A^k (A^k)^\dagger \\ &= A^D (A^{k+2})^* = A^D A^{k+2} = A^{k+1}. \end{aligned}$$

□

The equality $A^{\textcircled{D}} = (A^k)^\dagger$ is studied in the next result.

Theorem 4. *If $A \in \mathcal{B}(X)^D$ and $k = ind(A)$, then the following statements are equivalent:*

(i) $A^{\textcircled{D}} = (A^k)^\dagger$;
(ii) $A^D A^k = (A^k)^\dagger A^k$;
(iii) $A^k = (A^k)^\dagger A^{k+1}$;
(iv) $(A^k)^* = A^*(A^k)^\dagger A^k$;
(v) $A^D A^k (A^k)^* = (A^k)^*$;
(vi) $A^k (A^k A^D)^* = A^k$.

Proof. (i) $\Rightarrow$ (ii): Because $(A^k)^\dagger = A^{\textcircled{D}} = A^D A^k (A^k)^\dagger$, we have

$$(A^k)^\dagger A^k = A^D A^k (A^k)^\dagger A^k = A^D A^k.$$

(ii) $\Rightarrow$ (iii): The hypothesis $A^D A^k = (A^k)^\dagger A^k$ gives $A^k = (A^D A^k)A = (A^k)^\dagger A^{k+1}$.

(iii) $\Rightarrow$ (i): From $A^k = (A^k)^\dagger A^{k+1}$, we get

$$A^{\textcircled{D}} = A^D A^k (A^k)^\dagger = A^k A^D (A^k)^\dagger = (A^k)^\dagger (A^{k+1} A^D)(A^k)^\dagger = (A^k)^\dagger A^k (A^k)^\dagger = (A^k)^\dagger.$$

(iii) $\Leftrightarrow$ (iv) and (v) $\Leftrightarrow$ (vi): These equivalences follow by properties of the adjoint operator.

(ii) $\Rightarrow$ (v): The condition $A^D A^k = (A^k)^\dagger A^k$ yield $A^D A^k (A^k)^* = (A^k)^\dagger A^k (A^k)^* = (A^k)^*$.

(v) $\Rightarrow$ (ii): Multiplying $A^D A^k (A^k)^* = (A^k)^*$ by $(A^\dagger)^*$ from the right hand side, we obtain $A^D A^k = (A^k)^\dagger A^k$. □

We can consider the equality $A^{\textcircled{D}} = (A^k)^*$ too.

Theorem 5. *If $A \in \mathcal{B}(X)^D$ and $k = ind(A)$, then the following statements are equivalent:*

(i) $A^{\textcircled{D}} = (A^k)^*$;
(ii) $A^D A^k = (A^k)^* A^k$;
(iii) $A^D[(A^k)^\dagger]^* = (A^k)^\dagger A^k$.

Proof. (i) $\Rightarrow$ (ii)–(iii): Since $(A^k)^* = A^{\textcircled{D}} = A^D A^k (A^k)^\dagger$, we deduce that $(A^k)^* A^k = A^D A^k$ and

$$(A^k)^\dagger A^k = (A^k)^* [(A^k)^\dagger]^* = A^D A^k (A^k)^\dagger [(A^k)^\dagger]^* = A^D [(A^k)^\dagger]^*.$$

(ii) $\Rightarrow$ (i): Multiplying $A^D A^k = (A^k)^* A^k$ by $(A^k)^\dagger$ from the right hand side, we get $A^{\textcircled{D}} = (A^k)^*$.

(iii) $\Rightarrow$ (i): The assumption $A^D[(A^k)^\dagger]^* = (A^k)^\dagger A^k$ implies

$$A^{\textcircled{D}} = A^D A^k (A^k)^\dagger = A^D [(A^k)^\dagger]^* (A^k)^* = (A^k)^\dagger A^k (A^k)^* = (A^k)^*.$$

□

3. Application to the core inverse

Applying the results of Section 2, we obtain many characterizations for n-potency of the core inverse.

When $ind(A) = 1$ in Lemma 1, we have the next representations for the power of the core inverse.

Lemma 3. *If $A \in \mathcal{B}(X)^\#$, then*

$$(A^{\textcircled{\#}})^n = (A^\#)^n A A^\dagger = (A^\#)^n A A^{(1,3)},$$

for any $n \geq 1$ and $A^{(1,3)} \in A\{1,3\}$.

Taking $ind(A) = 1$ in Theorem 1 and Corollary 1, we characterize n-potency of the core inverse. Note that $A^{\textcircled{\#}}$ is n-potent if and only if A is n-potent.

Corollary 4. *If $A \in \mathcal{B}(X)^\#$ and $n \geq 2$, then the following statements are equivalent:*

(i) *$A^{\textcircled{\#}}$ is n-potent;*
(ii) *$(A^{\textcircled{\#}})^{n-1} A = A$;*
(iii) *A is n-potent;*
(iv) *$A^\# A = A^{n-1}$;*
(v) *$AA^{(1,3)} = (A^{\textcircled{\#}})^{n-1}$, for $A^{(1,3)} \in A\{1,3\}$;*
(v') *$AA^\dagger = (A^{\textcircled{\#}})^{n-1}$;*
(vi) *$A^\# (A^{\textcircled{\#}})^{n-1} = A^{\textcircled{\#}}$;*
(vii) *$AA^{\textcircled{\#}} = (A^{\textcircled{\#}})^{n-1}$;*
(viii) *$(A^{\textcircled{\#}})^{n-1}$ is orthogonal projector;*
(ix) *$(A^{\textcircled{\#}})^{n-1}$ is an idempotent;*
(x) *$A^\# A = (A^\#)^{n-1}$;*
(xi) *$A^\#$ is n-potent;*
(xii) *$A^* = A^* (A^{\textcircled{\#}})^{n-1}$;*
(xiii) *$(A^{\textcircled{\#}})^m = (A^{\textcircled{\#}})^{m+n-1}$, for some/any $m \geq 1$;*
(xiv) *$(A^\#)^m = (A^\#)^{m+n-1}$, for some/any $m \geq 1$;*
(xv) *$(A^{\textcircled{\#}})^n$ is an outer inverse of A;*
(xvi) *$A^\# (A^{\textcircled{\#}})^{n-1}$ is an outer inverse of A;*

(xvi) $(A^{\#})^n$ *is an outer inverse of* A*;*
(xvii) $A^{\#}A$ *is an outer inverse of* A^{n-1}*;*
(xviii) $A^{n-1} = A^{\textcircled{\#}}A$*;*
(xix) $A^2A^{\textcircled{\#}}$ *is n-potent and* $(I - A^{n-1})A(I - AA^{\textcircled{\#}}) = 0$*.*

Choosing $n = 2$ in Corollary 4, we get equivalent conditions for $A^{\textcircled{\#}}$ to be an idempotent. We can observe that $A^{\textcircled{\#}}$ is an idempotent if and only if A is an idempotent.

Corollary 5. *If* $A \in \mathcal{B}(X)^{\#}$*, then the following statements are equivalent:*

(i) $A^{\textcircled{\#}}$ *is an idempotent;*
(ii) $A^{\textcircled{\#}}A = A$*;*
(iii) A *is an idempotent;*
(iv) $A^{\#}A = A$*;*
(v) $AA^{(1,3)} = A^{\textcircled{\#}}$*, for* $A^{(1,3)} \in A\{1,3\}$*;*
(v') $AA^{\dagger} = A^{\textcircled{\#}}$*;*
(vi) $A^{\#}A^{\textcircled{\#}} = A^{\textcircled{\#}}$*;*
(vii) $AA^{\textcircled{\#}} = A^{\textcircled{\#}}$*;*
(viii) $A^{\textcircled{\#}}$ *is orthogonal projector;*
(ix) $A^{\#}A = A^{\#}$*;*
(x) $A^{\#}$ *is an idempotent;*
(xi) $A^* = A^*A^{\textcircled{\#}}$*;*
(xii) $(A^{\textcircled{\#}})^m = (A^{\textcircled{\#}})^{m+1}$*, for some/any* $m \geq 1$*;*
(xiii) $(A^{\#})^m = (A^{\#})^{m+1}$*, for some/any* $m \geq 1$*;*
(xiv) $(A^{\textcircled{\#}})^2$ *is an outer inverse of* A*;*
(xv) $A^{\#}A^{\textcircled{\#}}$ *is an outer inverse of* A*;*
(xvi) $(A^{\#})^2$ *is an outer inverse of* A*;*
(xvii) $A^{\#}A$ *is an outer inverse of* A*;*
(xviii) $A^2A^{\textcircled{\#}}$ *is an idempotent and* $(I - A)A(I - AA^{\textcircled{\#}}) = 0$*;*
(xix) $A^{\textcircled{\#}}A^2 = A^{\textcircled{\#}}A$*.*

Theorem 3 implies several characterizations for the equality $A^{\textcircled{\#}} = A^2$ to be satisfied.

Corollary 6. *If* $A \in \mathcal{B}(X)^{\#}$*, then the following statements are equivalent:*

(i) $A^{\textcircled{\#}} = A^2$*;*
(ii) $AA^{\dagger} = A^3$*;*
(ii') $AA^{(1,3)} = A^3$*, for* $A^{(1,3)} \in A\{1,3\}$*;*
(iii) A^3 *is orthogonal projector;*
(iv) $A = A^4$ *and* $(A^3)^* = A^3$*;*
(v) $A^{\#}A = A^3$ *and* $(A^3)^* = A^3$*.*

Recall that $A \in \mathcal{B}(X)^{\#}$ is an EP operator if $A^{\#} = A^{\dagger}$ [9, 14]. By Theorem 7, we obtain some well-known characterizations of an EP operator.

Corollary 7. *If* $A \in \mathcal{B}(X)^{\#}$*, then the following statements are equivalent:*

(i) $A^{\text{\textcircled{\#}}} = A^{\dagger}$;
(ii) $A^{\#}A = A^{\dagger}A$;
(iii) $A = A^{\dagger}A^2$;
(iv) $A^* = A^*A^{\dagger}A$;
(v) $A^{\#}AA^* = A^*$;
(vi) $A(AA^{\#})^* = A$;
(vii) *A is EP.*

Theorem 5 gives necessary and equivalent conditions for $A^{\text{\textcircled{\#}}} = A^*$ to hold.

Corollary 8. *If $A \in \mathcal{B}(X)^{\#}$, then the following statements are equivalent:*
(i) $A^{\text{\textcircled{\#}}} = A^*$;
(ii) $A^{\#}A = A^*A$;
(iii) $A^{\#}(A^{\dagger})^* = A^{\dagger}A$.

In addition, if any of statements (i)–(iii) *holds, then A is EP.*

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