Miskolc Mathematical Notes
Vol. 24 (2023), No. 3, pp. 1615–1635
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.4225

CURVATURE PROPERTIES OF PROJECTIVE SEMI-SYMMETRIC LINEAR CONNECTIONS

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Received 03 May, 2022

Abstract. We study a projective semi-symmetric linear connection on a differentiable manifold $\mathcal{M}$ endowed with a Riemannian metric g. We start with linearly independent curvature tensors $\underset{\theta}{R}$, $\theta = 0, 1, \ldots, 5$ and derive the tensors $\underset{\theta}{W}$ for $\theta = 0, 1, \ldots, 5$ that, as we show, coincide with the Weyl tensor of projective curvature W^g. This confirms the well-known fact that there does not exist a generalization of the Weyl projective curvature tensor W^g.

2010 *Mathematics Subject Classification:* 53A45; 53B05; 53C05

Keywords: semi-symmetric linear connection, projective connection, Riemannian metric, curvature tensor, Weyl tensor of projective curvature

1. Introduction

Let $\mathcal{M}$ be a differentiable manifold of dimension $n > 2$ endowed with a Riemannian metric g. Let us consider a linear connection $\overset{1}{\nabla}$ whose torsion tensor is given by

$$\overset{1}{T}(X,Y) = \overset{1}{\nabla}_X Y - \overset{1}{\nabla}_Y X - [X,Y].$$

For a linear connection $\overset{1}{\nabla}$ on a differentiable manifold $\mathcal{M}$ there exists a unique *dual (conjugate) linear connection* $\overset{2}{\nabla}$ given by

$$\overset{2}{\nabla}_X Y = \overset{1}{\nabla}_Y X + [X,Y], \tag{1.1}$$

where X and Y are smooth vector fields on $\mathcal{M}$. The last equation is equivalent with the equation

$$\overset{2}{\nabla}_X Y = \overset{1}{\nabla}_X Y - \overset{1}{T}(X,Y).$$

The financial support of this work by the projects of the Ministry of Science, Technological Development and Innovation of the Republic of Serbia (contract reg. no. 451-03-47/2023-01/200124 for Milan Lj. Zlatanović, no. 451-03-47/2023-01/200383 for Miloš Z. Petrović and no. 451-03-47/2023-01/200123 for Miroslav D. Maksimović) and by project of Faculty of Sciences and Mathematics, University of Priština in Kosovska Mitrovica (internal-junior project IJ-2303).

By virtue of the connection $\overset{1}{\nabla}$ and its dual $\overset{2}{\nabla}$, one can define a torsion-free (symmetric) linear connection (see for instance [16])

$$\overset{0}{\nabla}_X Y = \frac{1}{2}(\overset{1}{\nabla}_X Y + \overset{2}{\nabla}_X Y).$$

Consequently, the torsion-free linear connection $\overset{0}{\nabla}$ can be expressed via linear connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$, respectively, and the torsion tensor $\overset{1}{T}$ as follows:

$$\overset{0}{\nabla}_X Y = \overset{1}{\nabla}_X Y - \frac{1}{2}\overset{1}{T}(X,Y), \qquad \overset{0}{\nabla}_X Y = \overset{2}{\nabla}_X Y + \frac{1}{2}\overset{1}{T}(X,Y).$$

Let ∇^g stands for the Levi-Civita connection of metric g and R^g denotes corresponding Riemannian curvature tensor given by

$$R^g(X,Y)Z = \nabla^g_X \nabla^g_Y Z - \nabla^g_Y \nabla^g_X Z - \nabla^g_{[X,Y]} Z.$$

Let us note that in the available literature the linear connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ are sometimes denoted by $\nabla^{\pm}$. Since these linear connections have non-vanishing torsion tensors the following two curvature tensors can defined

$$\underset{1}{R}(X,Y)Z = \overset{1}{\nabla}_X \overset{1}{\nabla}_Y Z - \overset{1}{\nabla}_Y \overset{1}{\nabla}_X Z - \overset{1}{\nabla}_{[X,Y]} Z, \tag{1.2}$$

$$\underset{2}{R}(X,Y)Z = \overset{2}{\nabla}_X \overset{2}{\nabla}_Y Z - \overset{2}{\nabla}_Y \overset{2}{\nabla}_X Z - \overset{2}{\nabla}_{[X,Y]} Z. \tag{1.3}$$

U. P. Singh in [26] introduced the third curvature tensor given by

$$\underset{3}{R}(X,Y)Z = \overset{2}{\nabla}_X \overset{1}{\nabla}_Y Z - \overset{1}{\nabla}_Y \overset{2}{\nabla}_X Z + \overset{2}{\nabla}_{\overset{1}{\nabla}_Y X} Z - \overset{1}{\nabla}_{\overset{2}{\nabla}_X Y} Z. \tag{1.4}$$

M. Prvanović in [24] proved the existence of the fourth curvature tensor

$$\underset{4}{R}(X,Y)Z = \overset{2}{\nabla}_X \overset{1}{\nabla}_Y Z - \overset{1}{\nabla}_Y \overset{2}{\nabla}_X Z + \overset{2}{\nabla}_{\overset{2}{\nabla}_Y X} Z - \overset{1}{\nabla}_{\overset{1}{\nabla}_X Y} Z. \tag{1.5}$$

As shown in [15] there exist six linearly independent curvature tensors, while the rest can be expressed as linear combinations of these six curvature tensors. We will consider the following six linearly independent curvature tensors $\underset{1}{R}, \underset{2}{R}, \underset{3}{R}, \underset{4}{R}$ given by (1.2 - 1.5) and $\underset{0}{R}$ and $\underset{5}{R}$ that are respectively given by

$$\underset{0}{R}(X,Y)Z = \overset{0}{\nabla}_X \overset{0}{\nabla}_Y Z - \overset{0}{\nabla}_Y \overset{0}{\nabla}_X Z - \overset{0}{\nabla}_{[X,Y]} Z \tag{1.6}$$

and

$$\underset{5}{R}(X,Y)Z = \frac{1}{2}\big(\overset{1}{\nabla}_X \overset{1}{\nabla}_Y Z - \overset{2}{\nabla}_Y \overset{1}{\nabla}_X Z + \overset{2}{\nabla}_X \overset{2}{\nabla}_Y Z - \overset{1}{\nabla}_Y \overset{2}{\nabla}_X Z - \overset{1}{\nabla}_{[X,Y]} Z - \overset{2}{\nabla}_{[X,Y]} Z\big). \tag{1.7}$$

The curvature tensor $\underset{1}{R}$ determined only by the linear connection $\overset{1}{\nabla}$ with torsion $\overset{1}{T}$ has been widely studied in many papers devoted to investigation of linear connections with torsion. Linear connections with torsion are particularly important in study of Kähler, para-Kähler, Hermitian and para-Hermitian geometries and because of their applications in string theory (see for instance S. Ivanov [9–11]).

Let us consider a local coordinate frame $\{\frac{\partial}{\partial x_s}, s = 1,\dots,dim(\mathcal{M})\}$ and set

$$X = \frac{\partial}{\partial x_i}, \quad Y = \frac{\partial}{\partial x_j}, \quad Z = \frac{\partial}{\partial x_k}.$$

In the following sections, we will express the curvature tensors $\underset{\theta}{R}$, $\theta = 0,1,\dots,5$, in terms of the Riemannian curvature tensor R^g and the projective semi-symmetric connection.

As is well-known the curvature tensor $\underset{0}{R}$ is skew-symmetric with respect to the first two arguments, and satisfies the first Bianchi identity:

$$\underset{0}{R}(X,Y)Z = -\underset{0}{R}(Y,X)Z, \qquad \sigma_{XYZ}\underset{0}{R}(X,Y)Z = 0.$$

From the definitions of curvature tensors (1.2 - 1.7) one can conclude that [16]

$$\underset{\theta}{R}(X,Y)Z = -\underset{\theta}{R}(Y,X)Z, \quad \theta = 1,2, \quad \text{and} \quad \sigma_{XYZ}\underset{\theta}{R}(X,Y)Z = 0, \quad \theta = 4,5.$$

The (0,4) curvature tensors corresponding to $\underset{\theta}{R}(X,Y)Z$, $\theta = 0,1,\dots,5$, and $R^g(X,Y)Z$ are respectively given by

$$\underset{\theta}{R}(X,Y,Z,W) := g(\underset{\theta}{R}(X,Y)Z,W), \qquad \theta = 0,1,\dots,5$$

and

$$R^g(X,Y,Z,W) := g(R^g(X,Y)Z,W).$$

The corresponding Ricci tensors are given by

$$\underset{\theta}{Ric}(Y,Z) := \mathrm{trace}[X \to \underset{\theta}{R}(X,Y)Z] \qquad \theta = 0,1,\dots,5$$

and

$$Ric^g(Y,Z) := \mathrm{trace}[X \to R^g(X,Y)Z].$$

The scalar curvatures are defined as the trace of $\underset{\theta}{Ric}(Y,Z)$, $\theta = 0,1,\dots,5$, and $Ric^g(Y,Z)$, and will be denoted by $\underset{\theta}{s}$, $\theta = 0,1,\dots,5$, and s^g, respectively.

The Weyl projective curvature tensor W^g of a Riemannian space $(\mathcal{M},g)$ given by

$$W^g(X,Y)Z = R^g(X,Y)Z + \frac{1}{n-1}\big(Ric^g(X,Z)Y - Ric^g(Y,Z)X\big), \tag{1.8}$$

is invariant with respect to a geodesic mapping between Riemannian spaces (see for instance [13, 14]). If we consider a geodesic mapping between some special manifolds with non-symmetric linear connection one can prove that there exist analogous expressions of the Weyl projective curvature tensor with respect to the curvature tensors $\underset{\theta}{R}(X,Y)Z$, $\theta = 0,1,\dots,5$, but some assumptions are necessary [21]. Therefore, there does not exist any generalization of the Weyl projective curvature tensor, but the tensors that have a similar role might be useful as replacements in some applications. The aim of the present paper is to derive such tensors with respect to the metric semi-symmetric linear connection. Some authors thoroughly studied semi-symmetric linear connections, see for instance [3–5, 8, 19, 20, 23, 25, 32, 33, 35]. Recently, some applications of semi-symmetric metric connection in the theories of gravity and electromagnetism has been shown in [6]. As mentioned in [7], Y. X. Liang [12] proved that the projective curvature tensor of semi-symmetric metric connection coincides with the Weyl projective curvature tensor of Levi-Civita connection if and only if the characteristic vector is proportional to the Riemannian metric g. P. Zhao and H. Song in [36] introduced the projective semi-symmetric connection and obtained the invariant under transformation of the projective semi-symmetric connection. In [1] S. K. Chaubey et. al. studied projective semi-symmetric connection with a parallel unit vector field and proved that the projective curvature tensor of this connection coincides with the Weyl projective curvature tensor of Levi-Civita connection. In [7] Y. Han et. al. considered the projective transformation of the quarter-symmetric metric connection and found some invariants under that transformation.

2. Projective Semi-Symmetric Linear Connections

Let $(\mathcal{M},g)$ be a Riemannian space of dimension $n>2$ and $\overset{1}{\nabla}$ be a *projective semi-symmetric linear connection* on $\mathcal{M}$ given by [34, 36]

$$\overset{1}{\nabla}_X Y = \nabla^g_X Y + \psi(Y)X + \psi(X)Y + \sigma(Y)X - \sigma(X)Y, \tag{2.1}$$

where ∇^g is the Levi-Civita connection of metric g, whereas ψ and σ are 1-forms given by

$$\psi(X) = \frac{n-1}{2(n+1)}\pi(X), \qquad \sigma(X) = \frac{1}{2}\pi(X).$$

From (2.1) we have that the (symmetric) linear connection $\overset{0}{\nabla}$ is related with the Levi-Civita connection ∇^g by [14]

$$\overset{0}{\nabla}_X Y = \nabla^g_X Y + \psi(Y)X + \psi(X)Y.$$

As is well-known a linear connection $\overset{2}{\nabla}$ which is dual to $\overset{1}{\nabla}$ is given by (1.1) and via linear connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ one can determine the curvature tensors $\underset{\theta}{R}(X,Y)Z$,

$\theta = 0, 1, \ldots, 5$ that are related with the Riemannian curvature tensor $R^g(X,Y)Z$ as in Proposition 1.

Proposition 1. *Let $(\mathcal{M}, g)$ be a Riemannian space of dimension $n > 2$ and $\overset{1}{\nabla}$ be a projective semi-symmetric linear connection given by (2.1). The curvature tensors $\underset{\theta}{R}(X,Y)Z$, $\theta = 0, 1, 2, \ldots, 5$ are related with the curvature tensor $R^g(X,Y)Z$ by:*

$$\begin{aligned}\underset{0}{R}(X,Y)Z = R^g(X,Y)Z &+ \frac{n-1}{2(n+1)}\big((\nabla^g_X\pi)(Y) - (\nabla^g_Y\pi)(X)\big)Z \\ &+ \frac{n-1}{2(n+1)}\big((\nabla^g_X\pi)(Z)Y - (\nabla^g_Y\pi)(Z)X\big) \\ &+ \frac{(n-1)^2}{4(n+1)^2}\pi(Z)\big(\pi(Y)X - \pi(X)Y\big),\end{aligned} \tag{2.2}$$

$$\begin{aligned}\underset{1}{R}(X,Y)Z = R^g(X,Y)Z &- \frac{1}{n+1}\big((\nabla^g_X\pi)(Y) - (\nabla^g_Y\pi)(X)\big)Z \\ &+ \frac{n}{n+1}\big((\nabla^g_X\pi)(Z)Y - (\nabla^g_Y\pi)(Z)X\big) \\ &+ \frac{n^2}{(n+1)^2}\pi(Z)\big(\pi(Y)X - \pi(X)Y\big),\end{aligned} \tag{2.3}$$

$$\begin{aligned}\underset{2}{R}(X,Y)Z = R^g(X,Y)Z &+ \frac{n}{n+1}\big((\nabla^g_X\pi)(Y) - (\nabla^g_Y\pi)(X)\big)Z \\ &- \frac{1}{n+1}\big((\nabla^g_X\pi)(Z)Y - (\nabla^g_Y\pi)(Z)X\big) \\ &+ \frac{1}{(n+1)^2}\pi(Z)\big(\pi(Y)X - \pi(X)Y\big),\end{aligned} \tag{2.4}$$

$$\begin{aligned}\underset{3}{R}(X,Y)Z = R^g(X,Y)Z &+ \frac{n}{n+1}\big((\nabla^g_X\pi)(Z)Y - (\nabla^g_Y\pi)(X)Z\big) \\ &+ \frac{1}{n+1}\big((\nabla^g_Y\pi)(Z)X - (\nabla^g_X\pi)(Y)Z\big) - \frac{n^2}{(n+1)^2}\pi(X)\pi(Z)Y \\ &+ \frac{1}{(n+1)^2}\pi(Y)\pi(Z)X + \frac{n-1}{n+1}\pi(Y)\pi(X)Z,\end{aligned} \tag{2.5}$$

$$\begin{aligned}\underset{4}{R}(X,Y)Z = R^g(X,Y)Z &+ \frac{n}{n+1}\big((\nabla^g_X\pi)(Z)Y - (\nabla^g_Y\pi)(X)Z\big) \\ &+ \frac{1}{n+1}\big((\nabla^g_Y\pi)(Z)X - (\nabla^g_X\pi)(Y)Z\big) + \frac{2n+1}{(n+1)^2}\pi(X)\pi(Z)Y \\ &- \frac{n^2+2n}{(n+1)^2}\pi(Y)\pi(Z)X + \frac{n-1}{n+1}\pi(Y)\pi(X)Z,\end{aligned} \tag{2.6}$$

$$\underset{5}{R}(X,Y)Z = R^g(X,Y)Z + \frac{n-1}{2(n+1)}\big((\nabla^g_X\pi)(Y) - (\nabla^g_Y\pi)(X)\big)Z$$

$$
\begin{aligned}
&+\frac{n-1}{2(n+1)}\big((\nabla^g_X\pi)(Z)Y-(\nabla^g_Y\pi)(Z)X\big) \\
&+\frac{(n-1)^2}{4(n+1)^2}\pi(Z)\big(\pi(Y)X-\pi(X)Y\big) \\
&-\frac{1}{4}\pi(Z)\big(\pi(Y)X+\pi(X)Y\big)+\frac{1}{2}\pi(X)\pi(Y)Z.
\end{aligned} \tag{2.7}
$$

By taking the trace with respect to the vector field X in the equations (2.2 - 2.7) from Proposition 1. we obtain the corresponding relations between the Ricci tensors $\underset{\theta}{Ric}$, $\theta=0,1,\dots,5$ and Ric^g.

Proposition 2. *Let $(\mathcal{M},g)$ be a Riemannian space of dimension $n>2$ and $\overset{1}{\nabla}$ be a projective semi-symmetric linear connection given by (2.1). The Ricci tensors $\underset{\theta}{Ric}(Y,Z)$, $\theta=0,1,2,\dots,5$ are related with the Ricci tensor $Ric^g(Y,Z)$ by:*

$$
\begin{aligned}
\underset{0}{Ric}(Y,Z)&=Ric^g(Y,Z)-\frac{n(n-1)}{2(n+1)}(\nabla^g_Y\pi)(Z)+\frac{n-1}{2(n+1)}(\nabla^g_Z\pi)(Y) \\
&\quad+\frac{(n-1)^3}{4(n+1)^2}\pi(Y)\pi(Z), \\
\underset{1}{Ric}(Y,Z)&=Ric^g(Y,Z)-\frac{n^2-n-1}{n+1}(\nabla^g_Y\pi)(Z)-\frac{1}{n+1}(\nabla^g_Z\pi)(Y) \\
&\quad+\frac{n^2(n-1)}{(n+1)^2}\pi(Y)\pi(Z), \\
\underset{2}{Ric}(Y,Z)&=Ric^g(Y,Z)-\frac{1}{n+1}(\nabla^g_Y\pi)(Z)+\frac{n}{n+1}(\nabla^g_Z\pi)(Y) \\
&\quad+\frac{n-1}{(n+1)^2}\pi(Y)\pi(Z), \\
\underset{3}{Ric}(Y,Z)&=Ric^g(Y,Z)+\frac{n}{n+1}(\nabla^g_Y\pi)(Z)-\frac{1}{n+1}(\nabla^g_Z\pi)(Y) \\
&\quad+\frac{n-1}{(n+1)^2}\pi(Y)\pi(Z), \\
\underset{4}{Ric}(Y,Z)&=Ric^g(Y,Z)+\frac{n}{n+1}(\nabla^g_Y\pi)(Z)-\frac{1}{n+1}(\nabla^g_Z\pi)(Y) \\
&\quad-\frac{n(n^2+n-2)}{(n+1)^2}\pi(Y)\pi(Z), \\
\underset{5}{Ric}(Y,Z)&=Ric^g(Y,Z)-\frac{n(n-1)}{2(n+1)}(\nabla^g_Y\pi)(Z)+\frac{n-1}{2(n+1)}(\nabla^g_Z\pi)(Y) \\
&\quad+\frac{n(n-1)}{(n+1)^2}\pi(Y)\pi(Z).
\end{aligned}
$$

Symmetry properties of Ricci tensors $\underset{\theta}{Ric}$, $\theta = 0,1,2,\ldots,5$ are related with closedness of the 1-form π as given in Proposition 3.

Proposition 3 (see [22, Theorem 3.6]). *Let $(\mathcal{M},g)$ be a Riemannian space of dimension $n > 2$ and $\overset{1}{\nabla}$ be a projective semi-symmetric linear connection given by (2.1) then for arbitrary $\theta \in \{0,1,2,\ldots,5\}$ the Ricci tensor $\underset{\theta}{Ric}$ is symmetric if and only if the 1-form π is closed.*

Starting with a projective semi-symmetric linear connection $\overset{1}{\nabla}$ given by (2.1) we will apply the method for deriving the Weyl projective curvature tensors on the linearly independent curvature tensor $\underset{\theta}{R}(X,Y)Z$ for $\theta = 0,1,\ldots,5$ to derive the tensors $\underset{\theta}{W}(X,Y)Z$, $\theta = 0,1,\ldots,5$ corresponding to the projective semi-symmetric linear connection $\overset{1}{\nabla}$ and its dual connection $\overset{2}{\nabla}$.

2.1. *Curvature tensor of the zero kind*

The relation (2.2) of Proposition 1 can be written as

$$\underset{0}{R}(X,Y)Z = R^g(X,Y)Z + \underset{0}{\alpha}(X,Z)Y - \underset{0}{\alpha}(Y,Z)X + \underset{0}{\beta}(X,Y)Z, \tag{2.8}$$

where

$$\underset{0}{\alpha}(X,Y) = \frac{n-1}{2(n+1)}(\nabla^g_X \pi)(Y) - \frac{(n-1)^2}{4(n+1)^2}\pi(X)\pi(Y)$$

and

$$\underset{0}{\beta}(X,Y) = \underset{0}{\alpha}(X,Y) - \underset{0}{\alpha}(Y,X).$$

By taking the trace with respect to the vector field X in (2.8) we obtain

$$\underset{0}{Ric}(Y,Z) = Ric^g(Y,Z) - (n-1)\underset{0}{\alpha}(Y,Z) + \underset{0}{\beta}(Z,Y). \tag{2.9}$$

After skew-symmetrization in the last relation with respect to the vector fields Y and Z we obtain

$$\underset{0}{Ric}(Y,Z) - \underset{0}{Ric}(Z,Y) = -(n-1)\underset{0}{\beta}(Y,Z) + 2\underset{0}{\beta}(Z,Y),$$

i.e.

$$\underset{0}{\beta}(Z,Y) = \frac{1}{n+1}(\underset{0}{Ric}(Y,Z) - \underset{0}{Ric}(Z,Y)), \tag{2.10}$$

which after plugging into (2.9) yields

$$\underset{0}{\alpha}(Y,Z) = -\frac{n}{n^2-1}\underset{0}{Ric}(Y,Z) - \frac{1}{n^2-1}\underset{0}{Ric}(Z,Y) + \frac{1}{n-1}Ric^g(Y,Z). \tag{2.11}$$

Now, by using (2.10) and (2.11) into (2.8) we obtain

$$\underset{0}{W}(X,Y)Z = W^g(X,Y)Z,$$

where

$$\begin{aligned}\underset{0}{W}(X,Y)Z =& \underset{0}{R}(X,Y)Z + \frac{n}{n^2-1}\big(\underset{0}{Ric}(X,Z)Y - \underset{0}{Ric}(Y,Z)X\big)\\ &+ \frac{1}{n^2-1}\big(\underset{0}{Ric}(Z,X)Y - \underset{0}{Ric}(Z,Y)X\big)\\ &+ \frac{1}{n+1}\big(\underset{0}{Ric}(X,Y) - \underset{0}{Ric}(Y,X)\big)Z\end{aligned} \tag{2.12}$$

and $W^g(X,Y)Z$ is the Weyl projective curvature tensor given by (1.8). The tensor $\underset{0}{W}(X,Y)Z$ given by (2.12) is independent of a 1-form π and it is known as the Weyl projective curvature tensor of symmetric linear connection (for example see [14]).

Proposition 4. *Let $(\mathcal{M},g)$ be a Riemannian space of dimension $n>2$, ∇^g be the Levi-Civita connection of g and $\overset{1}{\nabla}$ be a projective semi-symmetric linear connection given by (2.1), $\underset{0}{W}(X,Y)Z$ be the Weyl projective curvature tensor given by (2.12) and $W^g(X,Y)Z$ be the Weyl projective curvature tensor given by (1.8) then:*

(i) $\underset{0}{W}(X,Y)Z = W^g(X,Y)Z$,

(ii) If the Ricci tensor $\underset{0}{Ric}(X,Y)$ vanishes then $\underset{0}{W}(X,Y)Z = \underset{0}{R}(X,Y)Z$,

(iii) If the Ricci tensor $\underset{0}{Ric}(X,Y)$ is symmetric then the tensor $\underset{0}{W}(X,Y)Z$ given by (2.12) takes form

$$\underset{0}{W}(X,Y)Z = \underset{0}{R}(X,Y)Z + \frac{1}{n-1}\big(\underset{0}{Ric}(X,Z)Y - \underset{0}{Ric}(Y,Z)X\big).$$

2.2. *Curvature tensor of the first kind*

The relation (2.3) of Proposition 1 can be written as

$$\underset{1}{R}(X,Y)Z = R^g(X,Y)Z + \underset{1}{\alpha}(X,Z)Y - \underset{1}{\alpha}(Y,Z)X + \underset{1}{\beta}(X,Y)Z \tag{2.13}$$

where $\underset{1}{\alpha}$ and $\underset{1}{\beta}$ are bilinear forms given by

$$\underset{1}{\alpha}(X,Y) = \frac{n}{n+1}(\nabla^g_X\pi)(Y) - \frac{n^2}{(n+1)^2}\pi(X)\pi(Y)$$

and

$$\underset{1}{\beta}(X,Y) = \frac{1}{n+1}\big((\nabla^g_Y\pi)(X) - (\nabla^g_X\pi)(Y)\big) = \frac{1}{n}\big(\underset{1}{\alpha}(Y,X) - \underset{1}{\alpha}(X,Y)\big). \tag{2.14}$$

Obviously, $\underset{1}{\beta}(X,Y)$ is a skew-symmetric bilinear form, i.e. $\underset{1}{\beta}(X,Y) = -\underset{1}{\beta}(Y,X)$ which will be used in what follows.

By taking the trace with respect to the vector field X in the equation (2.13) we get

$$\underset{1}{Ric}(Y,Z) = Ric^g(Y,Z) - (n-1)\underset{1}{\alpha}(Y,Z) + \underset{1}{\beta}(Z,Y), \tag{2.15}$$

which after skew-symmetrization with respect to the vector fields Y and Z reads

$$\underset{1}{Ric}(Y,Z) - \underset{1}{Ric}(Z,Y) = n\underset{1}{\beta}(Z,Y) - n^2\underset{1}{\beta}(Z,Y) + 2\underset{1}{\beta}(Z,Y), \tag{2.16}$$

where we used that the Ricci tensor $Ric^g(Y,Z)$ is symmetric and $\underset{1}{\beta}$ is a skew-symmetric bilinear form given by (2.14).

Further, the equation (2.16) can be rewritten as

$$\underset{1}{Ric}(Y,Z) - \underset{1}{Ric}(Z,Y) = (n^2-n-2)\underset{1}{\beta}(Y,Z), \tag{2.17}$$

which after plugging into (2.15) yields

$$\begin{aligned}(n-1)\underset{1}{\alpha}(Y,Z) =& Ric^g(Y,Z) - \underset{1}{Ric}(Y,Z)\\ &- \frac{1}{(n^2-n-2)}\big(\underset{1}{Ric}(Y,Z) - \underset{1}{Ric}(Z,Y)\big).\end{aligned} \tag{2.18}$$

By replacing (2.17) and (2.18) into (2.13) we get

$$\begin{aligned}\underset{1}{R}(X,Y)Z =& R^g(X,Y)Z + \frac{1}{n-1}\big(Ric^g(X,Z) - \underset{1}{Ric}(X,Z)\big)Y\\ &- \frac{1}{n-1}\big(Ric^g(Y,Z) - \underset{1}{Ric}(Y,Z)\big)X\\ &- \frac{1}{(n-1)(n^2-n-2)}\big(\underset{1}{Ric}(X,Z) - \underset{1}{Ric}(Z,X)\big)Y\\ &+ \frac{1}{(n-1)(n^2-n-2)}\big(\underset{1}{Ric}(Y,Z) - \underset{1}{Ric}(Z,Y)\big)X\\ &+ \frac{1}{n^2-n-2}\big(\underset{1}{Ric}(X,Y) - \underset{1}{Ric}(Y,X)\big)Z\end{aligned}$$

or in a simplified way

$$\underset{1}{W}(X,Y)Z = W^g(X,Y)Z,$$

where

$$\begin{aligned}\underset{1}{W}(X,Y)Z =& \underset{1}{R}(X,Y)Z + \frac{n^2-n-1}{(n-1)(n^2-n-2)}\big(\underset{1}{Ric}(X,Z)Y - \underset{1}{Ric}(Y,Z)X\big)\\ &- \frac{1}{(n-1)(n^2-n-2)}\big(\underset{1}{Ric}(Z,X)Y - \underset{1}{Ric}(Z,Y)X\big)\\ &- \frac{1}{n^2-n-2}\big(\underset{1}{Ric}(X,Y) - \underset{1}{Ric}(Y,X)\big)Z\end{aligned} \tag{2.19}$$

and $W^g(X,Y)Z$ is given by (1.8). We note that the tensor $\underset{1}{W}(X,Y)Z$ given by (2.19) is independent of a 1-form π and it is obtained by P. Zhao and H. Song in [36].

Proposition 5. *Let $(\mathcal{M},g)$ be a Riemannian space of dimension $n>2$, ∇^g be the Levi-Civita connection of g and $\overset{1}{\nabla}$ be a projective semi-symmetric linear connection given by (2.1), $\underset{1}{W}(X,Y)Z$ be the tensor given by (2.19) and $W^g(X,Y)Z$ be the Weyl projective curvature tensor given by (1.8) then:*

(i) $\underset{1}{W}(X,Y)Z = W^g(X,Y)Z$,

(ii) *If the Ricci tensor $\underset{1}{Ric}(X,Y)$ vanishes then $\underset{1}{W}(X,Y)Z = \underset{1}{R}(X,Y)Z$,*

(iii) *If the Ricci tensor $\underset{1}{Ric}(X,Y)$ is symmetric, then the tensor $\underset{1}{W}(X,Y)Z$ given by (2.19) takes form*

$$\underset{1}{W}(X,Y)Z = \underset{1}{R}(X,Y)Z + \frac{1}{n-1}\big(\underset{1}{Ric}(X,Z)Y - \underset{1}{Ric}(Y,Z)X\big).$$

2.3. *Curvature tensor of the second kind*

The relation (2.4) of Proposition 1 can be written as

$$\underset{2}{R}(X,Y)Z = R^g(X,Y)Z - \gamma(X,Z)Y + \gamma(Y,Z)X + n(\gamma(X,Y)-\gamma(Y,X))Z, \tag{2.20}$$

where γ is a bilinear form given by

$$\gamma(X,Y) = \frac{1}{n+1}(\nabla^g_X\pi)(Y) + \frac{1}{(n+1)^2}\pi(X)\pi(Y).$$

Contracting of the vector field X in the equation (2.20) yields the following relation between the corresponding Ricci tensors

$$\underset{2}{Ric}(Y,Z) = Ric^g(Y,Z) - \gamma(Y,Z) + n\gamma(Z,Y). \tag{2.21}$$

On the other side, contracting the vector field Z in the equation (2.20), we have

$$\frac{1}{n^2-1}\,{}'\underset{2}{R}(X,Y) = \gamma(X,Y) - \gamma(Y,X), \tag{2.22}$$

where we denoted ${}'\underset{2}{R}(X,Y) = \mathrm{trace}[Z\to \underset{2}{R}(X,Y)Z]$ and used $\mathrm{trace}[Z\to R^g(X,Y)Z] = 0$.

If we put $Y=X$ and $Z=Y$ in (2.21) we get

$$\underset{2}{Ric}(X,Y) = Ric^g(X,Y) - \gamma(X,Y) + n\gamma(Y,X). \tag{2.23}$$

Adding (2.22) to (2.23) we obtain

$$\gamma(Y,X) = \frac{1}{n-1}\left(\underset{2}{Ric}(X,Y) - Ric^g(X,Y) + \frac{1}{n^2-1}\,{}'\underset{2}{R}(X,Y)\right). \tag{2.24}$$

Replacing (2.22) and (2.24) into (2.20) we obtain

$$\underset{2}{W}(X,Y)Z = W^g(X,Y)Z,$$

where

$$\begin{aligned}\underset{2}{W}(X,Y)Z = \underset{2}{R}(X,Y)Z &+ \frac{1}{(n-1)(n^2-1)}\big((n^2-1)\underset{2}{Ric}(Z,X) + {}'\underset{2}{R}(Z,X)\big)Y \\ &- \frac{1}{(n-1)(n^2-1)}\big((n^2-1)\underset{2}{Ric}(Z,Y) + {}'\underset{2}{R}(Z,Y)\big)X \\ &- \frac{n}{n^2-1}{}'\underset{2}{R}(X,Y)Z\end{aligned} \tag{2.25}$$

and $W^g(X,Y)Z$ is given by (1.8). The tensor $\underset{2}{W}(X,Y)Z$ given by (2.25) is independent of a 1-form π.

By skew-symmetrization in the equation (2.21) with respect to the vector fields Y and Z we obtain

$$\underset{2}{Ric}(Y,Z) - \underset{2}{Ric}(Z,Y) = -(n+1)(\gamma(Y,Z) - \gamma(Z,Y))$$

and after comparing it with equation (2.22), we get

$$\underset{2}{Ric}(Y,Z) - \underset{2}{Ric}(Z,Y) = -\frac{1}{n-1}{}'\underset{2}{R}(Y,Z),$$

from which we see that the tensor ${}'\underset{2}{R}(Y,Z)$ is skew-symmetric. Finally, using the above, we can prove the following theorem.

Theorem 1. *Let $(\mathcal{M},g)$ be a Riemannian space of dimension $n>2$, ∇^g be the Levi-Civita connection of g and $\overset{1}{\nabla}$ be a projective semi-symmetric linear connection given by (2.1), $\underset{2}{W}(X,Y)Z$ be the tensor given by (2.25) and $W^g(X,Y)Z$ be the Weyl projective curvature tensor given by (1.8) then:*

(i) $\underset{2}{W}(X,Y)Z = W^g(X,Y)Z$,

(ii) *If the Ricci tensor $\underset{2}{Ric}(X,Y)$ vanishes then $\underset{2}{W}(X,Y)Z = \underset{2}{R}(X,Y)Z$,*

(iii) *If the Ricci tensor $\underset{2}{Ric}(X,Y)$ is symmetric then ${}'\underset{2}{R}(X,Y)=0$ and the tensor $\underset{2}{W}(X,Y)Z$ given by (2.25) takes form*

$$\underset{2}{W}(X,Y)Z = \underset{2}{R}(X,Y)Z + \frac{1}{n-1}\big(\underset{2}{Ric}(X,Z)Y - \underset{2}{Ric}(Y,Z)X\big).$$

2.4. *Curvature tensor of the third kind*

We start with the equation (2.5) of Proposition 1 which reads

$$\underset{3}{R}(X,Y)Z = R^g(X,Y)Z + \frac{n}{n+1}\big((\nabla^g_X\pi)(Z)Y - (\nabla^g_Y\pi)(X)Z\big)$$

$$
\begin{aligned}
&+\frac{1}{n+1}\big((\nabla^g_Y\pi)(Z)X-(\nabla^g_X\pi)(Y)Z\big)-\frac{n^2}{(n+1)^2}\pi(X)\pi(Z)Y \\
&+\frac{1}{(n+1)^2}\pi(Y)\pi(Z)X+\frac{n-1}{n+1}\pi(Y)\pi(X)Z.
\end{aligned}
\tag{2.26}
$$

Contracting of the vector field X in the equation (2.26) yields a relation between the corresponding Ricci tensors which is

$$
\begin{aligned}
\underset{3}{Ric}(Y,Z)=&Ric^g(Y,Z)+\frac{n}{n+1}(\nabla^g_Y\pi)(Z)-\frac{1}{n+1}(\nabla^g_Z\pi)(Y)\\
&+\frac{n-1}{(n+1)^2}\pi(Y)\pi(Z).
\end{aligned}
\tag{2.27}
$$

On the other side, contracting of the vector field Z in the equation (2.26), we have

$$
\begin{aligned}
{}'\underset{3}{R}(X,Y)&:=\operatorname{trace}[Z\to\underset{3}{R}(X,Y)Z]\\
&=(n-1)\Big(\frac{n-1}{n+1}\pi(X)\pi(Y)-(\nabla^g_Y\pi)(X)\Big).
\end{aligned}
\tag{2.28}
$$

Combining (2.27) and (2.28), we have

$$
\begin{aligned}
\frac{1}{n+1}(\nabla^g_Y\pi)(Z)=&\frac{1}{n}\Big(\underset{3}{Ric}(Y,Z)-Ric^g(Y,Z)-\frac{1}{n^2-1}{}'\underset{3}{R}(Y,Z)\Big),\\
\frac{1}{n+1}\pi(Y)\pi(Z)=&\frac{n+1}{n(n-1)}\underset{3}{Ric}(Y,Z)-\frac{n+1}{n(n-1)}Ric^g(Y,Z)-\frac{1}{n(n-1)^2}{}'\underset{3}{R}(Y,Z)\\
&+\frac{1}{(n-1)^2}{}'\underset{3}{R}(Z,Y).
\end{aligned}
$$

Substituting the last two equalities in (2.26), we obtain

$$
\underset{3}{W}(X,Y)Z=W^g(X,Y)Z,
$$

where

$$
\begin{aligned}
\underset{3}{W}(X,Y)Z=&\underset{3}{R}(X,Y)Z+\frac{1}{n-1}\big(\underset{3}{Ric}(X,Z)Y-\underset{3}{Ric}(Y,Z)X\big)\\
&-\big(\underset{3}{Ric}(X,Y)-\underset{3}{Ric}(Y,X)\big)Z\\
&-\frac{1}{(n^2-1)(n-1)}\big({}'\underset{3}{R}(X,Z)-n^2\,{}'\underset{3}{R}(Z,X)\big)Y\\
&+\frac{1}{(n^2-1)(n-1)}\big({}'\underset{3}{R}(Y,Z)-{}'\underset{3}{R}(Z,Y)\big)X\\
&+\frac{1}{n^2-1}\big({}'\underset{3}{R}(X,Y)-(n+2){}'\underset{3}{R}(Y,X)\big)Z
\end{aligned}
\tag{2.29}
$$

and $W^g(X,Y)Z$ is given by (1.8). The tensor $\underset{3}{W}(X,Y)Z$ given by (2.29) is independent of a 1-form π.

By comparing the results obtained by skew-symmetrization in the equation (2.27) and by skew-symmetrization in the equation (2.28), we get

$$\underset{3}{Ric}(Y,Z)-\underset{3}{Ric}(Z,Y)=\frac{1}{n-1}({}'\underset{3}{R}(Y,Z)-{}'\underset{3}{R}(Z,Y)).$$

These conclusions are stated in Theorem 2.

Theorem 2. *Let $(\mathcal{M},g)$ be a Riemannian space of dimension $n>2$, ∇^g be the Levi-Civita connection of g and $\overset{1}{\nabla}$ be a projective semi-symmetric linear connection given by (2.1), $\underset{3}{W}(X,Y)Z$ be the tensor given by (2.29) and $W^g(X,Y)Z$ be the Weyl projective curvature tensor given by (1.8) then:*

(i) $\underset{3}{W}(X,Y)Z=W^g(X,Y)Z$,

(ii) If the tensors $\underset{3}{Ric}(X,Y)$ and ${}'\underset{3}{Ric}(X,Y)$ vanish then $\underset{3}{W}(X,Y)Z=\underset{3}{R}(X,Y)Z$,

(iii) If the Ricci tensor $\underset{3}{Ric}(X,Y)$ is symmetric then the tensor ${}'\underset{3}{R}(X,Y)$ is symmetric and the tensor $\underset{3}{W}(X,Y)Z$ given by (2.29) takes form

$$\begin{aligned}\underset{3}{W}(X,Y)Z=&\underset{3}{R}(X,Y)Z+\frac{1}{n-1}\big(\underset{3}{Ric}(X,Z)Y-\underset{3}{Ric}(Y,Z)X\big)\\&+\frac{1}{n-1}\big({}'\underset{3}{R}(X,Z)Y-{}'\underset{3}{R}(X,Y)Z\big).\end{aligned}$$

2.5. *Curvature tensor of the fourth kind*

The relation (2.6) of Proposition 1 reads

$$\begin{aligned}\underset{4}{R}(X,Y)Z=&R^g(X,Y)Z+\frac{n}{n+1}\big((\nabla^g_X\pi)(Z)Y-(\nabla^g_Y\pi)(X)Z\big)\\&+\frac{1}{n+1}\big((\nabla^g_Y\pi)(Z)X-(\nabla^g_X\pi)(Y)Z\big)+\frac{2n+1}{(n+1)^2}\pi(X)\pi(Z)Y\\&-\frac{n^2+2n}{(n+1)^2}\pi(Y)\pi(Z)X+\frac{n-1}{n+1}\pi(Y)\pi(X)Z.\end{aligned}\tag{2.30}$$

Contracting the vector field X in the equation (2.30) we obtain the relation

$$\begin{aligned}\underset{4}{Ric}(Y,Z)=&Ric^g(Y,Z)+\frac{n}{n+1}(\nabla^g_Y\pi)(Z)-\frac{1}{n+1}(\nabla^g_Z\pi)(Y)\\&-\frac{n^3+n^2-2n}{(n+1)^2}\pi(Y)\pi(Z).\end{aligned}\tag{2.31}$$

Contracting of the vector field Z in the equation (2.30) we obtain

$${}'\underset{4}{R}(X,Y)=-(n-1)(\nabla^g_Y\pi)(X)+\frac{(n-1)^2}{n+1}\pi(X)\pi(Y),\tag{2.32}$$

where

$${}'\underset{4}{R}(X,Y)=\mathrm{trace}[Z\to\underset{4}{R}(X,Y)Z].$$

From equation (2.32) we have

$$(\nabla^g_Y \pi)(X) = -\frac{1}{n-1}\,{}'\underset{4}{R}(X,Y) + \frac{n-1}{n+1}\pi(X)\pi(Y) \tag{2.33}$$

and by substituting equation (2.33) into (2.31) we get

$$\begin{aligned}\pi(Y)\pi(Z) = \frac{(n+1)^2}{n^3-1}\Big(&Ric^g(Y,Z) - \underset{4}{Ric}(Y,Z)\\ &+ \frac{1}{n^2-1}\,{}'\underset{4}{R}(Y,Z) - \frac{n}{n^2-1}\,{}'\underset{4}{R}(Z,Y)\Big).\end{aligned} \tag{2.34}$$

Combining equations (2.33) and (2.34), we obtain

$$\begin{aligned}(\nabla^g_Y \pi)(Z) =& \frac{n+1}{n^2+n+1}Ric^g(Y,Z) - \frac{n+1}{n^2+n+1}\underset{4}{Ric}(Y,Z) + \frac{1}{n^3-1}\,{}'\underset{4}{R}(Y,Z)\\ &- \frac{(n+1)^2}{n^3-1}\,{}'\underset{4}{R}(Z,Y).\end{aligned}$$

Substituting the last two equalities in (2.30) we get

$$\underset{4}{W}(X,Y)Z = W^g(X,Y)Z,$$

where

$$\begin{aligned}\underset{4}{W}(X,Y)Z =& \underset{4}{R}(X,Y)Z + \frac{1}{n-1}\big(\underset{4}{Ric}(X,Z)Y - \underset{4}{Ric}(Y,Z)X\big)\\ &+ \frac{n}{n^2+n+1}\big(\underset{4}{Ric}(X,Y) - \underset{4}{Ric}(Y,X)\big)Z\\ &- \frac{1}{(n^2-1)(n-1)}\big({}'\underset{4}{R}(X,Z) - n^2\,{}'\underset{4}{R}(Z,X)\big)Y\\ &+ \frac{1}{(n^2-1)(n-1)}\big({}'\underset{4}{R}(Y,Z) - {}'\underset{4}{R}(Z,Y)\big)X\\ &- \frac{n(n^2+2n+2)}{(n^3-1)(n+1)}\,{}'\underset{4}{R}(X,Y)Z\\ &- \frac{1}{(n^3-1)(n+1)}\,{}'\underset{4}{R}(Y,X)Z\end{aligned} \tag{2.35}$$

and $W^g(X,Y)Z$ is given by (1.8). The tensor $\underset{4}{W}(X,Y)Z$ given by (2.35) is independent of a 1-form π.

By skew-symmetrization in the equation (2.34) with respect to the vector fields Y and Z we obtain

$$\underset{4}{Ric}(Y,Z) - \underset{4}{Ric}(Z,Y) = \frac{1}{n-1}\big({}'\underset{4}{R}(Y,Z) - {}'\underset{4}{R}(Z,Y)\big)$$

and if we replace the last equation in (2.35), then the tensor $\underset{4}{W}(X,Y)Z$ can be rewritten as

$$\begin{aligned}\underset{4}{W}(X,Y)Z =&\underset{4}{R}(X,Y)Z+\frac{1}{n-1}\big(\underset{4}{Ric}(X,Z)Y-\underset{4}{Ric}(Y,Z)X\big)\\&-\frac{1}{n^2-1}\big(n'\underset{4}{R}(X,Y)+'\underset{4}{R}(Y,X)\big)Z\\&-\frac{1}{(n^2-1)(n-1)}\big('\underset{4}{R}(X,Z)-n^{2\prime}\underset{4}{R}(Z,X)\big)Y\\&+\frac{1}{(n^2-1)(n-1)}\big('\underset{4}{R}(Y,Z)-'\underset{4}{R}(Z,Y)\big)X.\end{aligned}\tag{2.36}$$

These conclusions are stated in Theorem 3.

Theorem 3. *Let $(\mathcal{M},g)$ be a Riemannian space of dimension $n>2$, ∇^g be the Levi-Civita connection of g and $\overset{1}{\nabla}$ be a projective semi-symmetric linear connection given by (2.1), $\underset{4}{W}(X,Y)Z$ be the tensor given by (2.36) and $W^g(X,Y)Z$ be the Weyl projective curvature tensor given by (1.8) then:*

(*i*) *$\underset{4}{W}(X,Y)Z=W^g(X,Y)Z$,*

(*ii*) *If the tensors $\underset{4}{Ric}(X,Y)$ and $'\underset{4}{R}(X,Y)$ vanish, then $\underset{4}{W}(X,Y)Z=\underset{4}{R}(X,Y)Z$,*

(*iii*) *If the Ricci tensor $\underset{4}{Ric}(X,Y)$ is symmetric then $'\underset{4}{R}(X,Y)$ is symmetric tensor and the tensor $\underset{4}{W}(X,Y)Z$ given by (2.36) takes form*

$$\begin{aligned}\underset{4}{W}(X,Y)Z =&\underset{4}{R}(X,Y)Z+\frac{1}{n-1}\big(\underset{4}{Ric}(X,Z)Y-\underset{4}{Ric}(Y,Z)X\big)\\&+\frac{1}{n-1}\big('\underset{4}{R}(X,Z)Y-'\underset{4}{R}(X,Y)Z\big).\end{aligned}$$

2.6. *Curvature tensor of the fifth kind*

We start with the relation (2.7) of Proposition 1:

$$\begin{aligned}\underset{5}{R}(X,Y)Z =&R^g(X,Y)Z+\frac{n-1}{2(n+1)}\big((\nabla^g_X\pi)(Y)-(\nabla^g_Y\pi)(X)\big)Z\\&+\frac{n-1}{2(n+1)}\big((\nabla^g_X\pi)(Z)Y-(\nabla^g_Y\pi)(Z)X\big)\\&+\frac{(n-1)^2}{4(n+1)^2}\pi(Z)\big(\pi(Y)X-\pi(X)Y\big)\\&-\frac{1}{4}\pi(Z)\big(\pi(Y)X+\pi(X)Y\big)+\frac{1}{2}\pi(X)\pi(Y)Z.\end{aligned}\tag{2.37}$$

If we contract the last relation with respect to X, Y and Z, respectively, we obtain

$$\begin{aligned}\underset{5}{Ric}(Y,Z) =&Ric^g(Y,Z)+\frac{n-1}{2(n+1)}(\nabla^g_Z\pi)(Y)-\frac{n(n-1)}{2(n+1)}(\nabla^g_Y\pi)(Z)\\&-\frac{n(n-1)}{(n+1)^2}\pi(Y)\pi(Z),\end{aligned}\tag{2.38}$$

$$\begin{aligned}{}''\underset{5}{R}(X,Z) =&-Ric^g(X,Z)+\frac{n(n-1)}{2(n+1)}(\nabla^g_X\pi)(Z)-\frac{n-1}{2(n+1)}(\nabla^g_Z\pi)(X)\\&-\frac{(n-1)(n^2+1)}{2(n+1)^2}\pi(X)\pi(Z)\end{aligned}\tag{2.39}$$

and

$${}'\underset{5}{R}(X,Y)=\frac{n-1}{2}(\nabla^g_X\pi)(Y)-\frac{n-1}{2}(\nabla^g_Y\pi)(X)+\frac{n-1}{2}\pi(X)\pi(Y),\tag{2.40}$$

where ${}''\underset{5}{R}(X,Z)=\mathrm{trace}[Y\to\underset{5}{R}(X,Y)Z]$ and ${}'\underset{5}{R}(X,Y)=\mathrm{trace}[Z\to\underset{5}{R}(X,Y)Z]$.

If we choose $Y=X$ and $Z=Y$ into (2.38) we get

$$\begin{aligned}\underset{5}{Ric}(X,Y) =&Ric^g(X,Y)+\frac{n-1}{2(n+1)}(\nabla^g_Y\pi)(X)-\frac{n(n-1)}{2(n+1)}(\nabla^g_X\pi)(Y)\\&-\frac{n(n-1)}{(n+1)^2}\pi(X)\pi(Y).\end{aligned}\tag{2.41}$$

On the other hand, by putting $Z=Y$ into (2.39) we obtain

$$\begin{aligned}{}''\underset{5}{R}(X,Y) =&-Ric^g(X,Y)+\frac{n(n-1)}{2(n+1)}(\nabla^g_X\pi)(Y)-\frac{n-1}{2(n+1)}(\nabla^g_Y\pi)(X)\\&-\frac{(n-1)(n^2+1)}{2(n+1)^2}\pi(X)\pi(Y).\end{aligned}\tag{2.42}$$

Adding the last two relations we obtain

$$\frac{n-1}{2}\pi(X)\pi(Y)=-\underset{5}{Ric}(X,Y)-{}''\underset{5}{R}(X,Y).\tag{2.43}$$

By substituting equation (2.43) into (2.41) and (2.40) we have the next relations

$$\begin{aligned}\frac{n^2+1}{(n+1)^2}\underset{5}{Ric}(X,Y)-\frac{2n}{(n+1)^2}{}''\underset{5}{R}(X,Y) =&Ric^g(X,Y)-\frac{n(n-1)}{2(n+1)}(\nabla^g_X\pi)(Y)\\&+\frac{n-1}{2(n+1)}(\nabla^g_Y\pi)(X)\end{aligned}\tag{2.44}$$

and

$$\underset{5}{Ric}(X,Y)+{}''\underset{5}{R}(X,Y)+{}'\underset{5}{R}(X,Y)=\frac{n-1}{2}(\nabla^g_X\pi)(Y)-\frac{n-1}{2}(\nabla^g_Y\pi)(X).\tag{2.45}$$

Combining equations (2.44) and (2.45) we get

$$\begin{aligned}\frac{n-1}{2(n+1)}(\nabla^g_X\pi)(Y) =&\frac{1}{n-1}Ric^g(X,Y)-\frac{n^2+n+2}{(n^2-1)(n+1)}\underset{5}{Ric}(X,Y)\\&-\frac{1}{n^2-1}{}'\underset{5}{R}(X,Y)+\frac{1}{(n+1)^2}{}''\underset{5}{R}(X,Y).\end{aligned}\tag{2.46}$$

Finally, putting (2.43) and (2.46) into (2.37) we obtain

$$\underset{5}{W}(X,Y)Z=W^g(X,Y)Z,$$

where

$$\begin{aligned}\underset{5}{W}(X,Y)Z =&\underset{5}{R}(X,Y)Z+\frac{1}{n^2-1}\big(\underset{5}{Ric}(X,Z)Y-(n+2)\underset{5}{Ric}(Y,Z)X\big)\\&+\frac{1}{(n+1)^2}\big(\underset{5}{Ric}(Y,X)-{}''\underset{5}{R}(X,Y)\big)Z\\&+\frac{1}{n^2-1}\big({}'\underset{5}{R}(X,Y)-{}'\underset{5}{R}(Y,X)\big)Z\\&+\frac{1}{n^2-1}\big({}'\underset{5}{R}(X,Z)Y-{}'\underset{5}{R}(Y,Z)X\big)\\&+\frac{n^2+n+2}{(n^2-1)(n+1)}\underset{5}{Ric}(X,Y)Z\\&+\frac{n^2+3n}{(n^2-1)(n+1)}{}''\underset{5}{R}(Y,X)Z\\&-\frac{1}{n^2-1}\big(n{}''\underset{5}{R}(X,Z)Y+{}''\underset{5}{R}(Y,Z)X\big)\end{aligned}\tag{2.47}$$

and $W^g(X,Y)Z$ is the Weyl projective curvature tensor given by (1.8). The tensor $\underset{5}{W}(X,Y)Z$ given by (2.47) is independent of a 1-form π.

By using skew-symmetrization of equations (2.40), (2.41) and (2.42) we obtain

$$\underset{5}{Ric}(X,Y)-\underset{5}{Ric}(Y,X)=-\frac{1}{2}\big({}'\underset{5}{R}(X,Y)-{}'\underset{5}{R}(Y,X)\big)=-{}''\underset{5}{R}(X,Y)+{}''\underset{5}{R}(Y,X),$$

from which

$$\underset{5}{Ric}(X,Y)-{}''\underset{5}{R}(Y,X)=\underset{5}{Ric}(Y,X)-{}''\underset{5}{R}(X,Y).$$

If we use the last equation then the tensor $\underset{5}{W}(X,Y)Z$ given by (2.47) can be written in the form

$$\begin{aligned}\underset{5}{W}(X,Y)Z =&\underset{5}{R}(X,Y)Z+\frac{1}{n^2-1}\big(\underset{5}{Ric}(X,Z)Y-(n+2)\underset{5}{Ric}(Y,Z)X\big)\\&+\frac{1}{n-1}\big(\underset{5}{Ric}(X,Y)+{}''\underset{5}{R}(Y,X)\big)Z\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{n^2-1}\big({}'\underset{5}{R}(X,Y)-{}'\underset{5}{R}(Y,X)\big)Z \\
&+\frac{1}{n^2-1}\big({}'\underset{5}{R}(X,Z)Y-{}'\underset{5}{R}(Y,Z)X\big) \\
&-\frac{1}{n^2-1}\big(n{}''\underset{5}{R}(X,Z)Y+{}''\underset{5}{R}(Y,Z)X\big).
\end{aligned} \tag{2.48}
$$

The obtained conclusions will be formulated in the following theorem.

Theorem 4. *Let $(\mathcal{M},g)$ be a Riemannian space of dimension $n>2$, ∇^g be the Levi-Civita connection of g and $\overset{1}{\nabla}$ be a projective semi-symmetric linear connection given by (2.1), $\underset{5}{W}(X,Y)Z$ be the tensor given by (2.48) and $W^g(X,Y)Z$ be the Weyl projective curvature tensor given by (1.8) then:*

(*i*) *$\underset{5}{W}(X,Y)Z=W^g(X,Y)Z$,*

(*ii*) *If the tensors $\underset{5}{Ric}(X,Y)$, ${}'\underset{5}{R}(X,Y)$ and ${}''\underset{5}{R}(X,Y)$ vanish then $\underset{5}{W}(X,Y)Z=\underset{5}{R}(X,Y)Z$,*

(*iii*) *If the Ricci tensor $\underset{5}{Ric}(X,Y)$ is symmetric then the tensors ${}'\underset{5}{R}(X,Y)$, ${}''\underset{5}{R}(X,Y)$ are symmetric and the tensor $\underset{5}{W}(X,Y)Z$ given by (2.48) takes form*

$$
\begin{aligned}
\underset{5}{W}(X,Y)Z=&\underset{5}{R}(X,Y)Z+\frac{1}{n^2-1}\big(\underset{5}{Ric}(X,Z)Y-(n+2)\underset{5}{Ric}(Y,Z)X\big) \\
&+\frac{1}{n-1}\big(\underset{5}{Ric}(X,Y)+{}''\underset{5}{R}(Y,X)\big)Z+\frac{1}{n^2-1}\big({}'\underset{5}{R}(X,Z)Y-{}'\underset{5}{R}(Y,Z)X\big) \\
&-\frac{1}{n^2-1}\big(n{}''\underset{5}{R}(X,Z)Y+{}''\underset{5}{R}(Y,Z)X\big).
\end{aligned}
$$

3. CONCLUSION

The method for obtaining the Weyl projective curvature tensors did not allow us to make any additional assumptions in order to derive the tensors $\underset{\theta}{W}(X,Y)Z$, $\theta=0,1,\dots,5$. By comparing our results with the results of related papers that deal with projective semi-symmetric linear connections we concluded that our approach which is based on the curvature tensors $\underset{\theta}{R}(X,Y)Z$, for $\theta=3,4,5$ complements the existing results in the current literature. It would be interesting to see some applications of tensors $\underset{\theta}{W}(X,Y)Z$, $\theta=0,1,\dots,5$ in the theories of gravity that includes the torsion tensor. Also, we can conclude that the Riemannian metric g is projectively flat if and only if for arbitrary $\theta\in\{0,1,\dots,5\}$ the tensor $\underset{\theta}{W}(X,Y)Z$ identically vanishes.

Remark 1. In this paper, using the projective semi-symmetric linear connection, we obtained that the corresponding tensors $\underset{\theta}{W}$, $\theta=0,1,\dots,5$, are coincident with the

Weyl projective tensor W^g of the Levi-Civita connection. Also, papers [22, 23, 36] determined tensors coincident with the Weyl projective tensor W^g in a similar way. Papers [1, 2, 12] dealt with similar problems, as they also determined tensors coincident with Weyl projective tensor W^g. If we consider a geodesic mapping between some special manifolds with non-symmetric linear connection one can prove that there exist analogous expressions of the Weyl projective curvature tensor with respect to the curvature tensors $\underset{\theta}{R}(X,Y)Z$, $\theta = 0,1,\ldots,5$, but some assumptions are necessary [21]. In papers [17, 18, 27–31, 37], the authors obtained some geometric objects that are different than the Weyl tensor of projective curvature and invariant with respect to the so-called equitorsion geodesic mapping. These invariant geometric objects were called ET-projective parameters. In case when some of them were tensors, then they were called ET-projective curvature tensors. The relations between the ET-projective parameters and the Weyl projective tensor with respect to the symmetric connection are presented in the paper [28] (in the present paper, the tensor $\underset{0}{W}$ given by (2.12) is the Weyl projective tensor of symmetric connection). Using the idea from that paper, the relations between the tensors $\underset{\theta}{W}$, $\theta = 0,1,\ldots,5$ and the corresponding ET-projective parameters can be presented analogously.

Acknowledgements

The authors are very grateful to the anonymous reviewer for their valuable comments and suggestions.

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