Miskolc Mathematical Notes
Vol. 24 (2023), No. 3, pp. 1235–1245

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.4220

ON NEW VERSION OF HERMITE-JENSEN-MERCER INEQUALITIES FOR NEWLY DEFINED QUANTUM INTEGRAL

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Received 27 April, 2022

Abstract. In current study, we establish some new quantum Hermite Hadamard-Jensen-Mercer type integral inequalities by way of recently newly defined integral.Then we investigate the connections between our results and those in earlier works.Furthermore, we present some examples which satisfied our main outcomes. We expect that this innovative way opens many avenues for interested researchers will reconcile discovering further quantum approximations of Hermite-Hadamard type variants for other classes of convex functions, and, additionally, to discover uses in the aforementioned scientific disciplines.

2010 *Mathematics Subject Classification:* 26D10; 26D15

Keywords: Jensen-Mercer inequalities, Hermite-Hadamard inequalities, ${}_aT_q$-integral, ${}^bT_q$ integral

1. Quantum calculus

The investigation of quantum calculus is equivalent to the usual infinitesimal calculus without the concept of limits or the investigation of calculus without limits (quantum is from the Latin word "quantus" and literally it means how much, in Swedish "Kvant"). It has two main branches, q-calculus and h-calculus. And both of them were worked out by P. Cheung and V. Kac [14] in the early twentieth century. In the same era F.H. Jackson started working on quantum calculus or q-calculus, but Euler and Jacobi had already figured out this type of calculus. A number of studies have been recently conducted on the field of q-analysis, beginning from Euler, due to it vast applications in mathematical modelling and the existence of its connection in the framework between physics and mathematics. Tariboon and Ntouyas [22] proposed the quantum calculus concepts on finite intervals and obtained several q-analogues of classical mathematical objects. This inspired other researchers and, as a consequence, various novel results concerning quantum analogues of classical mathematical results have already been inaugurated in the literature. In [4], Alp et al. acquired some bonds for left-hand side of q-Hermite-Hadamard inequalities and

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quantum calculations by way of convex and quasi-convex functions for midpoint type inequalities. For more details, see [14, 18, 20–22] and the references cited therein.

In different mathematical fields, Quantum calculus has a broad area of application in number theory, combinatorics, orthogonal polynomials, simple hyper-geometric functions, and other sciences, quantum theory, physics and relativity theory. It has been discovered that quantum calculus is a sub field of the more general mathematical field of time scales calculus. New developments have recently been made in the research and methodology of dynamic derivatives on time scales. The research offers a unification and application of traditional differential and difference equations. Moreover, it is a alliance of the discrete theory with the continuous theory from a theoretical perspective. Time scales provide a unified framework for studying dynamic equations on both discrete and continuous domains. In studying quantum calculus, we are concerned with a specific time scale, called the q-time scale, defined as follows: $T := q^{N_0} := qt : t \in N_0$, see [2–5, 9–12] and the references cited therein.

On the other hand, the well known discrete Jensen inequality states that: let $0 < x_1 \leq x_2 \leq \ldots \leq x_n$ and let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ non negative weights such that $\Sigma_{i=1}^{n} \mu_i = 1$. If F is a convex function on the interval $[a, b]$, then

$$F\left(\sum_{i=1}^{n} \mu_i x_i\right) \leq \sum_{i=1}^{n} \mu_i F(x_i),$$

where for all $x_i \in [a, b]$ and $\mu_i \in [0, 1]$, $(i = \overline{1, n})$ [8]. The Hermite–Hadamard inequality states that, if a mapping $F : I \subseteq R \to R$ is a convex on I with $a, b \in I$ and $a < b$, then

$$F\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b F(x)dx \leq \frac{F(a)+F(b)}{2}.$$

The double inequality holds in the reversed direction if F is concave [7].

Theorem 1 ([17, Theoem 1.2]). *If F is convex function on $I = [a, b]$, then*

$$F\left(a+b-\sum_{i=1}^{n} \mu_i x_i\right) \leq F(a)+F(b)-\sum_{i=1}^{n} \mu_i F(x_i), \tag{1.1}$$

for each $x_i \in [a, b]$ and $\mu_i \in [0, 1]$, $(i = \overline{1, n})$ with $\sum_{i=1}^{n} \mu = 1$.

Inequality (1.1) is known as the Jensen–Mercer inequality. By using inequality (1.1), Kian and Moslehian prove the Hermite-Jensen-Mercer inequality in [16]. Recently several papers have devoted to generalization of Hermite-Jensen-Mercer inequality. For more recent and related results connected with Jensen-Mercer inequality and Hermite-Jensen–Mercer inequality, see [1, 6, 19, 23, 24].

In this research, we establish the Hermite-Jensen-Mercer inequalities for T_q integrals.We also present some examples to illustrate our results.

2. Preliminaries of q-Calculus and Some Inequalities

In this section we present some required definitions and related inequalities about q-calculus. Also, here and further we use the following notation(see [14]):

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+q^2+\ldots+q^{n-1}, \quad q\in(0,1).$$

In [13], Jackson gave the q-Jackson integral from 0 to b for $0<q<1$ as follows:

$$\int_0^b F(x)\ d_qx \ = (1-q)\,b\sum_{n=0}^{\infty} q^n F(bq^n)$$

provided the sum converge absolutely.
Jackson in [13] gave the q-Jackson integral in a generic interval $[a,b]$ as:

$$\int_a^b F(x)\ d_qx \ = \int_0^b F(x)\ d_qx \ - \int_0^a F(x)\ d_qx \ .$$

Definition 1 ([21, Definition 3.3])**.** Let $F:[a,b]\to R$ be a continuous function. Then, the q_a-definite integral on $[a,b]$ is defined as

$$\int_a^b F(x)_a\, d_qx = (1-q)(b-a)\sum_{n=0}^{\infty} q^n F(q^n b+(1-q^n)a)$$
$$= (b-a)\int_0^1 F((1-t)a+tb)\,d_qt.$$

Many of the most renowned inequalities in the classical analysis such as Hölder's inequality, Hermite-Hadamard's inequality and Ostrowski's inequality, Cauchy-Bunyakovsky-Schwarz, Gruss, Gruss-Cebysev and other integral inequalities have been proven and applied for q-calculus.
In [4], Alp et al. proved the following q_a-Hermite-Hadamard inequalities for convex functions in the setting of quantum calculus:

Theorem 2 ([4, Theoem 6])**.** *Let $F:[a,b]\to R$ be a convex differentiable function on $[a,b]$ and $0<q<1$. Then q-Hermite-Hadamard inequalities are as follows:*

$$F\left(\frac{qa+b}{1+q}\right) \le \frac{1}{b-a}\int_a^b F(x)\ {}_a d_qx \ \le \frac{qF(a)+F(b)}{1+q}. \tag{2.1}$$

In [18] and [4], authors established some bounds for left and right hand sides of the inequality (2.1).
On the other hand, Bermudo et al. gave the following new definition and related Hermite-Hadamard type inequalities:

Definition 2 ([5, Definition 6]). Let $F:[a,b]\to R$ be a continuous function. Then, the q^b-definite integral on $[a,b]$ is defined as

$$\int_a^b F(x)\ {}^b d_q x = (1-q)(b-a)\sum_{n=0}^{\infty} q^n F(q^n a+(1-q^n)b)$$
$$= (b-a)\int_0^1 F(ta+(1-t)b)\ d_q t.$$

Theorem 3 ([5, Theorem 11]). *Let $F:[a,b]\to R$ be a convex function on $[a,b]$ and $0<q<1$. Then, q-Hermite-Hadamard inequalities are as follows:*

$$F\left(\frac{a+qb}{1+q}\right)\le \frac{1}{b-a}\int_a^b F(x)\ {}^b d_q x \le \frac{F(a)+qF(b)}{1+q}.$$

From Theorem 1 and Theorem 3, one can the following inequalities:

Corollary 1 ([5, Corollary 14]). *For any convex function $F:[a,b]\to R$ and $0<q<1$, we have*

$$F\left(\frac{qa+b}{1+q}\right)+F\left(\frac{a+qb}{1+q}\right)\le \frac{1}{b-a}\left\{\int_a^b F(x)\ {}_a d_q x + \int_a^b F(x)\ {}^b d_q x\right\}$$
$$\le F(a)+F(b)$$

and

$$F\left(\frac{a+b}{2}\right)\le \frac{1}{2(b-a)}\left\{\int_a^b F(x)\ {}_a d_q x + \int_a^b F(x)\ {}^b d_q x\right\}\le \frac{F(a)+F(b)}{2}.$$

2.1. *T_q-Integrals and Related Inequalities*

In this subsection, we present definitions and properties given by using trapezoids.

Alp and Sarikaya, by using the area of trapezoids, introduced the following generalized quantum integral which is called ${}_aT_q$-integral in [2].

Definition 3 ([2, Definition 3.1]). Let $F:[a,b]\to R$ is continuous function. for $x\in[a,b]$

$$\int_a^b F(s)\ {}_a d_q^T s = \frac{(1-q)(b-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n F(q^n b+(1-q^n)a) - F(b)\right]$$

$$= (b-a)\int_0^1 F(tb+(1-t)a)_0 d_q^T t$$

where $0 < q < 1$.

Theorem 4 ([2, Theorem 4.1])**.** *Let $F:[a,b]\to R$ be a convex continuous function on $[a,b]$ and $0<q<1$. Then we have*

$$F\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b F(x)\ \ {}_a d_q^T x\ \le \frac{F(a)+F(b)}{2}$$

In [15], Kara et al. introduced the following generalized quantum integral which is called ${}^bT_q$-integral.

Definition 4 ([15, Definition 6])**.** Let $F:[a,b]\to R$ is continuous function. for $x\in[a,b]$,

$$\int_a^b F(s)\ \ {}^b d_q^T s\ = \frac{(1-q)(b-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n F(q^n a+(1-q^n)b) - f(a)\right]$$

$$= (b-a)\int_0^1 F(ta+(1-t)b)^1 d_q^T t$$

where $0 < q < 1$.

Theorem 5 ([15, Theorem 11])**.** *Let $F:[a,b]\to R$ be a convex continuous function on $[a,b]$ and $0<q<1$. Then we have*

$$F\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b F(x)\ \ {}^b d_q^T x\ \le \frac{F(a)+F(b)}{2}.$$

3. Generalized Hermite-Jensen-Mercer inequalities for Quantum Integrals

In this section, we establish some new Quantum Hermite-Jensen-Mercer type inequalities by way of T_q integrals.

Theorem 6. *Let $F:[a,b]\to R$ be a convex function on $[a,b]$ and $0<q<1$. Then we have the following quantum Jensen-Mercer inequalities*

$$\begin{aligned} F\left(a+b-\frac{x+y}{2}\right) &\le F(a)+F(b)-\frac{1}{2(y-x)}\left[\int_x^y F(w)\,{}_x d_q^T w+\int_x^y F(w)\,{}^y d_q^T w\right] \\ &\le F(a)+F(b)-F\left(\frac{x+y}{2}\right) \qquad (3.1) \end{aligned}$$

for all $x,y\in[a,b]$ with $x\le y$.

Proof. It follows from the Jensen-Mercer inequality, we have

$$F\left(a+b-\frac{u+v}{2}\right) \leq F(a)+F(b)-\frac{1}{2}[F(u)+F(v)]$$

for $u,v \in [a,b]$. Considering $u = tx+(1-t)y$ and $v = ty+(1-t)x$, where $t \in [0,1]$, we get

$$F\left(a+b-\frac{x+y}{2}\right) \leq F(a)+F(b)-\frac{1}{2}\left[F(tx+(1-t)y)+F(ty+(1-t)x)\right].$$

By ${}_aT_q$-integrating with respect to t over $[0,1]$ for $q \in (0,1)$, we have

$$\begin{aligned} &F\left(a+b-\frac{x+y}{2}\right) \\ &\leq F(a)+F(b)-\frac{1}{2}\left[\int_0^1 F(tx+(1-t)y)d_q^T t+\int_0^1 F(ty+(1-t)x)d_q^T t\right]. \end{aligned} \tag{3.2}$$

From Definition 3, and Theorem 4, we have

$$\int_0^1 F(tx+(1-t)y)\,d_q^T t = \frac{1}{y-x}\int_x^y F(w)\,{}_x d_q^T w \geq F\left(\frac{x+y}{2}\right) \tag{3.3}$$

and similarly by Definition 4 and Theorem 5, we have

$$\int_0^1 F(tx+(1-t)y)\,d_q^T t = \frac{1}{y-x}\int_x^y F(w)\,{}^y d_q^T w \geq F\left(\frac{x+y}{2}\right) \tag{3.4}$$

Using (3.3) and (3.4) in (3.2), we get the desired result (3.1). The proof is completed. □

Remark 1. If we take the limit $q \to 1$ in Theorem 6, then we have the following inequalities

$$\begin{aligned} F\left(a+b-\frac{x+y}{2}\right) &\leq F(a)+F(b)-\frac{1}{(y-x)}\int_x^y F(w)\,dw \\ &\leq F(a)+F(b)-F\left(\frac{x+y}{2}\right) \end{aligned}$$

which are given by Kian and Moslehian in [16, Theorem 2.1].

Corollary 2. *If we take $x = a$ and $y = b$ in (3.1) then we get*

$$\begin{aligned} F\left(\frac{a+b}{2}\right) &\leq F(a)+F(b)-\frac{1}{2(b-a)}\left[\int_a^b F(w)\,{}_a d_q^T w+\int_a^b F(w)\,{}^b d_q^T w\right] \\ &\leq F(a)+F(b)-F\left(\frac{a+b}{2}\right). \end{aligned}$$

Theorem 7. *Let $F : [a,b] \to R$ be a convex function on $[a,b]$ and $0 < q < 1$. Then we have the following quantum Jensen-Mercer inequalities*

$$F\left(a+b-\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_{a+b-y}^{a+b-x} F(w)\ {}_{(a+b-y)}d_q^T w \leq F(a)+F(b)-\frac{F(x)+F(y)}{2} \tag{3.5}$$

for all $x, y \in [a,b]$ with $x \leq y$.

Proof. From Definition 3 and Theorem 4, we have

$$\begin{aligned}
&\int_0^1 F(a+b-(tx+(1-t)y))_0 d_q^T t \\
&= \int_0^1 F(t(a+b-x)+(1-t)(a+b-y))_0 d_q^T t \\
&= \frac{1}{y-x}\int_{a+b-y}^{a+b-x} F(w)\ {}_{(a+b-y)}d_q^T w \\
&\geq F\left(a+b-\frac{x+y}{2}\right)
\end{aligned} \tag{3.6}$$

which gives proof of first inequality in (3.5). By Jensen-Mercer inequality, we obtain

$$\begin{aligned}
&\frac{1}{y-x}\int_{a+b-y}^{a+b-x} F(w)\ {}_{(a+b-y)}d_q^T w \\
&= \int_0^1 F(a+b-(tx+(1-t)y))_0 d_q^T t \\
&\leq F(a)+F(b)-F(x)\int_0^1 t\ {}_0d_q^T t - F(y)\int_0^1 (1-t)\ {}_0d_q^T t \\
&= F(a)+F(b)-\frac{F(x)+F(y)}{2}.
\end{aligned} \tag{3.7}$$

Combining the inequalities (3.6) and (3.7), we get the required result. □

Remark 2. If we take the limit $q \to 1$ in Theorem 7, then then we have the following inequalities

$$F\left(a+b-\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_{a+b-y}^{a+b-x} F(w)\,dw \leq F(a)+F(b)-\frac{F(x)+F(y)}{2} \tag{3.8}$$

which are proven by Kian and Moslehian in [16, Theorem 2.1].

Remark 3. If we choose $x = a$ and $y = b$ in Theorem 7, then Theorem 7 reduces to Theorem 4.

Theorem 8. *Let $F : [a,b] \to R$ be a convex function on $[a,b]$ and $0 < q < 1$. Then we have the following quantum Jensen-Mercer inequalities*

$$F\left(a+b-\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_{a+b-y}^{a+b-x} F(w)\, {}^{(a+b-x)}d_q^T w \tag{3.9}$$
$$\leq F(a)+F(b)-\frac{F(x)+F(y)}{2}$$

for all $x,y \in [a,b]$ with $x \leq y$.

Proof. From Definition 4 and Theorem 5, we have

$$\begin{aligned} &\int_0^1 F(a+b-(ty+(1-t)x))\,{}_0d_q^T t \\ &= \int_0^1 F(t(a+b-y)-(1-t)(a+b-x))\,{}_0d_q^T t \\ &= \frac{1}{y-x}\int_{a+b-y}^{a+b-x} F(w)\, {}^{(a+b-x)}d_q^T w \\ &\geq F\left(a+b-\frac{x+y}{2}\right) \end{aligned} \tag{3.10}$$

which proves the first inequality in (3.9). By Jensen-Mercer inequality, we have

$$\begin{aligned} &\frac{1}{y-x}\int_{a+b-y}^{a+b-x} F(w)\, {}^{(a+b-x)}d_q^T w \\ &= \int_0^1 F(a+b-(ty+(1-t)x))\,{}_0d_q^T t \\ &\leq F(a)+F(b)-F(y)\int_0^1 t\,{}_0d_q^T t - F(x)\int_0^1 (1-t)\,{}_0d_q^T t \\ &= F(a)+F(b)-\frac{F(x)+F(y)}{2}. \end{aligned} \tag{3.11}$$

Combining the inequalities (3.10) and (3.11), we get required result. □

Remark 4. By taking the limit $q \to 1$ in Theorem 8, then inequalities (3.9) reduce to inequalities (3.8).

Remark 5. If we assign $x = a$ and $y = b$ in Theorem 8, then Theorem 8 reduces to Theorem 5.

4. Examples

In this section,we present some examples which illustrates the validity of our new results.

Example 1. Define the the convex function $F(x)=x^2$ on $[0,2]$. By considering $q=\frac{1}{2}$, $x=0$, $y=1$ in Theorem 6, we have

$$F\left(a+b-\frac{x+y}{2}\right)=F\left(\frac{3}{2}\right)=\frac{9}{4},$$

$$\begin{aligned} F(a)+F(b)-\frac{1}{2(y-x)}\left[\int_x^y F(w)\,{}_x d_q^T w+\int_x^y F(w)\,{}^y d_q^T w\right] \\ =4-\frac{1}{2}\left[\int_0^1 w^2\,{}_0 d_q^T w+\int_0^1 w^2\,{}^1 d_q^T w\right]=4-\left[\frac{5}{28}+\frac{5}{28}\right]=\frac{51}{14}\end{aligned}$$

and

$$F(a)+F(b)-F\left(\frac{x+y}{2}\right)=4-\frac{1}{4}=\frac{15}{4}$$

Thus, the inequalities (3.1) are verified.

Example 2. Let consider the convex function $F(x)=x^2$ on $[-\frac{3}{2},2]$. If $q=\frac{1}{2}$, $x=-\frac{1}{2}$, $y=\frac{1}{2}$ in (3.5), then we have

$$F\left(a+b-\frac{x+y}{2}\right)=F\left(\frac{1}{2}\right)=\frac{1}{4},$$

$$\frac{1}{y-x}\int_{a+b-y}^{a+b-x} F(w)\,{}_{(a+b-y)}d_q^T w=\int_0^1 w^2\,{}_0 d_{\frac{1}{2}}^T w=\frac{5}{14},$$

and

$$F(a)+F(b)-\frac{F(x)+F(y)}{2}=\frac{25}{4}-\frac{1}{4}=6.$$

Thus, the inequalities (3.5) are verified.

Example 3. If we apply the Theorem 8 to the convex function $F(x)=x^2$ on $[0,2]$ for $q=\frac{1}{2}$, $x=0$, $y=1$, we have

$$F\left(a+b-\frac{x+y}{2}\right)=F\left(\frac{3}{2}\right)=\frac{9}{4},$$

$$\frac{1}{y-x}\int_{a+b-y}^{a+b-x} F(w)\,{}^{(a+b-x)}d_q^T w=\int_1^2 w^2\,{}^2 d_{\frac{1}{2}}^T w=\frac{33}{14},$$

and

$$F(a)+F(b)-\frac{F(x)+F(y)}{2}=4-\frac{1}{2}=\frac{7}{2}.$$

Thus, the inequalities (3.9) are verified.

5. Conclusion

In the present work,we proved some new Quantum Hermite-Jensen-Mercer inequalities by using the innovative idea of T_q integrals.Some newly established inequalities could be turned into previous results of literature.It is a very new and interesting inequalities for which researcher can obtain similar inequalities for other kind of convexities in their new work.

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