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FRACTIONAL HERMITE-HADAMARD INEQUALITY AND ERROR ESTIMATES FOR SIMPSON'S FORMULA THROUGH CONVEXITY WITH RESPECT TO A PAIR OF FUNCTIONS

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Abstract. In this article, we establish two new and different versions of fractional Hermite-Hadamard type inequality for the convex functions with respect to a pair of functions. Moreover, we establish a new Simpson's type inequalities for differentiable convex functions with respect to a pair of functions. We also prove two more Simpson's type inequalities for differentiable convex functions with respect to a pair of functions using the power mean and Hölder's inequalities. It is also shown that the newly established inequalities are the extension of some existing results. Finally, we add some mathematical examples and their graphs to show the validity of newly established results.

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1. INTRODUCTION

The interesting concept of inequalities has long been a source of debate among mathematicians. Only a few of the exciting applications include fractional calculus, quantum calculus, operator theory, numerical analysis, operator equations, network theory, and quantum information theory. Right now, this is a very dynamic research topic and the interaction between diverse fields have enriched it. Applied sciences need the use of numerical integration and definite integral estimate. Simpson's rules, which can be stated as follows, are fundamental among the numerical techniques:

(i) Simpson's $1/3$ rule

$$\int_{\omega_1}^{\omega_2} \mathfrak{F}(\varkappa)\, d\varkappa \approx \frac{\omega_2-\omega_1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right].$$

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Due to their importance in approximation theory, researchers have employed fractional calculus to establish many fractional integral inequalities. Examples of inequalities that can be used to discover the bounds of numerical integration formulas include Hermite-Hadamard, Simpson's, midpoint, Ostrowski's, and trapezoidal inequalities. The authors of [13] established the boundaries of the trapezoidal formula and inequality of the Hermite-Hadamard type using Riemann-Liouville fractional integrals. Set [15] derived fractional Ostrowski's type inequalities using differentiable convexity. Through Riemann-Liouville fractional integrals, İşcan and Wu [7] derived some bounds of numerical integration and inequality of Hermite-Hadamard type for reciprocal convex functions. Sarikaya and Yildrim established midpoint bounds and a new variant of fractional inequality of Hermite-Hadamard type in [14]. Sarikaya et al. [12] used general convexity and Riemann-Liouville fractional integral operators to find estimates for Simpson's $1/3$ formula. Using the Riemann-Liouville fractional integrals, the authors derived some new boundaries for Simpson's $1/3$ formula in [9]. An s-convexity was utilized by the authors of [3] to find various estimates for Simpson's $1/3$ formula. Sarikaya and Ertugral [11] introduced a new class of fractional integrals known as generalized fractional integrals in 2020, as well as Hermite-Hadamard type inequalities linked to the newly defined class of integrals. Zhao et al. used reciprocal convex functions and generalized fractional integral operators to establish various estimates for a trapezoidal formula in [18]. Budak et al. [2] used generalized fractional integrals to establish various estimates for Simpson's $1/3$ formula for differentiable convex functions.

Sitthiwirattham et al. [17] have discovered some estimates for Simpson's 3/8 formula using the Riemann-Liouville fractional integrals operators. For other inequalities involving fractional integral operators, see [1,6,16,19] and the references therein.

Motivated by the above-mentioned literature and applications of fractional inequalities, we use Riemann-Liouville fractional integrals to derive Hermite-Hadamard inequality for convex functions with respect to a pair of functions. In addition, we prove some important Simpson's type inequalities for differentiable convex functions with respect to a pair of functions. It is also shown that the results established in the paper are the generalization of the results proved in [4,10,11].

2. Fractional integrals and related inequalities

In this section, we review several fundamental fractional integral notations and concepts. We also address inequalities and various types of convexity. We use the following notations throughout the work for the sake of brevity:

- A convex subset of a real vector space X is denoted by C.
- I denotes an interval of $\mathbb{R}$.
- I° denotes the interior of I.

Definition 1. A mapping $\mathfrak{F} : C \to \mathbb{R}$ is called convex if

$$\mathfrak{F}(\tau\varkappa + (1-\tau)\gamma) \le \tau\mathfrak{F}(\varkappa) + (1-\tau)\mathfrak{F}(\gamma),$$

where $\varkappa, \gamma \in C$ and $\tau \in [0,1]$.

The Hermite-Hadamard inequality (double inequality) is basic in the convex functions theory, providing a two-sided estimate of the (integral) the mean value of a convex function. In other words, if $\mathfrak{F} : I \to \mathbb{R}$ is convex with $\omega_1, \omega_2 \in I$ and $\omega_1 < \omega_2$, then

$$\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) \le \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} \mathfrak{F}(\varkappa)\,d\varkappa \le \frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2}. \tag{2.1}$$

B. Samet defined a new class of convex functions with respect to a pair of functions in [10], as follows:

Definition 2. Let $g, h : C \to \mathbb{R}$ be two mappings. A mapping $\mathfrak{F} : C \to \mathbb{R}$ is called (g,h)-convex, if the following inequality holds for $M(\varkappa,\gamma) = g(\varkappa)h(\gamma) + g(\gamma)h(\varkappa)$:

$$\mathfrak{F}(\tau\varkappa + (1-\tau)\gamma) \le \tau^2\mathfrak{F}(\varkappa) + (1-\tau)^2\mathfrak{F}(\gamma) + \tau(1-\tau)M(\varkappa,\gamma) \tag{2.2}$$

for all $\tau \in [0,1]$ and $\varkappa, \gamma \in C$.

B. Samet also established the Hermite-Hadamard inequality (double inequality) for a new class of convex functions in terms of a pair of functions, which he expressed as:

Theorem 1 ([10]). *Let $\mathfrak{F} : I \subset \mathbb{R} \to \mathbb{R}$ be a (g,h)-convex function over $[\omega_1,\omega_2]$ with $g,h : [\omega_1,\omega_2] \to \mathbb{R}$. If $\mathfrak{F} \in L^1(\omega_1,\omega_2)$ and $g,h \in L^2(\omega_1,\omega_2)$, then we have the following fractional Hermite-Hadamard inequality:*

$$\begin{aligned}
&2\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} g(\omega_1+\omega_2-\varkappa)\,h(\varkappa)\,d\varkappa \\
&= 2\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} g(\varkappa)\,h(\omega_1+\omega_2-\varkappa)\,d\varkappa \\
&\le \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} \mathfrak{F}(\varkappa)\,d\varkappa \\
&\le \frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{3} + \frac{M(\omega_1,\omega_2)}{6}.
\end{aligned} \tag{2.3}$$

Definition 3 ([5,8]). Let $\mathfrak{F} \in L_1[\omega_1,\omega_2]$. The Riemann-Liouville fractional integrals (RLFIs) $J^{\alpha}_{\omega_1+}\mathfrak{F}$ and $J^{\alpha}_{\omega_2-}\mathfrak{F}$ of order $\alpha > 0$ with $\omega_1 \ge 0$ are defined as follows:

$$J^{\alpha}_{\omega_1+}\mathfrak{F}(\varkappa) = \frac{1}{\Gamma(\omega_1)}\int_{\omega_1}^{\varkappa} (\varkappa-\tau)^{\alpha-1}\mathfrak{F}(\tau)\,d\tau, \ \varkappa > \omega_1$$

and

$$J^{\alpha}_{\omega_2-}\mathfrak{F}(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{\omega_2} (\tau-\varkappa)^{\alpha-1}\mathfrak{F}(\tau)\,d\tau, \ \varkappa < \omega_2,$$

respectively, where Γ is the well-known Gamma function.

For the first time in 2013, Sarikaya et al. established the following fractional Hermite-Hadmard type inequality:

Theorem 2 ([13]). *For a positive convex function $\mathfrak{F}: I\subset\mathbb{R}\to\mathbb{R}$ with $\mathfrak{F}\in L_1[\omega_1,\omega_2]$ and $0\le\omega_1<\omega_2$, the following inequality holds:*

$$\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}(\omega_2)+J^{\alpha}_{\omega_2-}\mathfrak{F}(\omega_1)\right]\le\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2}.$$

The following modified version of the fractional Hermite-Hadamard inequality was then established by Sarikaya and Yildirim:

Theorem 3 ([14]). *For a positive convex function $\mathfrak{F}: I\subset\mathbb{R}\to\mathbb{R}$ with $\mathfrak{F}\in L_1[\omega_1,\omega_2]$, $0\le\omega_1<\omega_2$ and $\omega_1,\omega_2\in I$, the following inequality holds:*

$$\begin{aligned}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)&\le\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\omega_1+\omega_2}{2}\right)+}\mathfrak{F}(\omega_2)+J^{\alpha}_{\left(\frac{\omega_1+\omega_2}{2}\right)-}\mathfrak{F}(\omega_1)\right]\\&\le\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2}.\end{aligned}$$

Remark 1. If we set $\alpha=1$ in inequalities (2.1) and (2.2), then we obtain the traditional Hermite-Hadamard inequality:

$$\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\le\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(\varkappa)\,d\varkappa\le\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2}.$$

3. HERMITE-HADAMARD INEQUALITY

Using the Riemann-Liouville fractional integrals, we prove two new Hermite-Hadamard inequalities for (g,h)-convex functions in this section.

Theorem 4. *Let $\mathfrak{F}: I\subset\mathbb{R}\to\mathbb{R}$ be a (g,h)-convex function on $[\omega_1,\omega_2]$ with $g,h:[\omega_1,\omega_2]\to\mathbb{R}$. If $\mathfrak{F}\in L^1(\omega_1,\omega_2)$ and $g,h\in L^2(\omega_1,\omega_2)$, then we have the following fractional Hermite-Hadamard inequality:*

$$\begin{aligned}&2\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\Omega\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\Omega\left(\frac{\omega_1+\omega_2}{2}\right)\right]\\&\le\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\qquad(3.1)\\&\le\frac{[\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)](\alpha+1)}{2(\alpha+2)}+\frac{M(\omega_1,\omega_2)}{2(\alpha+2)},\end{aligned}$$

where $\Omega(\varkappa)=g(\varkappa)h(\omega_1+\omega_2-\varkappa)$.

Proof. Because (g,h)-convexity of $\mathfrak{F}$, we have

$$4\mathfrak{F}\left(\frac{\varkappa+\gamma}{2}\right)\le\mathfrak{F}(\varkappa)+\mathfrak{F}(\gamma)+h(\varkappa)g(\gamma)+h(\gamma)g(\varkappa).$$

By letting $\varkappa = \frac{1-\tau}{2}\omega_1 + \frac{1+\tau}{2}\omega_2$ and $\gamma = \frac{1+\tau}{2}\omega_2 + \frac{1-\tau}{2}\omega_1$, we have

$$\begin{aligned} 4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) &\leq \mathfrak{F}\left(\frac{1-\tau}{2}\omega_1 + \frac{1+\tau}{2}\omega_2\right) + \mathfrak{F}\left(\frac{1+\tau}{2}\omega_2 + \frac{1-\tau}{2}\omega_1\right) \\ &\quad + h\left(\frac{1-\tau}{2}\omega_1 + \frac{1+\tau}{2}\omega_2\right) g\left(\frac{1+\tau}{2}\omega_2 + \frac{1-\tau}{2}\omega_1\right) \\ &\quad + h\left(\frac{1+\tau}{2}\omega_2 + \frac{1-\tau}{2}\omega_1\right) g\left(\frac{1-\tau}{2}\omega_1 + \frac{1+\tau}{2}\omega_2\right). \end{aligned} \tag{3.2}$$

Multiplying the last inequality by $\tau^{\alpha-1}$ and then integrating with respect to t over $[0,1]$, we have

$$\begin{aligned} \frac{4}{\alpha}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) &\leq \frac{2^\alpha\Gamma(\alpha)}{(\omega_2-\omega_1)^\alpha}\left[J^\alpha_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) + J^\alpha_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right] \\ &\quad + \frac{2^\alpha\Gamma(\alpha)}{(\omega_2-\omega_1)^\alpha}\left[J^\alpha_{\omega_1+}\Omega\left(\frac{\omega_1+\omega_2}{2}\right) + J^\alpha_{\omega_2-}\Omega\left(\frac{\omega_1+\omega_2}{2}\right)\right], \end{aligned}$$

which implies that

$$\begin{aligned} &2\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^\alpha}\left[J^\alpha_{\omega_1+}\Omega\left(\frac{\omega_1+\omega_2}{2}\right) + J^\alpha_{\omega_2-}\Omega\left(\frac{\omega_1+\omega_2}{2}\right)\right] \\ &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^\alpha}\left[J^\alpha_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) + J^\alpha_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]. \end{aligned}$$

As a result, the first inequality in (3.1) is established. We again use the (g,h)-convexity to prove the second inequality as follows:

$$\mathfrak{F}\left(\frac{1-\tau}{2}\omega_1 + \frac{1+\tau}{2}\omega_2\right) \leq \frac{(1-\tau)^2}{4}\mathfrak{F}(\omega_1) + \frac{(1+\tau)^2}{4}\mathfrak{F}(\omega_2) + \frac{1-\tau^2}{4}M(\omega_1,\omega_2) \tag{3.3}$$

and

$$\mathfrak{F}\left(\frac{1+\tau}{2}\omega_1 + \frac{1-\tau}{2}\omega_2\right) \leq \frac{(1+\tau)^2}{4}\mathfrak{F}(\omega_1) + \frac{(1-\tau)^2}{4}\mathfrak{F}(\omega_2) + \frac{1-\tau^2}{4}M(\omega_1,\omega_2). \tag{3.4}$$

After multiplying the inequalities (3.3) and (3.4) with $\tau^{\alpha-1}$, we add them and integrate the resultant one with respect to t over $[0,1]$, we have

$$\begin{aligned} &\int_0^1 \tau^{\alpha-1}\mathfrak{F}\left(\frac{1-\tau}{2}\omega_1 + \frac{1+\tau}{2}\omega_2\right)d\tau + \int_0^1 \tau^{\alpha-1}\mathfrak{F}\left(\frac{1+\tau}{2}\omega_1 + \frac{1-\tau}{2}\omega_2\right)d\tau \\ &\leq \frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{4}\int_0^1 \tau^{\alpha-1}\left((1+\tau)^2 + (1-\tau)^2\right)d\tau \\ &\quad + \frac{M(\omega_1,\omega_2)}{2}\int_0^1 \tau^{\alpha-1}\left(1-\tau^2\right)d\tau. \end{aligned}$$

Thus, the required inequality is obtained with simple computations and changes in the integration variables. □

Remark 2. By setting $\alpha = 1$ in Theorem 4, we recapture the inequality presented in Theorem 1.

Remark 3. If we choose $h(\varkappa) = \mathfrak{F}(\varkappa)$ and $g(\varkappa) = 1$ or $h(\varkappa) = 1$ and $g(\varkappa) = \mathfrak{F}(\varkappa)$ in Theorem 4, then we obtain the Hermite-Hadamrd inequality given in Theorem 3.

Example 1. Let $[\omega_1, \omega_2] = [0,1]$ and define the functions $\mathfrak{F}, g, h : [0,1] \to \mathbb{R}$ by $\mathfrak{F}(\varkappa) = \varkappa^3$, $g(\varkappa) = \varkappa$ and $h(\varkappa) = \varkappa^2$ such that $\mathfrak{F}$ is (g,h)-convex on $[0,1]$. Under these assumptions, we have

$$
\begin{aligned}
&2\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\Omega\left(\frac{\omega_1+\omega_2}{2}\right) + J^{\alpha}_{\omega_2-}\Omega\left(\frac{\omega_1+\omega_2}{2}\right)\right]\\
&= \frac{1}{4} - \frac{\alpha}{2^{1-\alpha}}\left[\int_0^{\frac{1}{2}}\left(\frac{1}{2}-\varkappa\right)^{\alpha-1}\varkappa(1-\varkappa)^2\,d\varkappa + \int_{\frac{1}{2}}^{1}\left(\varkappa-\frac{1}{2}\right)^{\alpha-1}\varkappa(1-\varkappa)^2\,d\varkappa\right]\\
&= \frac{1}{4} - \frac{\alpha}{2^{1-\alpha}}\left[\frac{2\alpha^2+6\alpha+3}{2^{\alpha+2}\alpha(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{2\alpha+3}{2^{\alpha+2}\alpha(\alpha+1)(\alpha+2)(\alpha+3)}\right]\\
&= \frac{\alpha+1}{4(\alpha+2)},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) + J^{\alpha}_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\\
&= \frac{\alpha}{2^{1-\alpha}}\left[\int_0^{\frac{1}{2}}\left(\frac{1}{2}-\varkappa\right)^{\alpha-1}\varkappa^3\,d\varkappa + \int_{\frac{1}{2}}^{1}\left(\varkappa-\frac{1}{2}\right)^{\alpha-1}\varkappa^3\,d\varkappa\right]\\
&= \frac{\alpha}{2^{1-\alpha}}\left[\frac{3}{2^{\alpha+2}\alpha(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{(2\alpha+3)\left(2\alpha^2+6\alpha+3\right)}{2^{\alpha+2}\alpha(\alpha+1)(\alpha+2)(\alpha+3)}\right]\\
&= \frac{(2\alpha+3)\left(2\alpha^2+6\alpha+3\right)+3}{8(\alpha+1)(\alpha+2)(\alpha+3)},
\end{aligned}
$$

and

$$
\frac{[\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)](\alpha+1)}{2(\alpha+2)} + \frac{M(\omega_1,\omega_2)}{2(\alpha+2)} = \frac{\alpha+1}{2(\alpha+2)}.
$$

We use MATLAB software to plot the function image of the above three functions, as shown in Figure 1. From the position relationship of the image, we can see that the inequalities relationship in Theorem 4 is tenable.

Theorem 5. *Let $\mathfrak{F} : I \subset \mathbb{R} \to \mathbb{R}$ be a (g,h)-convex function on $[\omega_1, \omega_2]$ with $g, h : [\omega_1, \omega_2] \to \mathbb{R}$. If $\mathfrak{F} \in L^1(\omega_1, \omega_2)$ and $g, h \in L^2(\omega_1, \omega_2)$, then we have the following*

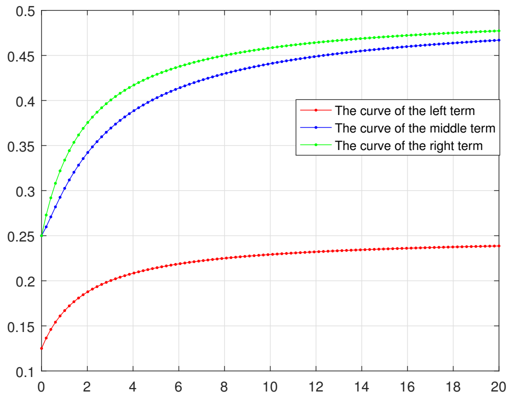

FIGURE 1. The image description for Theorem 4.

fractional Hermite-Hadamard inequality:

$$\begin{aligned} & 2\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\frac{\omega_1+\omega_2}{2}+}\Omega(\omega_2)+J^{\alpha}_{\frac{\omega_1+\omega_2}{2}-}\Omega(\omega_1)\right] \\ & \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\frac{\omega_1+\omega_2}{2}+}\mathfrak{F}(\omega_2)+J^{\alpha}_{\frac{\omega_1+\omega_2}{2}-}\mathfrak{F}(\omega_1)\right] \\ & \leq \frac{[\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)]}{4}\left[2-\frac{\alpha(\alpha+3)}{(\alpha+1)(\alpha+2)}\right]+\frac{M(\omega_1,\omega_2)\alpha(\alpha+3)}{4(\alpha+1)(\alpha+2)}, \end{aligned} \tag{3.5}$$

where $\Omega(\varkappa)=g(\varkappa)h(\omega_1+\omega_2-\varkappa)$.

Proof. If we follow the steps used in the last theorem for $\varkappa=\frac{2-\tau}{2}\omega_1+\frac{\tau}{2}\omega_2$ and $\gamma=\frac{\tau}{2}\omega_1+\frac{2-\tau}{2}\omega_2$, then one can obtain the desired inequality. □

Remark 4. By setting $\alpha=1$ in Theorem 5, we recapture the inequality presented in Theorem 1.

Remark 5. If we choose $h(\varkappa)=\mathfrak{F}(\varkappa)$ and $g(\varkappa)=1$ or $h(\varkappa)=1$ and $g(\varkappa)=\mathfrak{F}(\varkappa)$ in Theorem 5, then we have the following Hermite-Hadamrd inequality proved by Dragomir et al. in [4]:

$$\begin{aligned} \mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) & \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right] \\ & \leq \frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2}. \end{aligned}$$

Example 2. Let $[\omega_1,\omega_2]=[0,1]$ and define the functions $\mathfrak{F},g,h:[0,1]\to\mathbb{R}$ by $\mathfrak{F}(\varkappa)=\varkappa^3$, $g(\varkappa)=\varkappa$ and $h(\varkappa)=\varkappa^2$ such that $\mathfrak{F}$ is (g,h)-convex on $[0,1]$. Under these assumptions, we have

$$
\begin{aligned}
&2\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\frac{\omega_1+\omega_2}{2}+}\Omega(\omega_2)+J^{\alpha}_{\frac{\omega_1+\omega_2}{2}-}\Omega(\omega_1)\right]\\
&=\frac{1}{4}-\frac{\alpha}{2^{1-\alpha}}\left[\int_{\frac{1}{2}}^{1}(1-\varkappa)^{\alpha-1}\varkappa(1-\varkappa)^{2}d\varkappa+\int_{0}^{\frac{1}{2}}\varkappa^{\alpha-1}\varkappa(1-\varkappa)^{2}d\varkappa\right]\\
&=\frac{1}{4}-\frac{\alpha}{2^{1-\alpha}}\left[\frac{\alpha+4}{2^{\alpha+3}(\alpha+2)(\alpha+3)}+\frac{\alpha^{2}+7\alpha+14}{2^{\alpha+3}(\alpha+1)(\alpha+2)(\alpha+3)}\right]\\
&=\frac{1}{4}-\frac{\alpha}{16}\frac{2\alpha^{2}+12\alpha+18}{(\alpha+1)(\alpha+2)(\alpha+3)}\\
&=\frac{1}{4}-\frac{\alpha(\alpha+3)}{8(\alpha+1)(\alpha+2)},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\frac{\omega_1+\omega_2}{2}+}\mathfrak{F}(\omega_2)+J^{\alpha}_{\frac{\omega_1+\omega_2}{2}-}\mathfrak{F}(\omega_1)\right]\\
&=\frac{\alpha}{2^{1-\alpha}}\left[\int_{\frac{1}{2}}^{1}(1-\varkappa)^{\alpha-1}\varkappa^{3}d\varkappa+\int_{0}^{\frac{1}{2}}\varkappa^{\alpha-1}\varkappa^{3}d\varkappa\right]\\
&=\frac{\alpha}{2^{1-\alpha}}\left[\frac{(\alpha+4)\left(\alpha^{2}+5\alpha+12\right)}{2^{\alpha+3}\alpha(\alpha+1)(\alpha+2)(\alpha+3)}+\frac{1}{2^{\alpha+3}(\alpha+3)}\right]\\
&=\frac{\alpha^{3}+6\alpha^{2}+17\alpha+24}{8(\alpha+1)(\alpha+2)(\alpha+3)},
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{[\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)]}{4}\left[2-\frac{\alpha(\alpha+3)}{(\alpha+1)(\alpha+2)}\right]+\frac{M(\omega_1,\omega_2)\alpha(\alpha+3)}{4(\alpha+1)(\alpha+2)}\\
&=\frac{1}{4}\left[2-\frac{\alpha(\alpha+3)}{(\alpha+1)(\alpha+2)}\right].
\end{aligned}
$$

We use MATLAB software to plot the function image of the above three functions, as shown in Figure 2. From the position relationship of the image, we can see that the inequalities relationship in Theorem 5 is tenable.

4. SIMPSON'S TYPE INEQUALITIES

We prove new Simpson's type inequalities for differentiable convex functions with respect to a pair of functions in this section. The following result and notations will be used to prove the inequalities in this section that are dependent on $\varpi=\left(\frac{2}{3}\right)^{\frac{1}{\alpha}}$:

$$
A_1(\varpi,\alpha)=\int_0^1\left|\frac{\tau^{\alpha}}{2}-\frac{1}{3}\right|d\tau
$$

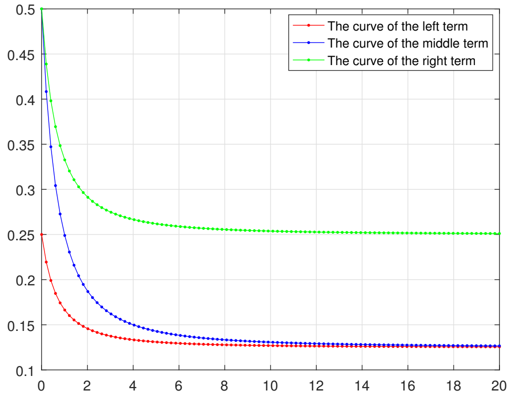

FIGURE 2. The image description for Theorem 5.

$$\begin{aligned}
&= \int_0^{\varpi} \left(\frac{1}{3} - \frac{\tau^{\alpha}}{2}\right) d\tau + \int_{\varpi}^{1} \left(\frac{\tau^{\alpha}}{2} - \frac{1}{3}\right) d\tau \\
&= \frac{2}{3}\varpi - \frac{\varpi^{\alpha+1}}{\alpha+1} + \frac{1}{2(\alpha+1)} - \frac{1}{3},
\end{aligned}$$

$$\begin{aligned}
A_2(\varpi,\alpha) &= \int_0^1 \tau \left|\frac{\tau^{\alpha}}{2} - \frac{1}{3}\right| d\tau \\
&= \int_0^{\varpi} \tau \left(\frac{1}{3} - \frac{\tau^{\alpha}}{2}\right) d\tau + \int_{\varpi}^{1} \tau \left(\frac{\tau^{\alpha}}{2} - \frac{1}{3}\right) d\tau \\
&= \frac{\varpi^2}{3} - \frac{\varpi^{\alpha+2}}{\alpha+2} + \frac{1}{2(\alpha+2)} - \frac{1}{6},
\end{aligned}$$

$$\begin{aligned}
A_3(\varpi,\alpha) &= \int_0^1 \tau \left|\frac{\tau^{\alpha}}{2} - \frac{1}{3}\right| d\tau \\
&= \int_0^{\varpi} \tau^2 \left(\frac{1}{3} - \frac{\tau^{\alpha}}{2}\right) d\tau + \int_{\varpi}^{1} \tau^2 \left(\frac{\tau^{\alpha}}{2} - \frac{1}{3}\right) d\tau \\
&= \frac{2}{9}\varpi^3 - \frac{\varpi^{\alpha+3}}{\alpha+3} + \frac{1}{2(\alpha+3)} - \frac{1}{9}.
\end{aligned}$$

Lemma 1 ([3]). *Let $\mathfrak{F} : [\omega_1, \omega_2] \to \mathbb{R}$ be a differentiable mapping over (ω_1, ω_2) and $\mathfrak{F}' \in L^1[\omega_1, \omega_2]$ with $\omega_1 < \omega_2$. Then, we have the following equality:*

$$\frac{1}{6}\left[\mathfrak{F}(\omega_1) + 4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) + \mathfrak{F}(\omega_2)\right]$$

$$-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right] \tag{4.1}$$
$$=\frac{(\omega_2-\omega_1)}{2}\left[\int_0^1\left(\frac{\tau^{\alpha}}{2}-\frac{1}{3}\right)\left[\begin{array}{c}\mathfrak{F}'\left(\frac{1+\tau}{2}\omega_2+\frac{1-\tau}{2}\omega_1\right)\\ -\mathfrak{F}'\left(\frac{1-\tau}{2}\omega_2+\frac{1+\tau}{2}\omega_1\right)\end{array}\right]d\tau\right].$$

Theorem 6. *In addition of Lemma 1, if $|\mathfrak{F}'|$ is (g,h)-mapping, then we have the following new fractional Simpson's inequality:*

$$\begin{aligned}&\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\right|\\&\leq\frac{\omega_2-\omega_1}{2}\left[\frac{|\mathfrak{F}'(\omega_1)|+|\mathfrak{F}'(\omega_2)|}{2}\left(A_1(\varpi,\alpha)+A_3(\varpi,\alpha)\right)\right.\\&\left.+\frac{M(\omega_1,\omega_2)}{2}\left(A_1(\varpi,\alpha)-A_3(\varpi,\alpha)\right)\right].\end{aligned} \tag{4.2}$$

Proof. From (4.1) and modulus properties, we have

$$\begin{aligned}&\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\right|\\&\leq\frac{(\omega_2-\omega_1)}{2}\left[\int_0^1\left|\frac{\tau^{\alpha}}{2}-\frac{1}{3}\right|\left[\begin{array}{c}\left|\mathfrak{F}'\left(\frac{1+\tau}{2}\omega_2+\frac{1-\tau}{2}\omega_1\right)\right|\\ +\left|\mathfrak{F}'\left(\frac{1-\tau}{2}\omega_2+\frac{1+\tau}{2}\omega_1\right)\right|\end{array}\right]d\tau\right].\end{aligned} \tag{4.3}$$

Using the (g,h)-convexity, we get

$$\begin{aligned}&\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\right|\\&\leq\frac{(\omega_2-\omega_1)}{2}\left[\int_0^1\left|\frac{\tau^{\alpha}}{2}-\frac{1}{3}\right|\left[\begin{array}{c}\frac{1+\tau^2+2\tau}{4}|\mathfrak{F}'(\omega_2)|\\ +\frac{1+\tau^2-2\tau}{4}|\mathfrak{F}'(\omega_1)|+\frac{1-\tau^2}{4}M(\omega_1,\omega_2)\end{array}\right]d\tau\right.\\&\left.+\int_0^1\left|\frac{\tau^{\alpha}}{2}-\frac{1}{3}\right|\left[\begin{array}{c}\frac{1+\tau^2+2\tau}{4}|\mathfrak{F}'(\omega_1)|\\ +\frac{1+\tau^2-2\tau}{4}|\mathfrak{F}'(\omega_2)|+\frac{1-\tau^2}{4}M(\omega_1,\omega_2)\end{array}\right]d\tau\right]\end{aligned}$$

$$= \frac{\omega_2-\omega_1}{2}\int_0^1 \left|\frac{\tau^\alpha}{2}-\frac{1}{3}\right|\left[\frac{|\mathfrak{F}'(\omega_1)|+|\mathfrak{F}'(\omega_2)|}{2}\left(1+\tau^2\right)+\frac{M(\omega_1,\omega_2)}{2}\left(1-\tau^2\right)\right]d\tau$$

$$= \frac{\omega_2-\omega_1}{2}\left[\frac{|\mathfrak{F}'(\omega_1)|+|\mathfrak{F}'(\omega_2)|}{2}\left(A_1(\varpi,\alpha)+A_3(\varpi,\alpha)\right)\right.$$
$$\left.+\frac{M(\omega_1,\omega_2)}{2}\left(A_1(\varpi,\alpha)-A_3(\varpi,\alpha)\right)\right].$$

Thus, the proof is finished. □

Remark 6. In Theorem 6, if we set $\alpha = 1$, then we recapture the following Simpson's inequality:

$$\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(\varkappa)\,d\varkappa\right|$$
$$\leq (\omega_2-\omega_1)\left[\frac{329}{7776}\left(\left|\mathfrak{F}'(\omega_1)\right|+\left|\mathfrak{F}'(\omega_2)\right|\right)+\frac{211}{7776}M(\omega_1,\omega_2)\right],$$

which is established by Samet in [10, Theorem 3.3].

Corollary 1. *If we choose $h(\varkappa) = |\mathfrak{F}'(\varkappa)|$ and $g(\varkappa) = 1$ or $h(\varkappa) = 1$ and $g(\varkappa) = |\mathfrak{F}'(\varkappa)|$ in Theorem 6, then we have the following inequality*

$$\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^\alpha}\left[J^\alpha_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^\alpha_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\right|$$
$$\leq \frac{\omega_2-\omega_1}{2}A_1(\varpi,\alpha)\left[\left|\mathfrak{F}'(\omega_1)\right|+\left|\mathfrak{F}'(\omega_2)\right|\right].$$

Remark 7. If we take $\alpha = 1$ in Corollary 1, then we have the following Simpson inequality

$$\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(\varkappa)\,d\varkappa\right|$$
$$\leq \frac{5(\omega_2-\omega_1)}{72}\left(\left|\mathfrak{F}'(\omega_1)\right|+\left|\mathfrak{F}'(\omega_2)\right|\right),$$

which is proved by Sarikaya et al. in [11].

Example 3. Let $[\omega_1,\omega_2] = [0,1]$ and define the functions $\mathfrak{F}, g, h : [0,1] \to \mathbb{R}$ by $\mathfrak{F}(\varkappa) = \frac{\varkappa^4}{4}$, $g(\varkappa) = \varkappa$ and $h(\varkappa) = \varkappa^2$ such that $\mathfrak{F}'(\varkappa) = \varkappa^3$ and $|\mathfrak{F}'|$ is (g,h)-convex on $[0,1]$. Under these assumptions, we have

$$\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]\right.$$

$$
\begin{aligned}
&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\Bigg| \\
&=\left|\frac{5}{96}-\frac{\alpha}{2^{3-\alpha}}\left[\int_0^{\frac{1}{2}}\left(\frac{1}{2}-\varkappa\right)^{\alpha-1}\varkappa^4 d\varkappa+\int_{\frac{1}{2}}^{1}\left(\varkappa-\frac{1}{2}\right)^{\alpha-1}\varkappa^4 d\varkappa\right]\right| \\
&=\left|\frac{5}{96}-\frac{\alpha}{2^{3-\alpha}}\left[\begin{array}{c}\frac{3}{2^{\alpha+1}(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} \\ +\frac{2\alpha(\alpha+4)(\alpha+2)^2+3}{2^{\alpha+1}\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}\end{array}\right]\right| \\
&=\left|\frac{5}{96}-\frac{\alpha(\alpha+4)(\alpha+2)^2+3}{8(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}\right|
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\omega_2-\omega_1}{2}\left[\frac{|\mathfrak{F}'(\omega_1)|+|\mathfrak{F}'(\omega_2)|}{2}\left(A_1(\varpi,\alpha)+A_3(\varpi,\alpha)\right)\right. \\
&\left.+\frac{M(\omega_1,\omega_2)}{2}\left(A_1(\varpi,\alpha)-A_3(\varpi,\alpha)\right)\right] \\
&=\frac{\left(A_1(\varpi,\alpha)+A_3(\varpi,\alpha)\right)}{4}.
\end{aligned}
$$

We use MATLAB software to plot the function image of the above three functions,

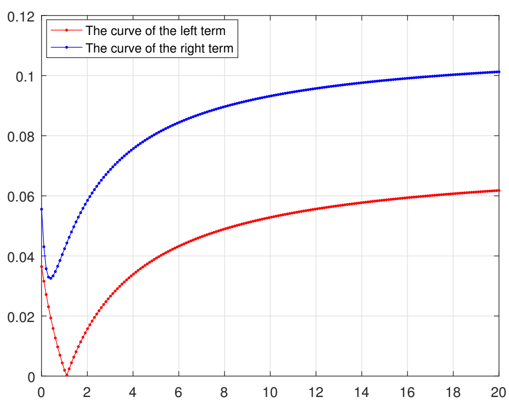

FIGURE 3. The image description for Theorem 6.

as shown in Figure 3. From the position relationship of the image, we can see that the inequalities relationship in Theorem 6 is tenable.

Theorem 7. *We assume that the conditions of Lemma 1. If* $|\mathfrak{F}'|^q$, $q>1$ *is* (g,h)*-mapping, then we have the following new fractional Simpson's inequality:*

$$\begin{aligned}&\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]\right.\\&\quad\left.-\frac{\Gamma(\alpha+1)}{2(\omega_2-\omega_1)^\alpha}\left[J^\alpha_{\omega_1+}\mathfrak{F}(\omega_2)+J^\alpha_{\omega_2-}\mathfrak{F}(\omega_1)\right]\right|\\&\leq\frac{(\omega_2-\omega_1)}{2}\Theta_1^{\frac{1}{p}}(\alpha,p)\times\left[\left(\frac{|\mathfrak{F}'(\omega_1)|^q+7|\mathfrak{F}'(\omega_2)|^q+2M(\omega_1,\omega_2)}{12}\right)^{\frac{1}{q}}\right.\\&\quad\left.+\frac{7|\mathfrak{F}'(\omega_1)|^q+|\mathfrak{F}'(\omega_2)|^q+2M(\omega_1,\omega_2)}{12}\right],\end{aligned}$$

where

$$\Theta_1(\alpha,p)=\int_0^1\left|\frac{\tau^\alpha}{2}-\frac{1}{3}\right|^p d\tau$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Taking modulus in (4.1) and using Hölder's inequality, we have

$$\begin{aligned}&\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]\right.\\&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^\alpha}\left[J^\alpha_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^\alpha_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\right|\\&\leq\frac{(\omega_2-\omega_1)}{2}\left[\int_0^1\left|\frac{\tau^\alpha}{2}-\frac{1}{3}\right|\left[\begin{array}{c}\left|\mathfrak{F}'\left(\frac{1+\tau}{2}\omega_2+\frac{1-\tau}{2}\omega_1\right)\right|\\+\left|\mathfrak{F}'\left(\frac{1-\tau}{2}\omega_2+\frac{1+\tau}{2}\omega_1\right)\right|\end{array}\right]d\tau\right]\\&\leq\frac{(\omega_2-\omega_1)}{2}\left(\int_0^1\left|\frac{\tau^\alpha}{2}-\frac{1}{3}\right|^p d\tau\right)^{\frac{1}{p}}\times\left[\begin{array}{c}\left(\int_0^1\left|\mathfrak{F}'\left(\frac{1+\tau}{2}\omega_2+\frac{1-\tau}{2}\omega_1\right)\right|^q\right)^{\frac{1}{q}}\\+\left(\int_0^1\left|\mathfrak{F}'\left(\frac{1-\tau}{2}\omega_2+\frac{1+\tau}{2}\omega_1\right)\right|^q\right)^{\frac{1}{q}}\end{array}\right].\end{aligned}$$

Applying convexity of $|\mathfrak{F}'|^q$, we obtain

$$\begin{aligned}&\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]\right.\\&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^\alpha}\left[J^\alpha_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^\alpha_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\right|\\&\leq\frac{(\omega_2-\omega_1)}{2}\left[\int_0^1\left|\frac{\tau^\alpha}{2}-\frac{1}{3}\right|\left[\begin{array}{c}\left|\mathfrak{F}'\left(\frac{1+\tau}{2}\omega_2+\frac{1-\tau}{2}\omega_1\right)\right|\\+\left|\mathfrak{F}'\left(\frac{1-\tau}{2}\omega_2+\frac{1+\tau}{2}\omega_1\right)\right|\end{array}\right]d\tau\right]\end{aligned}$$

$$\leq \frac{(\omega_2-\omega_1)}{2}\left(\int_0^1\left|\frac{\tau^\alpha}{2}-\frac{1}{3}\right|^p d\tau\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{(1+\tau)^2}{4}\left|\mathfrak{F}'(\omega_2)\right|^q+\frac{(1-\tau)^2}{4}\left|\mathfrak{F}'(\omega_1)\right|^q+\frac{1-\tau^2}{4}M(\omega_1,\omega_2)\right)\right.$$
$$\left.\left(\frac{(1-\tau)^2}{4}\left|\mathfrak{F}'(\omega_2)\right|^q+\frac{(1+\tau)^2}{4}\left|\mathfrak{F}'(\omega_1)\right|^q+\frac{1-\tau^2}{4}M(\omega_1,\omega_2)\right)\right]$$
$$\times\left[\begin{array}{c}\left(\int_0^1\left|\mathfrak{F}'\left(\frac{1+\tau}{2}\omega_2+\frac{1-\tau}{2}\omega_1\right)\right|^q\right)^{\frac{1}{q}}\\ +\left(\int_0^1\left|\mathfrak{F}'\left(\frac{1-\tau}{2}\omega_2+\frac{1+\tau}{2}\omega_1\right)\right|^q\right)^{\frac{1}{q}}\end{array}\right].$$

Thus, the proof is finished. □

Corollary 2. *In Theorem 7, if we set* $\alpha=1$*, then we obtain the following new Simpson's inequality:*

$$\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{2(\omega_2-\omega_1)^\alpha}\left[J^\alpha_{\omega_1+}\mathfrak{F}(\omega_2)+J^\alpha_{\omega_2-}\mathfrak{F}(\omega_1)\right]\right|$$
$$\leq\frac{\omega_2-\omega_1}{2^{1-\frac{1}{p}}}\left(\frac{1}{6^{p+1}(p+1)}+\frac{1}{3^{p+1}(p+1)}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{|\mathfrak{F}'(\omega_2)|^q+7|\mathfrak{F}'(\omega_1)|^q+2M(\omega_1,\omega_2)}{12}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{|\mathfrak{F}'(\omega_1)|^q+7|\mathfrak{F}'(\omega_2)|^q+2M(\omega_1,\omega_2)}{12}\right)^{\frac{1}{q}}\right].$$

5. Concluding Remarks

This paper introduces two fresh versions of Hermite-Hadamard type inequalities for convex functions in the context of fractional calculus, utilizing a pair of functions. Additionally, we have shown new Simpson's type inequalities for convex functions using fractional integrals. We have demonstrated through examples that these new inequalities are valid and a progression of existing ones. It is an intriguing challenge for future researchers to explore similar inequalities for other fractional integrals.

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