Miskolc Mathematical Notes
Vol. 24 (2023), No. 2, pp. 611–624
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.4207

CONSTRUCTION OF VECTOR LYAPUNOV FUNCTION FOR NONLINEAR LARGE-SCALE SYSTEM WITH PERIODIC SUBSYSTEMS

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Received 14 April, 2022

Abstract. A new approach for constructing vector Lyapunov function for nonlinear non-autonomous large-scale systems is proposed. It is assumed that independent subsystems are linear periodic systems. The components of the vector Lyapunov function are chosen as a quadratic form with a variable matrix. This matrix is an approximate solution of the Lyapunov matrix differential equation. This solution is constructed using the discretization method and the representation of the evolution operator proposed by Magnus. Sufficient conditions for the asymptotic stability of a trivial solution of a nonlinear large-scale system are established. The effectiveness of obtained results are illustrated by the example of stability investigation for coupled systems.

2010 *Mathematics Subject Classification:* 34D23; 34G20; 34K45

Keywords: vector Lyapunov function, asymptotic stability, discretization method, comparison method, nonlinear nonautonomous large-scale systems

1. INTRODUCTION

Vector Lyapunov functions (VLF) method is widely used to study the stability of complex nonlinear large-scale systems. The main reason for the emergence of this method is the complexity of constructing the Lyapunov function for systems of large dimension. In this situation, the VLF is a more convenient tool for studying the stability of systems than the scalar Lyapunov function, and has been first presented in [4, 12]. Method of VLF assumes a preliminary decomposition of a large-scale system into subsystems of smaller dimensions. Further, independent subsystems are singled out, for which the problem of constructing the Lyapunov function is much easier to solve than for the whole system. The constructed (scalar) Lyapunov functions for independent subsystems are combined into VLF. It should be noted that the application of VLF method is possible only if the independent subsystems are asymptotically stable.

This research was supported by the German Research Foundation (DFG), grant No. SL 343/1-1 and Deutscher Akademischer Austauschdienst (DAAD), Personal ref. no.: 91775148.

There are two approaches to study the stability of large-scale systems based on VLF. The first [12] approach involves obtaining a system of differential inequalities for VLF and applying the comparison principle. The second [13] one involves constructing a scalar function by convolution of the constructed VLF with a positive vector and then applying the Lyapunov theorem for scalar function.

The methods of constructing VLF for nonlinear systems with asymptotically stable linear subsystems are presented in [3]. Advances in the development of computational mathematics and computer calculations methods stimulated the development of discretization method for constructing Lyapunov functions. In [1], the discretization method is used to study the robust stability of a switched linear differential system. In [18], a continuous switched system with all unstable subsystems is considered. To study asymptotic stability, the direct Lyapunov method is used, based on the idea of time discretization of solutions of the Lyapunov matrix differential equation.

Vector Lyapunov functions method is widely used in various branches of control theory and systems theory. In [15], the concept of decentralized control is proposed for various classes of autonomous control systems and, based on the method of vector functions, algorithms for stabilization of nonlinear large-scale systems are proposed. The efficiency of these algorithms is primarily due to the fact that nonlinear systems with linear autonomous subsystems are considered, for which there is an explicit procedure for constructing the Lyapunov function.

In [14], a generalized vector Lyapunov function is developed to analyze the stability of nonlinear dynamical systems using the generalized comparison principle. In particular, a cascade of nonlinear dynamical system with continuous mapping is considered. A generalization of the inverse Lyapunov theorem, which establishes the existence of a Lyapunov vector function for the asymptotic stability of equilibrium of a nonlinear dynamical system, is presented. This result is used to establish an equivalence between asymptotic stabilizability and the existence of a vector control function. The problem of feedback control is also considered and the concept of a vector control function is introduced as a generalization of the Lyapunov control function.

In [2, 16], using the Matrosov comparison method, in combination with methods of convex geometry, the approaches of constructing VLF for studying the stability of fixed points of dynamical systems in the space of convex compact sets are proposed. As a result, for a dynamical system in the space of convex compact sets, a comparison system in a finite-dimensional space has been obtained and investigated. The cases of nonlinear systems with nilpotent and idempotent operators are also considered, and the case on the plane is studied in detail. A construction of an infinite-dimensional VLF for cascades in the space of convex compact sets is proposed in [2].

Vector Lyapunov function is widely used in the theory of nonlinear control systems [9], the main result of which shows that the existence of vector Lyapunov control function is a necessary and sufficient condition for the existence of a smooth globally stabilizing feedback.

Note also that VLF is applicable to the study of impulsive nonlinear systems [7, 8]. In [6], this method is used to study the critical equilibrium states of nonlinear impulsive large-scale systems in the case when continuous and discrete dynamics are both unstable.

In the theory of stability of autonomous nonlinear systems, the concept of input-to-state stability (ISS) and the based on it stability theorems for coupled nonlinear systems (small-gain theorems) (see [5]) is a reliable alternative to the method of vector Lyapunov functions. This approach as well as VLF method assumes the existence of the property of asymptotic stability of independent subsystems.

The main contribution of our article is to develop a new method for constructing the VLF for studying the stability of a nonlinear nonautonomous coupled system of differential equations with linear periodic subsystems. To study the stability of the zero solution, the method of VLF and the comparison principle are used.

For independent subsystems, a new method for constructing scalar Lyapunov functions, which are used as components of the vector Lyapunov function is proposed. The methods of commutator calculus [10, 11] in combination with the discretization method are used. Our approach develops the ideas from [17], which is proposed to use the identities of the commutator calculus to study the asymptotic stability of linear periodic systems.

A linear impulsive comparison system is obtained and investigated. The asymptotic stability of this comparison systems implies the stability of the equilibrium of the considered nonlinear system. An illustrative example of the study of asymptotic stability of nonlinear system consisting of two coupled subsystems is given. For each linear independent subsystem, the Lyapunov function is constructed as a quadratic form with a variable matrix. In this example, there is no Lyapunov function of a quadratic form with a constant matrix.

The paper consists of 8 Sections and is organized as follows. Section 2 provides basic notation and information from linear algebra. In Section 3, the problem statement of stability of the equilibrium of a nonlinear large-scale system, which consists of coupled linear periodic systems, is presented. In Section 4 a new method for constructing the vector Lyapunov function is proposed and an appropriate comparison system is constructed. The main result of the paper is formulated and proved in Section 5, and in Section 6, the problem of stability of linear impulsive comparison systems is discussed. Section 7 gives an example of constructing a vector Lyapunov function. Section 8 is devoted to a discussion of the results and their possible development.

2. PRELIMINARIES

Let $\mathbb{Z}$ be a set of integers, $\mathbb{Z}_+$ — set of non-negative integers, $\mathbb{R}$ — set of real numbers, $\mathbb{R}_+$ — set of real non-negative numbers.

Let $\mathbb{R}^n$ denote the n -dimensional Euclidean space with standard scalar product, $\mathbb{R}^{n\times m}$ is a linear space of $n\times m$ matrices, $\mathbb{R}^{n\times n}$ is the Banach algebra with a norm $\|A\| = \lambda_{\max}^{1/2}(A^{\mathrm{T}}A)$. Let $\mathbb{S}^n \subset \mathbb{R}^{n\times n}$ be the subspace of symmetric matrices, for $P\in\mathbb{S}^n$ the inequality $P\succ 0$ means that the matrix P is positive definite, let $\sigma(A)$ and $r_\sigma(A)$ denote the spectrum and spectral radius for the matrix $A\in\mathbb{R}^{n\times n}$ respectively. If for A the spectrum $\sigma(A)\subset\mathbb{R}$, then $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ are the smallest and largest eigenvalues, respectively, $I_n\in\mathbb{R}^{n\times n}$ is an identity matrix.

$$\mathbb{C}^- = \{z\in\mathbb{C}, \quad \mathrm{Re}\, z<0\}, \quad \mathbb{C}^+ = \{z\in\mathbb{C}, \quad \mathrm{Re}\, z>0\},$$

$\mathbb{C}$ is a set of complex numbers. The Kronecker symbol δ_{ij} is defined as $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ for $i\neq j$.

Next we present some information from the commutator calculus that is important for further research [11].

The commutator of two matrices $A,B\in\mathbb{R}^{n\times n}$ which is defined as

$$[A,B] = AB - BA$$

introduces the structure of a Lie algebra in $\mathbb{R}^{n\times n}$. The commutation operator ad_A, $A\in\mathbb{R}^{n\times n}$ is defined as a linear mapping

$$\mathbb{R}^{n\times n}\to\mathbb{R}^{n\times n}, \quad Y\mapsto \mathrm{ad}_A(Y) = [A,Y], \quad Y\in\mathbb{R}^{n\times n}.$$

Let X, Y and Z be the independent matrix variables, $F(X,Y)$ is a formal series of variables X and Y, $\lambda\in\mathbb{R}$, then the polarization identity

$$F(X+\lambda Z,Y) = F(X,Y)+\lambda F_1(X,Y,Z)+\lambda^2 F_2(X,Y,Z)+\ldots$$

defines the Hausdorff derivative $\left(Z\frac{\partial}{\partial X}\right)F(X,Y) \overset{df}{=} F_1(X,Y,Z)$.

We define recursively the following Lie polynomials of matrix variables X and Y (definition and more details about Lie elements see in [10])

$$\{Y,X^0\} = Y, \quad \{Y,X^{l+1}\} = [\{Y,X^l\},X], \quad l\in\mathbb{Z}_+.$$

It is easy to see that

$$\mathrm{ad}_X^l(Y) = (-1)^l\{Y,X^l\}.$$

For the further discussion we need the following F. Hausdorff's identities

$$\begin{aligned} e^{-X}\left(\left(Y\frac{\partial}{\partial X}\right)e^X\right) &= Y+\sum_{k=1}^{\infty}\frac{1}{(k+1)!}\{Y,X^k\},\\ \left(\left(Y\frac{\partial}{\partial X}\right)e^X\right)e^{-X} &= Y+\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\{Y,X^k\}. \end{aligned} \tag{2.1}$$

We also need a chain rule for differentiating the composition of mappings. Let $P(X)$ be a series of matrix variable X, $A\colon (a,b) \to \mathbb{R}^{n\times n}$ is a mapping differentiable at the point $t_0 \in (a,b)$ and $B(t) := P(A(t))$ exists in a neighborhood of the point t_0, then we obtain the chain rule

$$\frac{dB}{dt}\Big|_{t=t_0} = \Big(Y\frac{\partial}{\partial X}\Big)P(X), \quad Y = \frac{dA}{dt}\Big|_{t=t_0}, \quad X = A(t_0). \tag{2.2}$$

3. Problem statement

Consider a coupled nonautonomous system of differential equations consisting of r subsystems

$$\dot{\mathbf{x}}_i(t) = A_i(t)\mathbf{x}_i(t) + \mathbf{f}_i(t, \mathbf{x}_1(t), \ldots, \mathbf{x}_r(t)), \tag{3.1}$$

where $\mathbf{x}_i \in \mathbb{R}^{n_i}$, $i = 1,\ldots,r$, $A_i\colon \mathbb{R} \to \mathbb{R}^{n_i\times n_i}$ is a piecewise continuous mapping. Suppose $A_i(t)$ are θ -periodic functions, $A_i(t+\theta) = A_i(t)$ for all $t \in \mathbb{R}$, $\mathbf{f}_i : \mathbb{R}\times\Omega \to \mathbb{R}^{n_i}$, $\Omega \subset \mathbb{R}^n$ is an open connected set containing $\mathbf{x} = 0$. If

$$\mathbf{x} = (\mathbf{x}_1^{\mathrm{T}}, \ldots, \mathbf{x}_r^{\mathrm{T}})^{\mathrm{T}}, \quad \mathbf{f} = (\mathbf{f}_1^{\mathrm{T}}, \ldots, \mathbf{f}_r^{\mathrm{T}})^{\mathrm{T}},$$

then $\mathbf{f} : \mathbb{R}\times\Omega \to \mathbb{R}^n$, $n = \sum_{k=1}^r n_k$. Suppose that the mapping $\mathbf{f}$ satisfies the conditions guaranteeing the existence and uniqueness of solutions of the Cauchy problem for the system (3.1) with the initial conditions $\mathbf{x}_i(t_0) = \mathbf{x}_{i0}$, $i = 1,\ldots,r$ and $\mathbf{f}(t,0) \equiv 0$. This condition guarantees that $\mathbf{x} = 0$ is a solution of the system (3.1).

Let N be a fixed positive integer, $h = \frac{\theta}{N}$. Let us define the positive constants a_{mi}, b_{mi}, c_{mi}, where $m = 0,\ldots,N-1$, $i = 1,\ldots,r$, so that the following inequalities are satisfied

$$\|A_i(t)\| \le a_{mi}, \quad \Big\|\int_{mh}^{t} A_i(s)\,ds\Big\| \le c_{mi}(t-mh), \tag{3.2}$$

$$\Big\|\Big[A_i(t), \int_{mh}^{t} A_i(\tau)\,d\tau\Big]\Big\| \le b_{mi}(t-mh) \tag{3.3}$$

for all $t \in (mh,(m+1)h]$.

Regarding the function $\mathbf{f}$ let us make the following assumptions:

(i) there exist piecewise continuous functions $f_{ij} : \mathbb{R} \to \mathbb{R}_+$, $i,j = 1,\ldots,r$ such that for all $\mathbf{x} \in \Omega \subset \mathbb{R}^n$ the following inequalities hold

$$\|\mathbf{f}_i(t,\mathbf{x})\| \le \sum_{j=1}^{r} f_{ij}(t)\|\mathbf{x}_j\|; \tag{3.4}$$

(ii) there exist finite piecewise continuous functions $\widehat{f}_{ij} : (0,\theta] \to \mathbb{R}_+$ such that

$$\sup\{f_{ij}(t+k\theta) \;:\; k \in \mathbb{Z}_+\} \le \widehat{f}_{ij}(t). \tag{3.5}$$

Remark 1. The θ - periodic continuation of the functions $\widehat{f}_{ij}$ to $\mathbb{R}$ will also be denoted by $\widehat{f}_{ij}$.

Let $\mathbf{x}(t,t_0,\mathbf{x}_0)$ be a solution of the Cauchy problem for the system (3.1) with the initial condition $\mathbf{x}(t_0,t_0,\mathbf{x}_0)=\mathbf{x}_0$.

Definition 1. The equilibrium state $\mathbf{x}=0$ of the system of differential equations (3.1) is called

1) Lyapunov stable if for any $\varepsilon>0$, $t_0\in\mathbb{R}$ there exists $\delta=\delta(\varepsilon,t_0)>0$ such that inequality $\|\mathbf{x}_0\|<\delta$ implies $\|\mathbf{x}(t,t_0,\mathbf{x}_0)\|<\varepsilon$ for all $t\geq t_0$;
2) asymptotically stable in the sense of Lyapunov if it is stable and for any $t_0\in\mathbb{R}$ there exists a number $\rho=\rho(t_0)>0$ such that from condition $\|\mathbf{x}_0\|<\rho$ it follows that $\lim\limits_{t\to+\infty}\|\mathbf{x}(t,t_0,\mathbf{x}_0)\|=0$;
3) uniformly asymptotically stable, if $\delta=\delta(\varepsilon,t_0)$ and $\rho(t_0)$ can be chosen independent of t_0.

Remark 2. Using the θ - periodicity of the $A_i(t)$ and inequalities (3.4)–(3.5), one can show that the asymptotic stability of the equilibrium state $\mathbf{x}=0$ is equivalent to the uniform asymptotic stability of $\mathbf{x}=0$.

The aim of the paper is to prove sufficient conditions for the stability of the equilibrium state $\mathbf{x}=0$ of the large-scale system (3.1).

4. Construction of the vector Lyapunov function

Without loss of generality, it is assumed that $t_0=0$. To study the stability, we will use the VLF method. Let $v:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^r_+$ be a vector Lyapunov function. We choose the components of this function as quadratic forms $v_i(t,\mathbf{x}_i)=\mathbf{x}_i^{\mathrm{T}}P_i(t)\mathbf{x}_i$, where $P_i:\mathbb{R}\to\mathbb{S}^{n_i}$ is left continuous and θ -periodic mapping.

Obviously, it is sufficient to define the functions $P_i(t)$, $i=1,\dots,r$ on the period $(0,\theta]$. If N is certain positive integer, $h=\frac{\theta}{N}$, than for $t\in(mh,(m+1)h]$ we can define the matrices

$$\widehat{A}_{mi}(t)=\int\limits_{mh}^{t}A_i(s)\,ds,\quad \widetilde{A}_{mi}=\frac{1}{h}\widehat{A}_{mi}((m+1)h)=\frac{1}{h}\int\limits_{mh}^{(m+1)h}A_i(s)\,ds,$$

$$\Phi_i=e^{h\widetilde{A}_{N-1,i}}\dots e^{h\widetilde{A}_{0,i}}.$$

Suppose that the number N can be chosen so that $r_\sigma(\Phi_i)<1$. If this condition is satisfied, then we choose a symmetric positive definite matrix P_{0i} that satisfies the linear matrix inequality

$$\Phi_i^{\mathrm{T}}P_{0i}\Phi_i-P_{0i}\prec 0.$$

Next, we define the matrices

$$P_{m+1,i}=e^{-h\widetilde{A}_{mi}^{\mathrm{T}}}P_{mi}e^{-h\widetilde{A}_{mi}},\quad m=0,\dots,N-1$$

and, finally, the function $P_i(t)$ is defined as

$$P_i(t)=e^{-\widehat{A}_{mi}^{\mathrm{T}}(t)}P_{mi}e^{-\widehat{A}_{mi}(t)},\quad t\in(mh,(m+1)h].$$

Using the inequalities (3.2)– (3.5) we can obtain the estimate of the total derivative of the constructed vector function $v(t,\mathbf{x})$ along the trajectories of the large-scale system (3.1).

Consider the total derivative of the function $v_i(t,\mathbf{x}_i)$ along the trajectory of the large-scale system (3.1)

$$\dot{v}_i(t,\mathbf{x}_i(t)) = \mathbf{x}_i^{\mathrm{T}}(\dot{P}_i(t)+A_i^{\mathrm{T}}(t)P_i(t)+P_i(t)A_i(t))\mathbf{x}_i + 2\mathbf{x}_i^{\mathrm{T}}P_i(t)\mathbf{f}_i(t,\mathbf{x}).$$

For $t \in (mh,(m+1)h]$, we need to compute the value of the expressions $\dot{P}_i(t)+A_i^{\mathrm{T}}(t)P_i(t)+P_i(t)A_i(t)$, $i=1,\dots,r$. Let us first consider the expression for $\frac{d}{dt}e^{-\widehat{A}_{mi}(t)}$. Applying the chain rule of differentiation (2.2), we obtain

$$\begin{aligned}\frac{d}{dt}e^{-\widehat{A}_{mi}(t)} &= -\Big(\frac{d\widehat{A}_{mi}(t)}{dt}\frac{\partial}{\partial X}\Big)e^X\Big|_{X=-\widehat{A}_{mi}(t)}\\ &= -\Big(A(t)\frac{\partial}{\partial X}\Big)e^X\Big|_{X=-\widehat{A}_{mi}(t)}.\end{aligned}$$

From the Hausdorff identity (2.1), it follows

$$\frac{d}{dt}e^{-\widehat{A}_{mi}(t)} = -e^{-\widehat{A}_{mi}(t)}\Big(A_i(t)+\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\{A_i(t),\widehat{A}_{mi}^k(t)\}\Big).$$

Consequently, we obtain

$$\frac{d}{dt}e^{-\widehat{A}_{mi}^{\mathrm{T}}(t)} = -\Big(A_i^{\mathrm{T}}(t)+\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\{A_i(t),\widehat{A}_{mi}^k(t)\}^{\mathrm{T}}\Big)e^{-\widehat{A}_{mi}^{\mathrm{T}}(t)}$$

and therefore

$$\begin{aligned}\dot{P}_i(t)+A_i^{\mathrm{T}}(t)P_i(t)+P_i(t)A_i(t) = &-\Big(\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\{A_i(t),\widehat{A}_{mi}^k(t)\}\Big)^{\mathrm{T}}P_i(t)\\ &-P_i(t)\Big(\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\{A_i(t),\widehat{A}_{mi}^k(t)\}\Big).\end{aligned}$$

From (3.2)– (3.3) it follows the estimates for $t \in (mh,(m+1)h]$

$$\|\{A_i(t),\widehat{A}_{mi}^k(t)\}\| \le b_{mi}(2c_{mi})^{k-1}(t-mh)^k,$$

$$\Big\|\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\{A_i(t),\widehat{A}_{mi}^k(t)\}\Big\| \le b_{mi}\sum_{k=1}^{\infty}\frac{(2c_{mi})^{k-1}(t-mh)^k}{(k+1)!} \le \eta_{mi}(t-mh),$$

where $\eta_{mi} = b_{mi}\sum_{k=1}^{\infty}\frac{(2c_{mi}h)^{k-1}}{(k+1)!}$, so we obtain

$$\mathbf{x}_i^{\mathrm{T}}(\dot{P}_i(t)+A_i^{\mathrm{T}}(t)P(t)+P(t)A_i(t))\mathbf{x}_i \le 2\|P_i^{-1/2}(t)\|\|P_i^{1/2}(t)\|(t-mh)\eta_{mi}\mathbf{x}_i^{\mathrm{T}}P_i(t)\mathbf{x}_i(t).$$

Using the following estimates

$$\|P_i^{1/2}(t)\| = \lambda_{\max}^{1/2}(P_i(t)) = \|P_i(t)\|^{1/2} = \|e^{-\widehat{A}_{mi}^{\mathrm{T}}(t)}P_{mi}e^{-\widehat{A}_{mi}(t)}\|^{1/2}$$

$$
\begin{aligned}
&\le e^{c_{mi}(t-mh)}\|P_{mi}\|^{1/2},\\
\lambda_{\min}(P_i(t)) &= \min_{\|\mathbf{x}_i\|=1} \mathbf{x}_i^{\mathrm{T}} P_i(t)\mathbf{x}_i\\
&= \min_{\|\mathbf{x}_i\|=1} (e^{-\widehat{A}_{mi}(t)}\mathbf{x}_i)^{\mathrm{T}} P_{mi} e^{-\widehat{A}_{mi}(t)}\mathbf{x}_i \ge \lambda_{\min}(P_{mi}) \min_{\|\mathbf{x}_i\|=1} \|e^{-\widehat{A}_{mi}(t)}\mathbf{x}_i\|^2\\
&\ge \min_{\|\mathbf{x}_i\|=1} \lambda_{\min}(P_{mi}) e^{-2c_{mi}(t-mh)}\|\mathbf{x}_i\|^2 = \lambda_{\min}(P_{mi}) e^{-2c_{mi}(t-mh)}
\end{aligned}
$$

and

$$
\|P_i^{-1/2}(t)\| = \lambda_{\max}^{1/2}(P_i^{-1}(t)) = \lambda_{\min}^{-1/2}(P_i(t)) \le \lambda_{\min}^{-1/2}(P_{mi}) e^{c_{mi}(t-mh)},
$$

which are valid for $t \in (mh, (m+1)h]$, we obtain

$$
\mathbf{x}_i^{\mathrm{T}}(\dot{P}_i(t) + A_i^{\mathrm{T}}(t)P_i(t) + P_i(t)A_i(t))\mathbf{x}_i \le 2e^{2c_{mi}(t-mh)} \sqrt{\frac{\lambda_{\max}(P_{mi})}{\lambda_{\min}(P_{mi})}}(t-mh)\eta_{mi} v_i(t,\mathbf{x}_i)
$$

for all $t \in (mh, (m+1)h)$.

Let us now consider the problem of estimation the expression $\mathbf{x}_i^{\mathrm{T}} P_i(t)\mathbf{f}_i(t,\mathbf{x})$. Applying the Cauchy-Bunyakovsky inequality and the assumptions (3.4)– (3.5) about the function $\mathbf{f}$, for $t \in (mh, (m+1)h]$, we obtain

$$
\begin{aligned}
\mathbf{x}_i^{\mathrm{T}} P_i(t)\mathbf{f}_i(t,\mathbf{x}) &= (P_i^{1/2}(t)\mathbf{x}_i)^{\mathrm{T}} P_i^{1/2}(t)\mathbf{f}_i(t,\mathbf{x}) \le \sqrt{v_i(t,\mathbf{x}_i)}\|P_i^{1/2}(t)\| \sum_{j=1}^{r} f_{ij}(t)\|\mathbf{x}_j\|\\
&\le \sqrt{v_i(t,\mathbf{x}_i)}\|P_i^{1/2}(t)\| \sum_{j=1}^{r} \widehat{f}_{ij}(t)\|P_j^{-1/2}(t)\|\sqrt{v_j(t,\mathbf{x}_j)}\\
&\le \sqrt{v_i(t,\mathbf{x}_i)} \sum_{j=1}^{r} \widehat{f}_{ij}(t) e^{(c_{mi}+c_{mj})(t-mh)} \sqrt{\frac{\lambda_{\max}(P_{mi})}{\lambda_{\min}(P_{mj})}} \sqrt{v_j(t,\mathbf{x}_j)}.
\end{aligned}
$$

Let us define the matrix $\Gamma(t) = (\gamma_{ij}(t))_{i,j=1}^{r}$, where

$$
\gamma_{ij}(t) = e^{2c_{mi}(t-mh)} \sqrt{\frac{\lambda_{\max}(P_{mi})}{\lambda_{\min}(P_{mi})}}(t-mh)\eta_{mi}\delta_{ij} + \widehat{f}_{ij}(t) e^{(c_{mi}+c_{mj})(t-mh)} \sqrt{\frac{\lambda_{\max}(P_{mi})}{\lambda_{\min}(P_{mj})}}.
$$

Then we get the estimate

$$
\dot{v}_i(t,\mathbf{x}_i(t)) \le 2\gamma_{ii}(t)v_i(t,\mathbf{x}_i) + 2\sqrt{v_i(t,\mathbf{x}_i)} \sum_{j=1, j\ne i}^{r} \gamma_{ij}(t)\sqrt{v_j(t,\mathbf{x}_i)}
$$

for all $t \in (0,\theta]$. If $t = \theta$, then based on the periodicity of the functions $P_i(t)$, we obtain

$$
\begin{aligned}
v_i(\theta+0, \mathbf{x}_i(\theta+0)) &= \mathbf{x}_i^{\mathrm{T}}(\theta+0)P_i(\theta+0)\mathbf{x}_i(\theta+0) = \mathbf{x}_i^{\mathrm{T}}(\theta)P_i(0+0)\mathbf{x}_i(\theta)\\
&= \mathbf{x}_i^{\mathrm{T}}(\theta)P_{0i}\mathbf{x}_i(\theta) = (P_{Ni}^{1/2}\mathbf{x}_i(\theta))^{\mathrm{T}} P_{Ni}^{-1/2} P_{0i} P_{Ni}^{-1/2} P_{Ni}^{1/2}\mathbf{x}_i(\theta)\\
&\le \lambda_{\max}(P_{0i}P_{Ni}^{-1})\mathbf{x}_i^{\mathrm{T}}(\theta)P_{Ni}\mathbf{x}_i(\theta) = \lambda_{\max}(P_{0i}P_{Ni}^{-1})\mathbf{x}_i^{\mathrm{T}}(\theta)P_i(\theta)\mathbf{x}_i(\theta)
\end{aligned}
$$

$$= \lambda_{\max}(P_{0i}P_{Ni}^{-1})v_i(\theta, \mathbf{x}_i(\theta)) := \delta_i v_i(\theta, \mathbf{x}_i(\theta)).$$

Extending the function $\Gamma(t)$ to $\mathbb{R}_+$ in a periodic manner, we conclude that the components of the vector Lyapunov function satisfy the inequalities

$$\begin{gathered} \dot{v}_i(t, \mathbf{x}_i(t)) \le 2\gamma_{ii}(t)v_i(t, \mathbf{x}_i) + 2\sqrt{v_i(t, \mathbf{x}_i)} \sum_{j=1, j\ne i}^{r} \gamma_{ij}(t)\sqrt{v_j(t, \mathbf{x}_i)}, \quad t \ne k\theta, \\ v_i(t+0, \mathbf{x}_i(t+0)) \le \delta_i v_i(t, \mathbf{x}_i(t)), \quad t = k\theta. \end{gathered} \tag{4.1}$$

Applying comparison theorems for impulsive differential inequalities, we get the estimates

$$v_i(t, \mathbf{x}_i(t)) \le u_i^+(t), \quad t \ge 0, \quad i = 1, \dots, r,$$

where $u^+(t) = (u_1(t), \dots, u_r(t))^{\mathrm{T}}$ is the maximum solution of the Cauchy problem ($u_0 = (v_1(0+0, \mathbf{x}_{10}), \dots, v_r(0+0, \mathbf{x}_{r0}))^{\mathrm{T}}$) for a system of impulsive differential equations

$$\begin{gathered} \dot{u}_i(t) = 2\gamma_{ii}(t)u_i(t) + 2\sqrt{u_i(t)} \sum_{j=1, j\ne i}^{r} \gamma_{ij}(t)\sqrt{u_j(t)}, \quad t \ne k\theta, \\ u_i(t+0) = \delta_i u_i(t), \quad t = k\theta, \quad u(0+0) = u_0. \end{gathered} \tag{4.2}$$

It is easy to see that for comparison system (4.2) if $u_0 > 0$, $u_0 \ne 0$, then $u(t) > 0$ for $t \ge 0$.

In this case, by changing variables $\zeta_i = u_i^{1/2}$, $i = 1, \dots, r$ the system of comparison (4.2) is reduced to a linear system of impulsive differential equations

$$\begin{gathered} \dot{\zeta}(t) = \Gamma(t)\zeta(t), \quad t \ne k\theta, \\ \zeta(t+0) = \Delta^{1/2}\zeta(t), \quad t = k\theta, \end{gathered} \tag{4.3}$$

where $\zeta = (\zeta_1, \dots, \zeta_r)^{\mathrm{T}}$, $\Delta = \operatorname{diag}\{\delta_1, \dots, \delta_r\}$.

5. Stability Theorem

The VLF constructed in the previous section and the obtained estimates of its derivative allow us to reduce the study of the stability of the equilibrium state of a nonlinear nonautonomous system to the study of the stability of a linear impulsive system (4.3) (linear comparison system).

Theorem 1. *Assume the linear impulsive system (4.3) is asymptotically stable in the sense of Lyapunov. Then the equilibrium state $\mathbf{x} = 0$ of the system (3.1) is asymptotically stable in the sense of Lyapunov.*

Proof. Let $\mathbf{x}_0 = (\mathbf{x}_{10}^{\mathrm{T}}, \dots, \mathbf{x}_{r0}^{\mathrm{T}})^{\mathrm{T}}$ be an initial conditions for the solution $\mathbf{x}(t)$ of the system of differential equations (3.1). Denote by $\eta > 0$ an arbitrary number. Consider the solution of the Cauchy problem for the differential comparison equation (4.2) with the initial conditions $u_{i0} = v_i(0+0, \mathbf{x}_{i0}) + \eta$. Then, by the comparison principle, the following inequalities hold

$$v_i(t, \mathbf{x}_i(t)) \le u_i(t).$$

Since, $u_0 > 0$, then $u_i(t) = \zeta_i^2(t, \zeta_0)$, where $\zeta(t) = (\zeta_1(t), \dots, \zeta_r(t))^{\mathrm{T}}$ is the solution of the Cauchy problem for (4.3) with initial condition $\zeta(0+0) = \zeta_0$, $\zeta_0 = (u_{10}^{1/2}, \dots, u_{r0}^{1/2})^{\mathrm{T}}$ and

$$v_i(t, \mathbf{x}_i(t)) \le \zeta_i^2(t).$$

Passing to the limit $\eta \to 0+$ and taking into account that the solutions of the linear impulsive differential equation continuously depend on the initial conditions, we obtain the estimate

$$v_i(t, \mathbf{x}_i(t)) \le \hat{\zeta}_i^2(t), \quad \text{for all} \quad t \ge 0, \quad i = 1, \dots, r,$$

where $\hat{\zeta}_i^2(t)$ is the solution of the Cauchy problem for (4.2) with initial condition $\hat{\zeta}(0) = (\hat{\zeta}_{10}, \dots, \hat{\zeta}_{r0})^{\mathrm{T}}$, $\hat{\zeta}_{i0} = (\mathbf{x}_{i0}^{\mathrm{T}} P_{0i} \mathbf{x}_{i0})^{1/2}$.

In particular, for $t = k\theta + 0$ we obtain

$$\lambda_{\min}(P_{0i})\|\mathbf{x}_i(k\theta)\|^2 \le v_i(k\theta+0, \mathbf{x}_i(k\theta+0)) \le \hat{\zeta}_i^2(k\theta+0)$$
$$\text{for all} \quad k \in \mathbb{Z}_+, \quad i = 1, \dots, r.$$

From asymptotic stability of the comparison system (4.3) it follows that for a given $\varepsilon_1 > 0$ there exists $\delta_1(\varepsilon_1) > 0$ such that inequalities $\hat{\zeta}_{i0} < \delta_1(\varepsilon_1)$ imply $\hat{\zeta}_i(t) < \varepsilon_1$ for all $t \ge 0$. Let ε be an arbitrary positive number,

$$\varepsilon_1 = \frac{1}{\sqrt{r}} \varepsilon \min_{i=1,\dots,r} \{\sqrt{\lambda_{\min}(P_{0i})}\}, \quad \delta(\varepsilon) = \min_{i=1,\dots,r} \left\{ \frac{\delta_1(\varepsilon_1)}{\sqrt{\lambda_{\max}(P_{0i})}} \right\}.$$

If $\|\mathbf{x}_{i0}\| < \delta(\varepsilon)$, then $\sqrt{\mathbf{x}_{i0}^{\mathrm{T}} P_{0i} \mathbf{x}_{i0}} < \delta_1(\varepsilon_1)$ and

$$\lambda_{\min}(P_{0i})\|\mathbf{x}_i(k\theta)\|^2 \le v_i(k\theta+0, \mathbf{x}_i(k\theta+0)) \le \hat{\zeta}_i^2(k\theta+0) \le \frac{\varepsilon^2}{r}\lambda_{\min}(P_{0i})$$
$$\text{for all} \quad k \in \mathbb{Z}_+, \quad i = 1, \dots, r.$$

Hence, we have the estimate $\|\mathbf{x}_i(k\theta)\| < \frac{\varepsilon}{\sqrt{r}}$ for all $k \in \mathbb{Z}_+$ and $\|\mathbf{x}(k\theta)\| < \varepsilon$.

Using the Cauchy-Bunyakovsky inequality, we obtain

$$\begin{aligned} \frac{d}{dt}\|\mathbf{x}_i(t)\|^2 &= 2\mathbf{x}_i^{\mathrm{T}}(t) A_i(t) \mathbf{x}_i(t) + 2\mathbf{x}_i^{\mathrm{T}}(t) f_i(t, \mathbf{x}_1(t), \dots, \mathbf{x}_r(t)) \\ &\le 2\|A_i(t)\|\|\mathbf{x}_i(t)\|^2 + 2\|\mathbf{x}_i(t)\| \sum_{j=1}^r \widehat{f}_{ij}(t)\|\mathbf{x}_j(t)\|. \end{aligned}$$

Denote

$$a = \max_{i=1,\dots,r} \sup_{t \in [0,\theta]} \|A_i(t)\|, \quad f^* = \max_{i=1,\dots,r} \sup_{t \in [0,\theta]} \sqrt{\sum_{k=1}^r \widehat{f}_{ij}^2},$$

then, again applying the Cauchy-Bunyakovsky inequality, we get

$$\frac{d}{dt}\|\mathbf{x}_i(t)\|^2 \le 2a\|\mathbf{x}_i(t)\|^2 + 2f^*\|\mathbf{x}_i(t)\|\|\mathbf{x}(t)\| \le (2a+f^*)\|\mathbf{x}_i(t)\|^2 + f^*\|\mathbf{x}(t)\|^2.$$

Hence, it follows that

$$\frac{d}{dt}\|\mathbf{x}(t)\|^2 \le (2a+f^*)\|\mathbf{x}(t)\|^2 + rf^*\|\mathbf{x}(t)\|^2 = (2a+(r+1)f^*)\|\mathbf{x}(t)\|^2 := L\|\mathbf{x}(t)\|^2.$$

And therefore, we have the estimate

$$\|\mathbf{x}(t)\| \le \|\mathbf{x}(k\theta)\|e^{\frac{L}{2}(t-k\theta)} \le e^{\frac{L}{2}\theta}\|\mathbf{x}(k\theta)\|, \quad \text{for all} \quad t \in [k\theta,(k+1)\theta].$$

As proved above, for $\varepsilon e^{-L\theta}$ there exists $\delta = \delta(\varepsilon) > 0$ such that from inequality $\|\mathbf{x}_0\| < \delta$ it follows $\sup\limits_{k\in\mathbb{Z}_+} \|\mathbf{x}(k\theta)\| < \varepsilon e^{-\frac{L}{2}\theta}$. So, we obtain the following estimate

$$\sup_{t\ge 0}\|\mathbf{x}(t)\| \le e^{\frac{L}{2}\theta}\sup_{k\in\mathbb{Z}_+}\|\mathbf{x}(k\theta)\| < \varepsilon,$$

which proves the stability of the equilibrium state $\mathbf{x} = 0$. To prove the attraction property of the equilibrium state $\mathbf{x} = 0$, note that from inequality

$$\lambda_{\min}(P_{0i})\|\mathbf{x}_i(k\theta)\|^2 \le \hat{\zeta}_i^2(k\theta+0) \quad \text{for all} \quad k \in \mathbb{Z}_+, \quad i = 1,\dots,r$$

it follows that $\lim_{k\to\infty}\|\mathbf{x}(k\theta)\| = 0$, and therefore

$$\lim_{t\to\infty}\|\mathbf{x}(t)\| \le e^{\frac{L}{2}\theta}\lim_{k\to\infty}\|\mathbf{x}(k\theta)\| = 0.$$

□

6. Stability of comparison system

Theorem 1 reduces the problem of stability of the equilibrium state $\mathbf{x} = 0$ of the original nonautonomous system (3.1) to the study of the linear comparison system (4.3). The study of the comparison system is easier, since it, as a rule, has a lower dimension and has an important property of positivity with respect to the cone $\mathcal{K} = \mathbb{R}_+^r$. The last property makes it possible to significantly simplify the study of this system and to obtain sufficient conditions for the stability of the linear comparison system (4.3).

Let us define the matrices $\Gamma_m = (\gamma_{ij}^{(m)})_{i,j=1}^r$, $m = 0,\dots,N-1$, where

$$\gamma_{ij}^{(m)} = e^{2c_{mi}h}\sqrt{\frac{\lambda_{\max}(P_{mi})}{\lambda_{\min}(P_{mi})}}h\eta_{mi}\delta_{ij} + \sup_{t\in(mh,(m+1)h]}\widehat{f}_{ij}(t)e^{(c_{mi}+c_{mj})h}\sqrt{\frac{\lambda_{\max}(P_{mi})}{\lambda_{\min}(P_{mj})}}.$$

It is obvious that $\Gamma(t) \le \Gamma_m$ for all $t \in (mh,(m+1)h]$. So, we get

$$\zeta(t) \le \widetilde{\zeta}(t), \tag{6.1}$$

where $\zeta(t)$ is a solution of the Cauchy problem for the linear comparison system (4.3) with the initial condition $\zeta(0) = \zeta_0 \geq 0$, $\widetilde{\zeta}(t)$ is a solution of Cauchy problem for system

$$\begin{gathered}\frac{d\widetilde{\zeta}}{dt} = \widetilde{\Gamma}(t)\widetilde{\zeta}, \quad t \neq k\theta, \quad \widetilde{\zeta}(0) = \zeta_0, \\ \widetilde{\zeta}(t+0) = \Delta^{1/2}\widetilde{\zeta}(t), \quad t = k\theta,\end{gathered} \tag{6.2}$$

where $\widetilde{\Gamma}(t) = \Gamma_m$ for $t \in (mh, (m+1)h]$. From the inequality (6.1) it follows that the asymptotic stability of the linear comparison system (6.2) implies the asymptotic stability of the linear comparison system (4.3) and the equilibrium state $\mathbf{x} = 0$ of the nonlinear nonautonomous system (3.1). Thus, we obtain the following sufficient conditions for the asymptotic stability of the equilibrium state $\mathbf{x} = 0$ of the nonlinear nonautonomous system (3.1).

Proposition 1. *Assume that the positive integer N is chosen large enough so that $r_\sigma(\Phi_i) < 1$ and the following inequality hold*

$$r_\sigma\Big(\prod_{k=0}^{N-1} e^{\Gamma_k h}\Delta^{1/2}\Big) < 1.$$

Then the equilibrium state $\mathbf{x} = 0$ of the nonlinear nonautonomous system (3.1) is asymptotically stable.

For the *proof* it is necessary to use the main result of the Floquet-Lyapunov theory.

7. Numerical example

Consider a nonlinear coupled system

$$\begin{aligned}\dot{x}_1(t) &= (-0.8 + \cos(20\pi t))x_1(t) - \sin(20\pi t)x_2 + f_1(t, \mathbf{x}_2(t)), \\ \dot{x}_2(t) &= -\sin(20\pi t)x_1 + (-0.8 - \cos(20\pi t))x_2(t) + f_2(t, \mathbf{x}_2(t)), \\ \dot{x}_3(t) &= (-0.8 + \cos(20\pi t))x_3(t) - \sin(20\pi t)x_4 + f_3(t, \mathbf{x}_1(t)), \\ \dot{x}_4(t) &= -\sin(20\pi t)x_3 + (-0.8 - \cos(20\pi t))x_4(t) + f_4(t, \mathbf{x}_1(t)),\end{aligned} \tag{7.1}$$

where $x_i \in \mathbb{R}$, $\mathbf{x} = (x_1, x_2, x_3, x_4)^{\mathrm{T}}$, $\mathbf{x}_1 = (x_1, x_2)^{\mathrm{T}}$, $\mathbf{x}_2 = (x_3, x_4)^{\mathrm{T}}$ and the functions $\mathbf{f}_1 = (f_1, f_2)^{\mathrm{T}}$, $\mathbf{f}_2 = (f_3, f_4)^{\mathrm{T}}$ satisfy the estimates

$$\|\mathbf{f}_1(t, \mathbf{x}_2)\| \leq \widehat{f}_{12}\|\mathbf{x}_2\|, \quad \|\mathbf{f}_2(t, \mathbf{x}_1)\| \leq \widehat{f}_{21}\|\mathbf{x}_1\|.$$

If $N = 1$, $P_{0i} = I$, $i = 1, 2$, then the matrices Γ_0 and Δ have the form

$$\Gamma = \begin{pmatrix} 0.6432 & 1.5834\widehat{f}_{12} \\ 1.5834\widehat{f}_{21} & 0.64312 \end{pmatrix},$$

$$\Delta = 0.9231I.$$

The sufficient conditions for the asymptotic stability of the equilibrium state $\mathbf{x} = 0$ of the nonlinear system (7.1) are of the form $\widehat{f}_{12}\widehat{f}_{21} < 0.00981$.

Let $\widehat{f}_{12} = \widehat{f}_{21} = 0.6$, $N = 10$, $P_{10} = P_{20} = I_2$, $a_{mi} = 2.2978$, $b_{mi} = 6.9282$, $\eta_{mi} = 3.5178$, $\delta_i = 0.8578$, $m = 0,\dots,9$, $i = 1,2$, $r_\sigma(\prod_{k=0}^{9} e^{\Gamma_k h}\Delta^{1/2}) = 0.99276 < 1$. Therefore, the equilibrium state $\mathbf{x} = 0$ of the nonlinear system (7.1) is asymptotically stable.

It should be noted that matrices

$$A_{11}(t) = A_{22}(t) = \begin{pmatrix} -0.8+\cos(20\pi t) & -\sin(20\pi t) \\ -\sin(20\pi t) & -0.8-\cos(20\pi t) \end{pmatrix}$$

have a spectrum $\sigma(A_{ii}(t)) = \{-1.8, 0.2\}$. Therefore, for each independent subsystem there is no Lyapunov function of quadratic form with a constant (independent of t) matrix.

8. Conclusion

The example given in Section 7 shows that the choice of the Lyapunov function for independent subsystems of a large-scale system is far from obvious. However, the proposed algorithm allows one to construct Lyapunov functions for independent subsystems and obtain sufficient stability conditions for large-scale system. These conditions are analogous to the well-known small-gain conditions in the theory of stability of coupled systems. Algorithm for constructing the Lyapunov function can be easily implemented using modern computing tools such as Maple or Matlab. It is important to note that the implementation of this algorithm requires calculating of a finite number of matrices and checking a finite number of inequalities. For high frequency systems, it requires little computation. In what follows, it is of interest to generalize the results obtained for some classes of infinite-dimensional systems and to study the possibility of abandoning the requirement for asymptotic stability of independent subsystems.

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