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WELL-POSEDNESS CHARACTERIZATIONS FOR SYSTEM OF MIXED HEMIVARIATIONAL INEQUALITIES

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Abstract. In this paper, we concern with the concept of well-posedness and well-posedness in generalized sense for a system of mixed hemivariational inequality (SMHVI) with perturbations. We establish several metric characterizations and equivalent conditions of well-posedness for SMHVI. Our main results improve and extend some announced work in the literature.

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1. INTRODUCTION

The theory of well-posedness in optimization problems and various comprehensive problems correlated to it such as variational inequalities, fixed point problems, equilibrium problems, saddle point problems, and inclusion problems has been paid deep impact attention by researchers due to their essential contributions to physics, mechanics, engineering, and economics. Tykhonov [20] first proposed the concept of well-posedness for the unconstrained global minimization problems, i.e., a minimization problem is well-posed, if it possesses a unique minimizer and the limit of every subsequence of the minimizing sequence is the unique minimizer. Indeed, an optimization problem may not have a unique solution. Hence, the notion of generalized well-posedness was introduced, which enhances the precise solution of the problem, under some mild conditions, by ensuring the convergence of every approximating solutions to the exact solution. Accordingly, the approximating approach captured the concept of well-posedness that plays a vital aspect in well-posedness theory. For more related research works concerning to the well-posedness results

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of optimization problems, we refer to [11, 14]. Later, the notion of well-posedness turned toward variational inequalities in different aspects. Lucchetti and Patrone [17] first studied and developed the concept of well-posedness for variational inequalities by Ekeland's technique. From then on, researchers are dedicated to investigate the well-posedness for variational inequalities and build their links with the well-posedness of inclusion and minimization problems. Lignola [13] and Lignola and Morgan [14] coined the notion of $L-$well-posedness for quasivariational inequalities and parametric well-posedness for variational inequalities, respectively. Ceng et al. [3] explored various necessary and sufficient conditions of the well-posedness for the mixed quasivariational-like inequalities. Meanwhile, Fang et al. [8, 9] demonstrated different forms of well-posedness results for mixed variational inequalities, which were further generalized by Ceng et al. [2]. In the works of [4, 8, 9], the authors discussed various necessary and sufficient metric characterizations of well-posedness. There, the authors also elaborated on the corresponding inclusion problems as well as the corresponding fixed point problems and established the equivalent relations between the well-posedness of mixed variational inequalities and their corresponding inclusion problems.

In the last decade, the study of well-posedness devoted special attention to hemivariational inequality problem (HVIP), which is a powerful and important generalization of variational inequalities. The HVIP was first investigated and explored by Panagiotopoulos [19] at the beginning of 1980s to formulate several classes of unilateral mechanical problems with nonsmooth and nonconvex energy functional. The HVIP is described through Clarke's generalized gradient for nonconvex and nondifferentiable functions. It is straightforward to see that if the involved functions are convex, then the HVIP weakens to the variational inequality (see [6]). Consequently, several extensive well-posedness results (for e.g. see, [2, 5, 10, 15, 22–24]) have been investigated and widely discussed for HVIP. Goeleven and Mentagui [10] first considered the well-posedness consequences for HVIP. Xiao et al. [23] discussed the well-posedness characterizations for a class of HVIP. Recently, Xiao et al. [24] also explored the well-posedness results for hemivariational inequalities by utilizing the notion of strongly relaxed monotonicity. By introducing $\alpha-\eta$ monotone mappings, Liu et al. [15] established $L-\alpha$ well-posedness for mixed quasivariational hemivariational inequality through Mosco convergence of sets. Xiao and Huang [22] proposed the well-posedness results for variational-hemivariational inequalities with perturbations. In 2013, Costea and Varga [7] considered the existence results for the system of hemivariational inequality and also derived the Nash generalized derivative points. Recently, Wang et al. [21] generalized the concept of well-posedness to a system of hemivariational inequalities in product Banach space. Besides, the authors in [21–23] also built the equivalence results between the well-posedness of the inequality problems and the well-posedness of its derived inclusion problems. Further, Ceng et al. [5]

studied the well-posedness results for system of time-dependent hemivariational inequalities. However, the research works based on the well-posedness of solutions for the system of hemivariational inequalities are quite limited due to the complexity of its structure. The aim of this work is to develop the theoretical framework on well-posedness for a new class of system of mixed hemivariational inequality with perturbations.

Motivated by the aforesaid research works on well-posedness, in this work, our aim is to introduce and generalize the concept of well-posedness for the SMHVI with perturbations, which includes the HVIP as a special case. Section 2 briefly revisits some preliminary definitions and notations, which will be used to achieve the main results. Section 3 formulates the SMHVI with perturbations and the concept of approximating sequence as well as strong (resp., weak) well-posedness. Section 4 explores the necessary theorems and the metric characterizations for the well-posedness of SMHVI.

2. Preliminaries

Let X be a Banach space with dual space X^*. We signify $\langle \cdot,\cdot\rangle_{X^*\times X}$ as the duality pairing between X^* and X. We denote $\to, \rightharpoonup$ and $\xrightarrow{w^*}$ as the strong, weak and weak* convergence, respectively, in the appointed space.

Definition 1 ([18, Definition 1.31])**.** A sequence $\{x_n\} \subset X$ is weakly convergent iff there exists $x \in X$ s.t.

$$\langle x^*, x_n\rangle_{X^*\times X} \to \langle x^*, x\rangle_{X^*\times X}, \qquad \forall\, x^* \in X^*.$$

Definition 2 ([18, Definition 1.42])**.** A sequence of functional $\{x_n^*\} \subset X^*$ is called weakly* convergent to $x^* \in X^*$ iff

$$\langle x_n^*, x\rangle_{X^*\times X} \to \langle x^*, x\rangle_{X^*\times X}, \qquad \forall\, x \in X.$$

Theorem 1 ([18, Proposition 1.37 and Proposition 1.45])**.** *If $\{x_n\} \subset X$ and $\{x_n^*\} \subset X^*$ with $x_n \to x$ (resp. $x_n \rightharpoonup x$) in X and $x_n^* \xrightarrow{w^*} x^*$ (resp. $x_n^* \to x^*$) in X^*, then*

$$\langle x_n^*, x_n\rangle_{X^*\times X} \to \langle x^*, x\rangle_{X^*\times X}.$$

A function $\phi\colon X \to \mathbb{R}\cup\{\infty\}$ is called lower semicontinuous (l.s.c.) if it fulfills

$$\phi(x) \le \liminf_{n\to\infty} \phi(x_n)$$

for all sequence $x_n \in X$ with $x_n \to x$.

Definition 3 ([21, Definition 2.13])**.** We say that a functional $J\colon X \to \mathbb{R}$ is locally Lipschitz, if for every $z \in X$, there exists a neighborhood V of z and a constant $K_z > 0$ s.t. $|J(x) - J(y)| \le K_z \|x - y\|_X$ for all $x, y \in V$.

Definition 4 ([21, Definition 2.16])**.** Let $J\colon X\to\mathbb{R}$ be a locally Lipschitz functional. We denote $J^{\circ}(x;y)$ and $\bar{\partial}J(x)$ as the Clarke's generalized directional derivative at the point $x\in X$ in the direction $y\in X$ and Clarke's generalized gradient of J at $x\in X$, respectively, which are defined as:

$$J^{\circ}(x;y)=\limsup_{w\to x,\varepsilon\to 0^{+}}\frac{J(w+\varepsilon y)-J(w)}{\varepsilon},$$

and $\bar{\partial}J(x)=\{\xi\in X^{*}:\langle\xi,y\rangle_{X^{*}\times X}\leq J^{\circ}(x;y),\forall\, y\in X\}$, respectively.

Lemma 1 ([6, Proposition 2.1.1 and Proposition 2.1.2])**.** *Let $J\colon X\to\mathbb{R}$ be a locally Lipschitz functional of rank L. Then*

(i) the function $y\to J^{\circ}(x;y)$ is finite, homogenous, subadditive on X and satisfies $|J^{\circ}(x;y)|\leq L\|y\|_{X}$;
(ii) the functional $(x,y)\to J^{\circ}(x,y)$ is upper semicontinuous;
(iii) $J^{\circ}(x;-y)=(-J)^{\circ}(x;y)$;
(iv) for each $x\in X,\bar{\partial}J(x)$ is a nonempty, convex, weak*-compact subset of X^{*} and $\|\xi\|_{X^{*}}\leq L$ for all $\xi\in\bar{\partial}J(x)$;
(v) for each $y\in X$, one has $J^{\circ}(x;y)=\max_{\xi\in\bar{\partial}J(x)}\langle\xi,y\rangle_{X^{*}\times X}$.

Definition 5 ([22, Definition 2.3])**.** The measure of non-compactness for the set $D\subset X$ is defined by

$$\mu(D)=\inf\left\{\gamma>0:D\subset\bigcup_{i=1}^{n}D_{i},\ \text{diam}|D_{i}|<\gamma,\ i=1,2,\ldots,n\right\},$$

where diam$|D_i|$ denotes the diameter of the set D_i.

If D is a bounded subset of X then $\mu(D)=0$ iff $\overline{D}$ is compact. Also, if D_1 is a subset of D_2 then $\mu(D_1)\leq\mu(D_2)$.

Definition 6 ([16, Definition 8.1.7])**.** Let C and D be two nonempty subsets of X. The Hausdroff distance $\mathscr{H}(\cdot,\cdot)$ between C and D is defined by

$$\mathscr{H}(C,D)=\max\{e(C,D),e(D,C)\},$$

where $e(C,D)=\sup_{c\in C}d(c,D)$ with $d(c,D)=\inf_{d\in D}\|c-d\|_{X}$.

Here, we want to emphasize that if $\{C_n\}$ be a sequence of nonempty subsets of X. Then we say that C_n converges to C with respect to $\mathscr{H}(\cdot,\cdot)$ iff $\mathscr{H}(C_n,C)\to 0$.

Definition 7 ([22, Definition 2.1 and Definition 2.2])**.** Let $G\colon X\to X^{*}$ be a single-valued operator, then G is

(i) hemicontinuous, iff for all $x,y\in X$, the function $\varepsilon\to\langle G(x+\varepsilon(y-x)),y-x\rangle_{X^{*}\times X}$ from $[0,1]$ into R is continuous at 0^{+};
(ii) monotone, iff $\langle G(x)-G(y),x-y\rangle_{X^{*}\times X}\geq 0,\ \forall\, x,y\in X$.

3. Formulation of the problem and approximating sequence

Let $k \in \mathbb{N}$. For each $i \in \{1,\dots,k\}$, let $(X_i, \|\cdot\|_{X_i})$ be a real Banach space with dual space $(X_i^*, \|\cdot\|_{X_i^*})$. It is well-known that $X_1 \times \cdots \times X_i \times \cdots \times X_k$ is also a Banach space with the product norm $\|x\| = \sum_{i=1}^{k} \|x_i\|_{X_i}$, $\forall\, x = (x_1,\dots,x_i,\dots,x_k) \in X_1 \times \cdots \times X_i \times \cdots \times X_k$. For $i \in \{1,\dots,k\}$, let $A_i\colon X_1 \times \cdots \times X_i \times \cdots \times X_k \to X_i^*$ be a mapping (nonlinear), $S_i\colon X_1 \times \cdots \times X_i \times \cdots \times X_k \to X_i^*$ be a perturbation, $\phi_i\colon X_i \to \mathbb{R} \cup \{\infty\}$ be a proper, convex, and l.s.c. functional, $f_i \in X_i^*$, and $J_i\colon X_i \to \mathbb{R}$ be a locally Lipschitz functional. The effective domain of ϕ_i is specified as:

$$\mathrm{dom}(\phi_i) = \{x_i \in X_i\colon \phi_i(x_i) < \infty\} \neq \varnothing,$$

for $i \in \{1,\dots,k\}$.

In this work, we are going to establish the well-posedness of the following SMHVI, which consists in finding $x = (x_1,\dots,x_i,\dots,x_k) \in X_1 \times \cdots \times X_i \times \cdots \times X_k$ s.t.

$$\text{(SMHVI)} \begin{cases} \langle A_1(x) + S_1(x) - f_1, y_1 - x_1\rangle_{X_1^* \times X_1} + J_1^\circ(x_1; y_1 - x_1) \\ +\phi_1(y_1) - \phi_1(x_1) \geq 0, \qquad \forall\, y_1 \in X_1, \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ \langle A_i(x) + S_i(x) - f_i, y_i - x_i\rangle_{X_i^* \times X_i} + J_i^\circ(x_i; y_i - x_i) \\ +\phi_i(y_i) - \phi_i(x_i) \geq 0, \qquad \forall\, y_i \in X_i, \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ \langle A_k(x) + S_k(x) - f_k, y_k - x_k\rangle_{X_k^* \times X_k} + J_k^\circ(x_k; y_k - x_k) \\ +\phi_k(y_k) - \phi_k(x_k) \geq 0, \qquad \forall\, y_k \in X_k, \end{cases}$$

where $J_i^\circ(x_i; y_i - x_i)$ is the Clarke's generalized directional derivative at $x_i \in X_i$ in the direction $y_i - x_i \in X_i$. We denote θ_i as the zero element of X_i^* for $i \in \{1,\dots,k\}$ and Q as the set of all solutions to SMHVI.

If $k = 1$, then the SMHVI weakens to the variational-hemivariational inequality, whose well-posedness was considered by Xiao and Huang [22]. Furthermore, if $k = 1$ with $S_1 = \theta_1$ and $\phi_1 = 0$, then the SMHVI is equivalent to the hemivariational inequality, whose well-posedness was given by Xiao et al. [23]. Moreover, letting $k = 1$ with $S_1 = \theta_1, J_1 = 0$, and $\phi_1 = 0$, SMHVI collapses to the classical variational inequality, i.e.,

$$\langle A_1(x_1) - f_1, y_1 - x_1\rangle \geq 0, \qquad \forall\, y_1 \in X_1.$$

Also, if $k = 1$ with $A_1 = \theta_1, S_1 = \theta_1, J_1 = 0$ and $f_1 = 0$ then SMHVI reduces to the global minimization problem, i.e.,

$$\min_{x_1 \in X_1} \phi(x_1).$$

To investigate the well-posedness of SMHVI, let us review some primitive definitions and results.

Definition 8. The sequence $x^n = (x_1^n, \ldots, x_i^n, \ldots, x_k^n) \in X_1 \times \cdots \times X_i \times \cdots \times X_k$ is an approximating sequence for SMHVI, if there exists a sequence $\{\rho_n\} \subset \mathbb{R}_+$ with $\rho_n \to 0$ as $n \to \infty$ s.t.

$$\begin{cases} \langle A_1(x^n) + S_1(x^n) - f_1, y_1 - x_1^n \rangle_{X_1^* \times X_1} + J_1^\circ(x_1^n; y_1 - x_1^n) + \phi_1(y_1) - \phi_1(x_1^n) \\ \geq -\rho_n \|y_1 - x_1^n\|_{X_1}, \qquad \forall\, y_1 \in X_1, \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ \langle A_i(x^n) + S_i(x^n) - f_i, y_i - x_i^n \rangle_{X_i^* \times X_i} + J_i^\circ(x_i^n; y_i - x_i^n) + \phi_i(y_i) - \phi_i(x_i^n) \\ \geq -\rho_n \|y_i - x_i^n\|_{X_i}, \qquad \forall\, y_i \in X_i, \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ \langle A_k(x^n) + S_k(x^n) - f_k, y_k - x_k^n \rangle_{X_k^* \times X_k} + J_k^\circ(x_k^n; y_k - x_k^n) + \phi_k(y_k) - \phi_k(x_k^n) \\ \geq -\rho_n \|y_k - x_k^n\|_{X_k}, \qquad \forall\, y_k \in X_k. \end{cases} \tag{3.1}$$

Definition 9. The SMHVI is said to be strongly (resp., weakly) well-posed iff it has a unique solution and every approximating sequence converges strongly (resp., weakly) to the unique solution.

Definition 10. The SMHVI is said to be strongly (resp., weakly) well-posed in generalized sense iff Q is nonempty and every approximating sequence has a subsequence which converges strongly (resp., weakly) to some point of Q.

Remark 1. Definition 9 and Definition 10 extend the definition of well-posedness studied in [8, 10, 21–24]. Also, it is worth noting that strong well-posedness (in the generalized sense) implies the weak well-posedness (in the generalized sense), however, the converse is not true in general.

For any $\rho > 0$, we define the sets $\Upsilon(\rho)$ and $\Psi(\rho)$ in $X_1 \times \cdots \times X_i \times \cdots \times X_k$ as:

$$\Upsilon(\rho) = \begin{cases} (x_1, \ldots, x_i, \ldots, x_k) \in X_1 \times \cdots \times X_i \times \cdots \times X_k : \\ \langle A_1(x_1, \ldots, x_i, \ldots, x_k) + S_1(x_1, \ldots, x_i, \ldots, x_k) - f_1, y_1 - x_1 \rangle_{X_1^* \times X_1} \\ +J_1^\circ(x_1; y_1 - x_1) + \phi_1(y_1) - \phi_1(x_1) \geq -\rho \|y_1 - x_1\|_{X_1}, \qquad \forall\, y_1 \in X_1, \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ \langle A_i(x_1, \ldots, x_i, \ldots, x_k) + S_i(x_1, \ldots, x_i, \ldots, x_k) - f_i, y_i - x_i \rangle_{X_i^* \times X_i} \\ +J_i^\circ(x_i; y_i - x_i) + \phi_i(y_i) - \phi_i(x_i) \geq -\rho \|y_i - x_i\|_{X_i}, \qquad \forall\, y_i \in X_i, \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ \langle A_k(x_1, \ldots, x_i, \ldots, x_k) + S_k(x_1, \ldots, x_i, \ldots, x_k) - f_k, y_k - x_k \rangle_{X_k^* \times X_k} \\ +J_k^\circ(x_k; y_k - x_k) + \phi_k(y_k) - \phi_k(x_k) \geq -\rho \|y_k - x_k\|_{X_k}, \qquad \forall\, y_k \in X_k, \end{cases}$$

and

$$\Psi(\rho)=\begin{cases}(x_1,\dots,x_i,\dots,x_k)\in X_1\times\cdots\times X_i\times\cdots\times X_k:\\ \langle A_1(y_1,\dots,x_i,\dots,x_k)+S_1(x_1,\dots,x_i,\dots,x_k)-f_1,y_1-x_1\rangle_{X_1^*\times X_1}\\ +J_1^\circ(x_1;y_1-x_1)+\phi_1(y_1)-\phi_1(x_1)\geq-\rho\|y_1-x_1\|_{X_1}, \qquad \forall\, y_1\in X_1,\\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\\ \langle A_i(x_1,\dots,y_i,\dots,x_k)+S_i(x_1,\dots,x_i,\dots,x_k)-f_i,y_i-x_i\rangle_{X_i^*\times X_i}\\ +J_i^\circ(x_i;y_i-x_i)+\phi_i(y_i)-\phi_i(x_i)\geq-\rho\|y_i-x_i\|_{X_i}, \qquad \forall\, y_i\in X_i,\\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\\ \langle A_k(x_1,\dots,x_i,\dots,y_k)+S_k(x_1,\dots,x_i,\dots,x_k)-f_k,y_k-x_k\rangle_{X_k^*\times X_k}\\ +J_k^\circ(x_k;y_k-x_k)+\phi_k(y_k)-\phi_k(x_k)\geq-\rho\|y_k-x_k\|_{X_k}, \qquad \forall\, y_k\in X_k.\end{cases}$$

Motivated by Remark 2.4 of [1], we have the following results.

Theorem 2. *If SMHVI is strongly well-posed in generalized sense and admits a unique solution then SMHVI is strongly well-posed.*

Proof. Let $x^n=(x_1^n,\dots,x_i^n,.,.,x_k^n)\in X_1\times\cdots\times X_i\times\cdots\times X_k$ be an approximating sequence for SMHVI. Accordingly, there exists $\{\rho_n\}\subset\mathbb{R}_+$ with $\{\rho_n\}\to 0$ as $n\to\infty$ and $x^n\in\Upsilon(\rho_n)\ \forall\, n\in\mathbb{N}$. Since SMHVI is strongly well-posed in generalized sense and has a unique solution $x=(x_1,\dots,x_i,\dots,x_k)$, then $\{x^n\}$ has a subsequence $\{x^{n_l}\}$, which converges strongly to x. It suffices to prove that $\{x^n\}$ converges strongly to x. Arguing by contradiction, let $x^n\not\to x$ which leads to $\|x^n-x\|\not\to 0$ as $n\to\infty$. Thus there exists $\eta>0$ and a subsequence $\{x^{m_l}\}$ of $\{x^n\}$ s.t.

$$\|x^{m_l}-x\|\geq\eta. \tag{3.2}$$

Now, it is simple to see that $\{x^{m_l}\}$ is an approximating sequence for SMHVI. Since SMHVI is strongly well-posed in generalized sense and also has a unique solution x, therefore, there exists a subsequence $\{x^{m_{l_q}}\}$ of $\{x^{m_l}\}$ s.t. $x^{m_{l_q}}\to x$ as $q\to\infty$, i.e.,

$$\|x^{m_{l_q}}-x\|<\frac{\eta}{2}.$$

Letting $m_{l_q}=m_{l'}$, we obtain

$$\|x^{m_{l'}}-x\|<\frac{\eta}{2},$$

which contradicts (3.2). This confirms that $x^n\to x$. □

Let $A_i\colon X_1\times\cdots\times X_i\times\cdots\times X_k\to X_i^*$. We say A_i is norm to weak*-sequentially continuous iff $x^n=(x_1^n,\dots,x_i^n,\dots,x_k^n)\to x=(x_1,\dots,x_i,\dots,x_k)$ in $X_1\times\cdots\times X_i\times\cdots\times X_k$ implies $A_i(x^n)\xrightarrow{w^*}A_i(x)$ in X_i^*. Further, we also need the following assumptions throughout this paper. For each $i\in\{1,\dots,k\}$,

(H_m) $A_i\colon X_1\times\cdots\times X_i\times\cdots\times X_k\to X_i^*$ is monotone with respect to X_i,
(H_a) $A_i\colon X_1\times\cdots\times X_i\times\cdots\times X_k\to X_i^*$ is norm to weak*-sequentially continuous,

- (H_j) $J_i \colon X_i \to \mathbb{R}$ is locally Lipschitz functional,
- (H_ϕ) $\phi_i \colon X_i \to \mathbb{R} \cup \{\infty\}$ is proper, convex and l.s.c. map with effective domain: $\operatorname{dom}(\phi_i) = \{x_i \in X_i : \phi_i(x_i) < \infty\} \neq \varnothing$,
- (H_c) $S_i \colon X_i \times \cdots \times X_i \times \cdots \times X_k \to X_i^*$ is continuous on $X_1 \times \cdots \times X_i \times \cdots \times X_k$,
- (H_h) $A_i \colon X_1 \times \cdots \times X_i \times \cdots \times X_k \to X_i^*$ is hemicontinuous with respect to X_i.

We end this section by addressing the following equivalence results between the sets $\Upsilon(\rho)$ and $\Psi(\rho)$ for any $\rho > 0$.

Lemma 2. *For each $i \in \{1,\ldots,k\}$, suppose that $A_i \colon X_1 \times \cdots \times X_i \times \cdots \times X_k \to X_i^*$ satisfies the hypothesis (H_m) and (H_h). Further assume that for each $i \in \{1,\ldots,k\}$, $\phi_i \colon X_i \to \mathbb{R} \cup \{\infty\}$ is a proper and convex functional, and $J_i \colon X_i \to \mathbb{R}$ satisfies the hypothesis (H_j). Then $\Upsilon(\rho) = \Psi(\rho)$ for any $\rho > 0$.*

Proof. Let $x = (x_1,\ldots,x_i,\ldots,x_k) \in X_1 \times \cdots \times X_i \times \cdots \times X_k$ and $x \in \Upsilon(\rho)$. Thus, for each $i \in \{1,\ldots,k\}$, we have

$$\langle A_i(x_1,\ldots,x_i,\ldots,x_k) + S_i(x_1,\ldots,x_i,\ldots,x_k) - f_i, y_i - x_i \rangle_{X_i^* \times X_i} + J_i^\circ(x_i; y_i - x_i) \\ + \phi_i(y_i) - \phi_i(x_i) \geq -\rho \|y_i - x_i\|_{X_i}, \qquad \forall\, y_i \in X_i. \tag{3.3}$$

Since A_i is monotone with respect to X_i (by H_m). We have

$$\langle A_i(x_1,\ldots,y_i,\ldots,x_k) - A_i(x_1,\ldots,x_i,\ldots,x_k), y_i - x_i \rangle_{X_i^* \times X_i} \geq 0, \qquad \forall\, y_i, x_i \in X_i,$$

i.e.,

$$\langle A_i(x_1,\ldots,y_i,\ldots,x_k), y_i - x_i \rangle_{X_i^* \times X_i} \\ \geq \langle A_i(x_1,\ldots,x_i,\ldots,x_k), y_i - x_i \rangle_{X_i^* \times X_i}, \qquad \forall\, x_i, y_i \in X_i. \tag{3.4}$$

By combining (3.3) and (3.4), we get

$$\langle A_i(x_1,\ldots,y_i,\ldots,x_k) + S_i(x_1,\ldots,x_i,\ldots,x_k) - f_i, y_i - x_i \rangle_{X_i^* \times X_i} + J_i^\circ(x_i; y_i - x_i) \\ + \phi_i(y_i) - \phi_i(x_i) \geq -\rho \|y_i - x_i\|_{X_i}, \qquad \forall\, y_i \in X_i.$$

i.e., $x = (x_1,\ldots,x_i,\ldots,x_k) \in \Psi(\rho)$, so, $\Upsilon(\rho) \subset \Psi(\rho)$.

Conversely, let $x = (x_1,\ldots,x_i,\ldots,x_k) \in X_1 \times \cdots \times X_i \times \cdots \times X_k$ and $x \in \Psi(\rho)$ for any $\rho > 0$. So, we yield that for each $i \in \{1,\ldots,k\}$

$$\langle A_i(x_1,\ldots,y_i,\ldots,x_k) + S_i(x_1,\ldots,x_i,\ldots,x_k) - f_i, y_i - x_i \rangle_{X_i^* \times X_i} + J_i^\circ(x_i; y_i - x_i) \\ + \phi_i(y_i) - \phi_i(x_i) \geq -\rho \|y_i - x_i\|_{X_i}, \qquad \forall\, y_i \in X_i. \tag{3.5}$$

Now for any $w = (w_1,\ldots,w_i,\ldots,w_k) \in X_1 \times \cdots \times X_i \times \cdots \times X_k$ and $t \in (0,1)$, let

$$z_i = x_i + t(w_i - x_i) = t w_i + (1-t) x_i \in X_i, \qquad i \in \{1,\ldots,k\}.$$

Inserting z_i in (3.5), we have that for each $i \in \{1,\ldots,k\}$

$$\langle A_i(x_1,\ldots,x_i + t(w_i - x_i),\ldots,x_k) + S_i(x_1,\ldots,x_i,\ldots,x_k) - f_i, t(w_i - x_i) \rangle_{X_i^* \times X_i} \\ + J_i^\circ(x_i; t(w_i - x_i)) + \phi_i(x_i + t(w_i - x_i)) - \phi_i(x_i) \geq -\rho t \|w_i - x_i\|_{X_i}.$$

By the convexity of ϕ_i and Lemma 1, we estimate that for each $i \in \{1, \ldots, k\}$

$$\langle A_i(x_1, \ldots, x_i + t(w_i - x_i), \ldots, x_k) + S_i(x_1, \ldots, x_i, \ldots, x_k) - f_i, w_i - x_i \rangle_{X_i^* \times X_i} + J_i^\circ(x_i; w_i - x_i) + \phi_i(w_i) - \phi_i(x_i) \geq -\rho \|w_i - x_i\|_{X_i}. \tag{3.6}$$

From (H_h), A_i is hemicontinuous with respect to X_i for each $i \in \{1, \ldots, k\}$. Taking the limit as $t \to 0$ in (3.6), we achieve

$$\langle A_i(x_1, \ldots, x_i, \ldots, x_k) + S_i(x_1, \ldots, x_i, \ldots, x_k) - f_i, w_i - x_i \rangle_{X_i^* \times X_i} + J_i^\circ(x_i; w_i - x_i) + \phi_i(w_i) - \phi_i(x_i) \geq -\rho \|w_i - x_i\|_{X_i}, \qquad \forall\, w_i \in X_i.$$

Hence, the arbitrariness of w implies that $x \in \Upsilon(\rho)$. Thus, $\Psi(\rho) \subset \Upsilon(\rho)\ \forall\, \rho > 0$. □

Remark 2. Lemma 2 improves and generalizes Lemma 3.1 in [22] and Lemma 3.1 in [23].

4. Main results

This section presents the metric characterizations and deduces some criteria, under which, SMHVI is strongly well-posed.

Proposition 1. *For each $i \in \{1, .., k\}$, let X_i be a real Banach space with dual space X_i^*. Further, assume that for each $i \in \{1, \ldots, k\}$, $A_i \colon X_1 \times \cdots \times X_i \times \cdots \times X_k \to X_i^*$, $S_i \colon X_1 \times \cdots \times X_i \times \cdots \times X_k \to X_i^*$, $J_i \colon X_i \to \mathbb{R}$ and $\phi_i \colon X_i \to \mathbb{R} \cup \{\infty\}$ satisfy the hypothesis (H_a), (H_c), (H_j) and (H_ϕ), respectively. Then, $\Psi(\rho)$ is closed in $X_1 \times \cdots \times X_i \times \cdots \times X_k$ for any $\rho > 0$.*

Proof. Let $x^n = (x_1^n, \ldots, x_i^n, \ldots, x_k^n) \in \Psi(\rho)$ be a sequence s.t. $(x_1^n, \ldots, x_i^n, \ldots, x_k^n) \to x = (x_1, \ldots, x_i, \ldots, x_k)$. Then for each $i \in \{1, \ldots, k\}$,

$$\langle A_i(x_1^n, \ldots, y_i, \ldots, x_k^n) + S_i(x_1^n, \ldots, x_i^n, \ldots, x_k^n) - f_i, y_i - x_i^n \rangle_{X_i^* \times X_i} + J_i^\circ(x_i^n; y_i - x_i^n) + \phi_i(y_i) - \phi_i(x_i^n) \geq -\rho \|y_i - x_i^n\|_{X_i}, \qquad \forall\, y_i \in X_i. \tag{4.1}$$

Since A_i, S_i, J_i, and ϕ_i, satisfy $(H_a), (H_c), (H_j)$ and (H_ϕ), respectively. Thus, for each $i \in \{1, \ldots, k\}$, by using (H_a) and Theorem 1, we obtain that

$$\langle A_i(x_1^n, \ldots, y_i, \ldots, x_k^n), y_i - x_i^n \rangle_{X_i^* \times X_i} \to \langle A_i(x_1, \ldots, y_i, \ldots, x_k), y_i - x_i \rangle_{X_i^* \times X_i}. \tag{4.2}$$

Also, taking the upper limit as $n \to \infty$ in (4.1), it follows from $(H_c), (H_j), (H_\phi)$ and (4.2) that for each $i \in \{1, 2 \ldots, k\}$

$$\langle A_i(x_1, \ldots, y_i, \ldots, x_k) + S_i(x_1, \ldots, x_i, \ldots, x_k) - f_i, y_i - x_i \rangle_{X_i^* \times X_i} + J_i^\circ(x_i; y_i - x_i) + \phi_i(y_i) - \phi_i(x_i) \geq -\rho \|y_i - x_i\|_{X_i}, \qquad \forall\, y_i \in X_i,$$

which confirms that $x = (x_1, \ldots, x_i, \ldots, x_k) \in \Psi(\rho)$. Therefore, $\Psi(\rho)$ is closed. □

The following result can be obtained by applying Lemma 2 and Proposition 1.

Corollary 1. *For each $i \in \{1,..,k\}$, let X_i be real Banach space with dual space X_i^* and assume that $A_i\colon X_1 \times \cdots \times X_i \times \cdots \times X_k \to X_i^*$ satisfies the hypothesis $(H_a),(H_h)$ and (H_m). Further, suppose that for each $i \in \{1,\ldots,k\}, S_i\colon X_1 \times \cdots \times X_i \times \cdots \times X_k \to X_i^*, J_i\colon X_i \to \mathbb{R}$ and $\phi_i\colon X_i \to \mathbb{R} \cup \{\infty\}$ satisfy the hypothesis (H_c), (H_j) and (H_ϕ), respectively. Then $\Upsilon(\rho)$ is closed for any $\rho > 0$.*

We now present the main results concerning to the strongly well-posedness of SMHVI.

Theorem 3. *For each $i \in \{1,..,k\}$, let X_i be a real Banach space with dual space X_i^* and assume that $A_i\colon X_1 \times \cdots \times X_i \times \cdots \times X_k \to X_i^*$ satisfies the hypothesis (H_a), (H_m) and (H_h). Further, suppose that for each $i \in \{1,\ldots,k\}, S_i\colon X_1 \times \cdots \times X_i \times \cdots \times X_k \to X_i^*, J_i\colon X_i \to \mathbb{R}$ and $\phi_i\colon X_i \to \mathbb{R} \cup \{\infty\}$ satisfy the hypothesis (H_c), (H_j) and (H_ϕ), respectively. Then SMHVI is strongly well-posedness iff*

$$\Upsilon(\rho) \neq \varnothing \quad \forall\, \rho > 0 \quad \textit{and} \quad diam(\Upsilon(\rho)) \to 0 \quad \textit{as}\ \rho \to 0.$$

Proof. Let SMHVI be strongly well-posed. Then SMHVI has a unique solution, say, $x = (x_1,\ldots,x_i,\ldots,x_k)$ and hence $x \in \Upsilon(\rho)\ \forall\, \rho > 0$, which relates to $Q \neq \varnothing$ and $\Upsilon(\rho) \neq \varnothing\ \forall\, \rho > 0$. Also, suppose that $diam(\Upsilon(\rho)) \not\to 0$ as $\rho \to 0$. Thus, there exists $\eta > 0$ and a sequence $\{\rho_n\} \subset \mathbb{R}_+$ with $\rho_n \to 0$ as $n \to \infty$ s.t. for each $n \in \mathbb{N}$, there exists $w^n = (w_1^n,\ldots,w_i^n,\ldots,w_k^n)$ and $z^n = (z_1^n,\ldots,z_i^n,\ldots,z_k^n)$ with $w^n, z^n \in \Upsilon(\rho_n)$ s.t.

$$\|w^n - z^n\| > \eta. \tag{4.3}$$

On the other hand, since $w^n, z^n \in \Upsilon(\rho_n)$, we have that the sequences $\{w^n\}$ and $\{z^n\}$ are both approximating sequences for SMHVI. Further, SMHVI is strongly well-posed, so the sequence $\{w^n\}$ and $\{z^n\}$ converges strongly to $x \in Q$, which contradicts (4.3).

In order to prove the sufficiency, let $x^n = (x_1^n,\ldots,x_i^n,\ldots,x_k^n)$ be an approximating sequence for SMHVI. Thus, there exists a sequence $\{\rho_n\} \subset \mathbb{R}_+$ with $\rho_n \to 0$ s.t. (3.1) holds, i.e., $x^n \in \Upsilon(\rho_n)$. Since $diam(\Upsilon(\rho_n)) \to 0$ as $\rho_n \to 0$, it is easy to draw that $\{x^n\}$ is a Cauchy sequence in $X_1 \times \cdots \times X_i \times \cdots \times X_k$. Since $X_1 \times \cdots \times X_i \times \cdots \times X_k$ is complete, and so $\{x^n\}$ converges strongly to some point $x = (x_1,\ldots,x_i,\ldots,x_k) \in X_1 \times \cdots \times X_i \times \cdots \times X_k$. Using the hypothesis (H_a), (H_j), (H_c), (H_ϕ) and Lemma 1, it follows that for each $i \in \{1,\ldots,k\}$,

$$\begin{aligned}
&\langle A_i(x_1,\ldots,y_i,\ldots,x_k) + S_i(x_1,\ldots,x_i,\ldots,x_k) - f_i, y_i - x_i\rangle_{X_i^* \times X_i} + J_i^\circ(x_i; y_i - x_i) \\
&+ \phi_i(y_i) - \phi_i(x_i), \\
&\geq \limsup_{n\to\infty} \Big[\langle A_i(x_1^n,\ldots,y_i,\ldots,x_k^n) + S_i(x_1^n,\ldots,x_i^n,\ldots,x_k^n) - f_i, y_i - x_i^n\rangle_{X_i^* \times X_i} + \\
&\qquad J_i^\circ(x_i^n; y_i - x_i^n) + \phi_i(y_i) - \phi_i(x_i^n) \Big],
\end{aligned}$$

$$
\begin{aligned}
&\geq \limsup_{n\to\infty}\Bigg[\langle A_i(x_1^n,\ldots,x_i^n,\ldots,x_k^n)+S_i(x_1^n,\ldots,x_i^n,\ldots,x_k^n)-f_i,y_i-x_i^n\rangle_{X_i^*\times X_i}+\\
&\qquad J_i^{\circ}(x_i^n;y_i-x_i^n)+\phi_i(y_i)-\phi_i(x_i^n)\Bigg], \quad [\text{by} \quad (H_m)]\\
&\geq \limsup_{n\to\infty}\Bigg[-\rho_n\|y_i-x_i^n\|_{X_i}\Bigg], \quad [\text{by} \quad (3.1)],\\
&=0, \qquad \forall\, y_i\in X_i.
\end{aligned}
$$

Now, due to the convexity of $\phi_i, (H_h)$ and Lemma 1, we can derive (by invoking the analogous proof outline of Lemma 2) that

$$
\begin{aligned}
&\langle A_i(x_1,\ldots,x_i,\ldots,x_k)+S_i(x_1,\ldots,x_i,\ldots,x_k)-f_i,y_i-x_i\rangle_{X_i^*\times X_i}+J_i^{\circ}(x_i;y_i-x_i)\\
&+\phi_i(y_i)-\phi_i(x_i)\geq 0, \qquad \text{for each } i\in\{1,\ldots,k\},
\end{aligned}
$$

which means that $x\in Q$. For the completeness of the proof, let us assume by contradiction that the SMHVI has two distinct solutions, say, $x,z\in Q$. Since $Q\subset\Upsilon(\rho)\ \forall\,\rho>0$, so as $x,z\in\Upsilon(\rho)\ \forall\,\rho>0$. Also, since $diam(\Upsilon(\rho))\to 0$ as $\rho\to 0$, it is straightforward to deduce that

$$\|x-z\|\leq\ diam(\Upsilon(\rho))\to 0,$$

which contradicts $x\neq z$, and this confirms the uniqueness of the solution. □

Theorem 4. *For each $i\in\{1,..,k\}$, let X_i be a real Banach space with dual space X_i^* and assume that $A_i\colon X_1\times\cdots\times X_i\times\cdots\times X_k\to X_i^*$ satisfies the hypothesis (H_a), (H_h) and (H_m). Further, suppose that for each $i\in\{1,\ldots,k\}, S_i\colon X_1\times\cdots\times X_i\times\cdots\times X_k\to X_i^*, J_i\colon X_i\to\mathbb{R}$ and $\phi_i\colon X_i\to\mathbb{R}\cup\{\infty\}$ satisfy the hypothesis (H_c), (H_j) and (H_ϕ), respectively. Then SMHVI is strongly well-posedness in generalized sense iff*

$$\Upsilon(\rho)\neq\varnothing \quad \forall\,\rho>0 \quad and \quad \mu(\Upsilon(\rho))\to 0 \quad as\ \rho\to 0. \tag{4.4}$$

Proof. Assume that SMHVI is strongly well-posed in generalized sense. For any $\rho>0$, then $Q\neq\varnothing$ and $Q\subset\Upsilon(\rho)$. Moreover, we claim that the set Q is compact. For any sequence, say, $x^n\in X_1\times\cdots\times X_i\times\cdots\times X_k$ s.t. $x^n\in Q$ and so $x^n\in\Upsilon(\rho)$ for all $\rho>0$. For any $\rho>0$, it follows from $Q\subset\Upsilon(\rho)$ that $\{x^n\}$ is an approximating sequence for SMHVI. Since the SMHVI is strongly well-posed in generalized sense, therefore $\{x^n\}$ has a subsequence that converges to some point of the set Q, which indicates that the set Q is compact and hence $\mu(Q)=0$. Now, we claim that $\mu(\Upsilon(\rho))\to 0$. In fact, for any $\rho>0, Q\subset\Upsilon(\rho)$ implies that

$$\mathscr{H}(\Upsilon(\rho),Q)=max\{e(\Upsilon(\rho),Q),e(Q,\Upsilon(\rho))\}=e(\Upsilon(\rho),Q). \tag{4.5}$$

Also, from the compactness of Q and (4.5), we obtain

$$\mu(\Upsilon(\rho))\leq 2\mathscr{H}(\Upsilon(\rho),Q)+\mu(Q)\leq 2\mathscr{H}(\Upsilon(\rho),Q)=2e(\Upsilon(\rho),Q).$$

Hence, to prove $\mu(\Upsilon(\rho)) \to 0$ as $\rho \to 0$, it is enough to show that $e(\Upsilon(\rho), Q) \to 0$ as $\rho \to 0$. Arguing by contradiction, we assume that $e(\Upsilon(\rho), Q) \not\to 0$ as $\rho \to 0$. Thus, there exists a constant $\varepsilon > 0$ and a sequence $\{\rho_n\} \subset \mathbb{R}_+$ with $\rho_n \to 0$ as $n \to \infty$ and $w^n \in \Upsilon(\rho_n)$ satisfying

$$d(w^n, Q) > \varepsilon, \qquad \forall\, n \in \mathbb{N}. \tag{4.6}$$

Since $\{w^n\}$ is an approximating sequence for SMHVI and SMHVI is well-posed in generalized sense, therefore, there exists a subsequence $\{w^{n_l}\}$ of $\{w^n\}$, which converges to some point of Q. So we get

$$0 < \varepsilon < d(w^{n_l}, Q) \to 0,$$

which contradicts (4.6).

Conversely, assume that (4.4) holds ($\Upsilon(\rho) \neq \varnothing$). By Corollary 1, we have that $\Upsilon(\rho)$ is closed for all $\rho > 0$. Next, we claim that

$$Q = \bigcap_{\rho>0} \Upsilon(\rho).$$

It is evident that $Q \subset \bigcap_{\rho>0} \Upsilon(\rho)$. Conversely, let $x = (x_1, \dots, x_i, \dots, x_k) \in \bigcap_{\rho>0} \Upsilon(\rho)$. So, there exists $\rho_n > 0$ s.t. $\rho_n \to 0$, which leads to $x = (x_1, \dots, x_i, \dots, x_k) \in \Upsilon(\rho_n)$. Thus for each $i \in \{1, \dots, k\}$, we get

$$\begin{aligned} &\langle A_i(x) + S_i(x) - f_i, y_i - x_i \rangle_{X_i^* \times X_i} + J_i^\circ(x_i; y_i - x_i) + \phi_i(y_i) - \phi_i(x_i) \\ &\geq -\rho_n \|y_i - x_i\|_{X_i}, \qquad \forall\, y_i \in X_i. \end{aligned} \tag{4.7}$$

Letting $n \to \infty$ in (4.7), we get

$$\langle A_i(x) + S_i(x) - f_i, y_i - x_i \rangle_{X_i^* \times X_i} + J_i^\circ(x_i; y_i - x_i) + \phi_i(y_i) - \phi_i(x_i) \geq 0, \qquad \forall\, y_i \in X_i.$$

For any $z = (z_1, \dots, z_i, \dots, z_k) \in X_1 \times \cdots \times X_i \times \cdots \times X_k$ and $t \in (0,1)$, let $y_i = x_i + t(z_i - x_i) \in X_i, i \in \{1, \dots, k\}$. Now using the convexity of ϕ_i and Lemma 1, it is easy to draw that $x \in Q$ and hence $\bigcap_{\rho>0} \Upsilon(\rho) \subset Q$. This shows that $Q = \bigcap_{\rho>0} \Upsilon(\rho)$. Since $\mu(\Upsilon(\rho)) \to 0$ as $\rho \to 0$. By generalized Cantor theorem (see [12]), it follows that Q is nonempty and compact with $\Upsilon(\rho) \to Q$ as $\rho \to 0$ w.r.t. $\mathscr{H}$, i.e.,

$$\mathscr{H}(\Upsilon(\rho), Q) \to 0 \quad \text{as } \rho \to 0. \tag{4.8}$$

As,

$$e(\Upsilon(\rho), Q) = \mathscr{H}(\Upsilon(\rho), Q), \quad \text{for } \rho > 0.$$

Hence, from (4.8), we get

$$e(\Upsilon(\rho), Q) \to 0 \quad \text{as } \rho \to 0. \tag{4.9}$$

In order to complete the proof, let $x^n = (x_1^n, \dots, x_i^n, \dots, x_k^n) \in X_1 \times \cdots \times X_i \times \cdots \times X_k$ be an approximating sequence for SMHVI, thus there exists $\{\rho_n\} \subset \mathbb{R}_+$ with $\rho_n \to 0$ s.t. for each $i \in \{1, \dots, k\}$, we have

$$\begin{aligned}&\langle A_i(x^n) + S_i(x^n) - f_i, y_i - x_i^n \rangle_{X_i^* \times X_i} + J_i^\circ(x_i^n; y_i - x_i^n) + \phi_i(y_i) - \phi_i(x_i^n) \\ &\geq -\rho_n \|y_i - x_i^n\|_{X_i}, \qquad \forall\, y_i \in X_i,\end{aligned}$$

i.e., $x^n \in \Upsilon(\rho_n)$. It ensures from (4.9) that

$$d(x^n, Q) \leq e(\Upsilon(\rho_n), Q) \to 0.$$

As the solution set Q is compact, so there exists $w^n = (w_1^n, \dots, w_i^n, \dots, w_k^n) \in Q$ s.t.

$$\|x^n - w^n\| = d(x^n, Q) \to 0. \tag{4.10}$$

Also, from the compactness of Q, the sequence $\{w^n\}$ has a subsequence $\{w^{n_l}\}$ s.t. $w^{n_l} \to w'$, (for some $w' \in Q$). Hence, it follows from (4.10) that

$$\|x^{n_l} - w'\| \leq \|x^{n_l} - w^{n_l}\| + \|w^{n_l} - w'\| \to 0,$$

which estimates $x^{n_l} \to w'$. This derives the strongly well-posedness of the SMHVI. □

The next theorem is arranged by setting $X_i = \mathbb{R}$ for each $i \in \{1, \dots, k\}$. Hence $X_1 \times \cdots \times X_i \times \cdots \times X_k = \mathbb{R}^k$. Here, we present some conditions by which the SMHVI is well-posed in generalized sense.

Theorem 5. *For each $i \in \{1, \dots, k\}$, let $A_i \colon \mathbb{R}^k \to \mathbb{R}$ be a continuous mapping satisfying the hypothesis (H_h) and (H_m). Further, assume that for each $i \in \{1, \dots, k\}$, $S_i \colon \mathbb{R}^k \to \mathbb{R}$, $J_i \colon \mathbb{R} \to \mathbb{R}$ and $\phi_i \colon \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ satisfy the hypothesis (H_c), (H_j) and (H_ϕ), respectively. If there exist $n_0 \in \mathbb{N}$ and $\rho > 0$ s.t. $\Upsilon(\rho)$ is nonempty, bounded and for every approximating sequence $\{x^n\}$ of SMHVI $x^n \in \Upsilon(\rho)\ \forall\, n > n_0$. Then SMHVI is strongly well-posedness in generalized sense.*

Proof. Let $x^n = (x_1^n, \dots, x_i^n, \dots, x_k^n) \in \mathbb{R}^k$ be an approximating sequence of the SMHVI. Then there exists $\{\rho_n\} \subset \mathbb{R}_+$ s.t. $\rho_n \to 0$ as $n \to \infty$. Thus for each $i \in \{1, \dots, k\}$,

$$\begin{aligned}&\langle A_i(x^n) + S_i(x^n) - f_i, y_i - x_i^n \rangle + J_i^\circ(x_i^n; y_i - x_i^n) + \phi_i(y_i) - \phi_i(x_i^n) \\ &\geq -\rho_n \|y_i - x_i^n\|_{\mathbb{R}}, \qquad \forall\, y_i \in \mathbb{R},\end{aligned} \tag{4.11}$$

On the other hand, let $\rho > 0$ be s.t. $\Upsilon(\rho)$ is nonempty and bounded. Accordingly, there exists $n_0 \in \mathbb{N}$ s.t. $x^n \in \Upsilon(\rho)\ \forall\, n > n_0$. This formulates that $\{x^n\}$ is bounded. Thus, there exists a subsequence $\{x^{n_l}\}$ of $\{x^n\}$ s.t. $x^{n_l} = (x_1^{n_l}, \dots, x_i^{n_l}, \dots, x_k^{n_l}) \to \hat{x} = (\hat{x}_1, \dots, \hat{x}_i, \dots, \hat{x}_k)$. Further, due to the hypothesis $(H_c), (H_m), (H_j)$ and the lower semicontinuity of ϕ_i along with Lemma 1, we have the following estimates

$$\begin{aligned}&\langle A_i(\hat{x}_1, \dots, y_i, \dots, \hat{x}_k) + S_i(\hat{x}_1, \dots, \hat{x}_i, \dots, \hat{x}_k) - f_i, y_i - \hat{x}_i \rangle \\ &+ J_i^\circ(\hat{x}_i; y_i - \hat{x}_i) + \phi_i(y_i) - \phi_i(\hat{x}_i),\end{aligned}$$

$$
\begin{aligned}
&\geq \limsup_{l\to\infty}\Bigg[\langle A_i(x_1^{n_l},\ldots,y_i,\ldots,x_k^{n_l})+S_i(x_1^{n_l},\ldots,x_i^{n_l},\ldots,x_k^{n_l})-f_i,y_i-x_i^{n_l}\rangle \\
&\qquad\qquad +J_i^{\circ}(x_i^{n_l};y_i-x_i^{n_l})+\phi_i(y_i)-\phi_i(x_i^{n_l})\Bigg], \\
&\geq \limsup_{l\to\infty}\Bigg[\langle A_i(x_1^{n_l},\ldots,x_i^{n_l},\ldots,x_k^{n_l})+S_i(x_1^{n_l},\ldots,x_i^{n_l},\ldots,x_k^{n_l})-f_i,y_i-x_i^{n_l}\rangle \\
&\qquad\qquad +J_i^{\circ}(x_i^{n_l};y_i-x_i^{n_l})+\phi_i(y_i)-\phi_i(x_i^{n_l})\Bigg], \\
&\geq \limsup_{l\to\infty}[-\rho_n\|y_i-x_i^{n_l}\|_{\mathbb{R}}], \qquad [\text{by} \quad (4.11)], \\
&=0.
\end{aligned}
$$

By following the arguments similar to Lemma 2, we can deduce (by hemicontinuity of A_i and the convexity of ϕ_i) that

$$
\begin{aligned}
&\langle A_i(\hat{x}_1,\ldots,\hat{x}_i,\ldots,\hat{x}_k)+S_i(\hat{x}_1,\ldots,\hat{x}_i,\ldots,\hat{x}_k)-f_i,y_i-\hat{x}_i\rangle \\
&+J_i^{\circ}(\hat{x}_i;y_i-\hat{x}_i)+\phi_i(y_i)-\phi_i(\hat{x}_i)\geq 0,
\end{aligned}
$$

which means that $(\hat{x}_1,\ldots,\hat{x}_i,\ldots,\hat{x}_k)\in Q$. Thus, the SMHVI is strongly well-posed in generalized sense. □

Remark 3. Proposition 1 generalizes Lemma 3.2 of [22] and Lemma 3.2 of [23]. Further, Theorem 3 and Theorem 4 generalize Theorem 3.1 of [22] and Theorem 3.2 of [22], respectively. Also, if $k=1$ with $S_1=\theta_1$ and $\phi_1=0$ then Theorem 3 and Theorem 4 reduce to Theorem 3.1 of [23] and Theorem 3.2 of [23], respectively.

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