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A NEW q-HERMITE-HADAMARD'S INEQUALITY AND ESTIMATES FOR MIDPOINT TYPE INEQUALITIES FOR CONVEX FUNCTIONS

THANIN SITTHIWIRATTHAM, MUHAMMAD AAMIR ALI, ASGHAR ALI, AND HÜSEYIN BUDAK

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Abstract. This paper proves a new q-Hermite-Hadamard inequality for convex functions using quantum integrals. We also prove some new midpoint-type inequalities for q-differentiable convex functions. Moreover, we present some examples to illustrate our established results, supplemented with graphs.

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1. Introduction

A function $\varphi : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$ is called convex if it satisfies the inequality

$$\varphi(tx+(1-t)y) \le t\varphi(x)+(1-t)\varphi(y)$$

where $x,y \in I$ and $t \in [0,1]$.

It is also well known that φ is convex if and only if it satisfies the Hermite-Hadamard inequality, stated below (see, [6]):

$$\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right) \le \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(x)dx \le \frac{\varphi(\alpha_1)+\varphi(\alpha_2)}{2}, \tag{1.1}$$

where $\varphi : I \to \mathbb{R}$ is a convex function and $\alpha_1, \alpha_2 \in I$ with $\alpha_1 < \alpha_2$. In [7, 11], authors proved some bounds for the right and left sides of the inequality (1.1) which are called trapezoid and midpoint inequalities, respectively.

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On the other hand, in [2], Alp et al. proved the following version of quantum Hermite-Hadamard type for convex functions using the left quantum integrals:

$$\varphi\left(\frac{q\alpha_1+\alpha_2}{[2]_q}\right) \leq \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(x)\ {}_{\alpha_1}d_qx \leq \frac{q\varphi(\alpha_1)+\varphi(\alpha_2)}{[2]_q}. \tag{1.2}$$

Recently, Bermudo et al. [3] used the right quantum integrals and proved the following variant of Hermite-Hadamard type inequality for convex functions:

$$\varphi\left(\frac{\alpha_1+q\alpha_2}{[2]_q}\right) \leq \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(x)\ {}^{\alpha_2}d_qx \leq \frac{\varphi(\alpha_1)+q\varphi(\alpha_2)}{[2]_q}. \tag{1.3}$$

Several papers are devoted to finding bounds for the left and right sides of inequalities (1.2) and (1.3). In [12], Noor et al. proved several bounds for the right-hand side of the inequality of (1.2). Alp et al., by using convex functions, established some new results for the left side of the inequality (1.2) in [2]. In [1, 4], quantum Simpson's and Newton's type inequalities for convex and co-ordinated convex functions were proved. For more recent results, one can consult [5, 10].

Inspired by the ongoing studies, we prove a new version of quantum Hermite–Hadamard inequalities for convex functions. We also prove some new corresponding quantum midpoint inequalities for q-differentiable convex functions. We present many examples to illustrate our results, supplemented with graphs.

2. Basics of q-calculus

In this section, we recall some basics of quantum calculus, and throughout this paper, let $0<q<1$ be a constant.

The q-number or q-analogue of $n\in\mathbb{N}$ is given by (see [9])

$$[n]_q=\frac{1-q^n}{1-q}=1+q+q^2+\cdots+q^{n-1}. \tag{2.1}$$

The q-Jackson integral for the function φ over $[0,\alpha_1]$ is defined as (see, [8]):

$$\int_0^{\alpha_2}\varphi(x)\,d_qx=(1-q)\,\alpha_2\sum_{n=0}^{\infty}q^n\varphi(\alpha_2q^n) \tag{2.2}$$

and q-Jackson integral for a function φ over $[\alpha_1,\alpha_2]$ is as follows (see, [8]):

$$\int_{\alpha_1}^{\alpha_2}\varphi(x)\,d_qx=\int_0^{\alpha_2}\varphi(x)\,d_qx-\int_0^{\alpha_1}\varphi(x)\,d_qx. \tag{2.3}$$

Definition 1 ([15]). Let $\varphi:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a continuous function. Then the left q-derivative of function φ at $x\in[\alpha_1,\alpha_2]$ is defined by

$${}_{\alpha_1}D_q\varphi(x)=\begin{cases}\dfrac{\varphi(x)-\varphi(qx+(1-q)\alpha_1)}{(1-q)(x-\alpha_1)}, & \text{if } x\neq\alpha_1;\\ \lim\limits_{x\to\alpha_1}\ {}_{\alpha_1}D_q\varphi(x), & \text{if } x=\alpha_1.\end{cases} \tag{2.4}$$

The function φ is said to be q-differentiable function on $[\alpha_1,\alpha_2]$ if ${}_{\alpha_1}D_q\varphi(x)$ exists for all $x\in[\alpha_1,\alpha_2]$.

Definition 2 ([15]). Let $\varphi:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a continuous function. Then the left q-integral of function φ at $z\in[\alpha_1,\alpha_2]$ is defined by

$$\int_{\alpha_1}^{z}\varphi(x)_{\alpha_1}d_qx=(1-q)(z-\alpha_1)\sum_{n=0}^{\infty}q^n\varphi(q^nz+(1-q^n)\alpha_1). \tag{2.5}$$

The function φ is said to be q-integrable function on $[\alpha_1,\alpha_2]$ if $\int_{\alpha_1}^{z}\varphi(x)_{\alpha_1}d_qx$ exists for all $z\in[\alpha_1,\alpha_2]$.

On the other hand, Bermudo et al. defined new quantum derivatives and quantum integral which are called right q-derivative and right q-integral:

Definition 3 ([3]). The right q-derivative of mapping $\varphi:[\alpha_1,\alpha_2]\to\mathbb{R}$ is defined as:

$${}^{\alpha_2}D_q\varphi(x)=\frac{\varphi(qx+(1-q)\alpha_2)-\varphi(x)}{(1-q)(\alpha_2-x)},\quad x\neq\alpha_2.$$

If $x=\alpha_2$, we define ${}^{\alpha_2}D_q\varphi(\alpha_2)=\lim_{x\to\alpha_2}{}^{\alpha_2}D_q\varphi(x)$ if it exists and it is finite.

Definition 4 ([3]). The right q-definite integral of mapping $\varphi:[\alpha_1,\alpha_2]\to\mathbb{R}$ at $z\in[\alpha_1,\alpha_2]$ is defined as:

$$\int_{z}^{\alpha_2}\varphi(x)^{\alpha_2}d_qx=(1-q)(\alpha_2-z)\sum_{k=0}^{\infty}q^k\varphi\left(q^kz+\left(1-q^k\right)\alpha_2\right).$$

Lemma 1 ([13]). *For continuous functions $\varphi,h:[\alpha_1,\alpha_2]\to\mathbb{R}$, the following equality is true:*

$$\begin{aligned}&\int_0^c h(t)\ {}^{\alpha_2}D_q\varphi(t\alpha_1+(1-t)\alpha_2)\,d_qt\\&=\frac{1}{\alpha_2-\alpha_1}\int_0^c D_qh(t)\varphi(qt\alpha_1+(1-qt)\alpha_2)\,d_qt-\left.\frac{h(t)\varphi(t\alpha_1+(1-t)\alpha_2)}{\alpha_2-\alpha_1}\right|_0^c.\end{aligned}$$

Lemma 2 ([14]). *For continuous functions $\varphi,h:[\alpha_1,\alpha_2]\to\mathbb{R}$, the following equality is true:*

$$\begin{aligned}&\int_0^c h(t)\,{}_{\alpha_1}D_q\varphi(t\alpha_2+(1-t)\alpha_1)\,d_qt\\&=\left.\frac{h(t)\varphi(t\alpha_2+(1-t)\alpha_1)}{\alpha_2-\alpha_1}\right|_0^c-\frac{1}{\alpha_2-\alpha_1}\int_0^c D_qh(t)\varphi(qt\alpha_2+(1-qt)\alpha_1)\,d_qt.\end{aligned}$$

3. q-HERMITE-HADAMARD INEQUALITY

In this section, we prove the new quantum Hermite-Hadamard inequality and give an example to illustrate the obtained inequality.

Theorem 1. *Let $\varphi:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a convex mapping, then we have the following inequality:*

$$\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\leq\frac{1}{\alpha_2-\alpha_1}\left[\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx+\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\right]\leq\frac{\varphi(\alpha_1)+\varphi(\alpha_2)}{2}. \tag{3.1}$$

Proof. Since φ is a convex function, we have

$$\begin{aligned}\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)&=\varphi\left(\frac{1}{2}\left(\frac{1-t}{2}\alpha_1+\frac{1+t}{2}\alpha_2\right)+\frac{1}{2}\left(\frac{1+t}{2}\alpha_1+\frac{1-t}{2}\alpha_2\right)\right)\\&\leq\frac{1}{2}\left[\varphi\left(\frac{1-t}{2}\alpha_1+\frac{1+t}{2}\alpha_2\right)+\varphi\left(\frac{1+t}{2}\alpha_1+\frac{1-t}{2}\alpha_2\right)\right]\\&\leq\frac{\varphi(\alpha_1)+\varphi(\alpha_2)}{2}.\end{aligned} \tag{3.2}$$

q-integrating the inequality (3.2) over $[0,1]$, we have

$$\begin{aligned}\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)&\leq\frac{1}{2}\int_0^1\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt\\&\quad+\frac{1}{2}\int_0^1\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_1-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt\\&=\frac{1}{\alpha_2-\alpha_1}\left[\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx+\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\right].\end{aligned}$$

It is obvious from Definition 2 and Definition 4 that

$$\int_0^1\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt=\frac{2}{\alpha_2-\alpha_1}\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx$$

and

$$\int_0^1\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_1-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt=\frac{2}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx.$$

This completes the proof. □

Remark 1. In Theorem 1, if we set the limit as $q\to1^-$, then we obtain the classical Hermite-Hadamard inequality (1.1).

Example 1. The function $\varphi(x) = e^x$ is convex on $\mathbb{R}$ so that it also convex on $[0,1]$. For this convex function, we have

$$\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2} \varphi(x) \ _{\frac{\alpha_1+\alpha_2}{2}}d_qx = \int_{\frac{1}{2}}^{1} e^x \ _{\frac{1}{2}}d_qx = \frac{(1-q)\,e^{1/2}}{2} \sum_{n=0}^{\infty} q^n e^{\frac{q^n}{2}}$$

and

$$\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}} \varphi(x) \ ^{\frac{\alpha_1+\alpha_2}{2}}d_qx = \int_{0}^{\frac{1}{2}} e^x \ ^{\frac{1}{2}}d_qx = \frac{(1-q)\,e^{1/2}}{2} \sum_{n=0}^{\infty} q^n e^{-\frac{q^n}{2}}.$$

Hence, we have

$$\frac{1}{\alpha_2-\alpha_1}\left[\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2} \varphi(x) \ _{\frac{\alpha_1+\alpha_2}{2}}d_qx + \int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}} \varphi(x) \ ^{\frac{\alpha_1+\alpha_2}{2}}d_qx\right]$$
$$= \frac{(1-q)\,e^{1/2}}{2}\left[\sum_{n=0}^{\infty} q^n e^{\frac{q^n}{2}} + \sum_{n=0}^{\infty} q^n e^{-\frac{q^n}{2}}\right] := \Upsilon_{mid}.$$

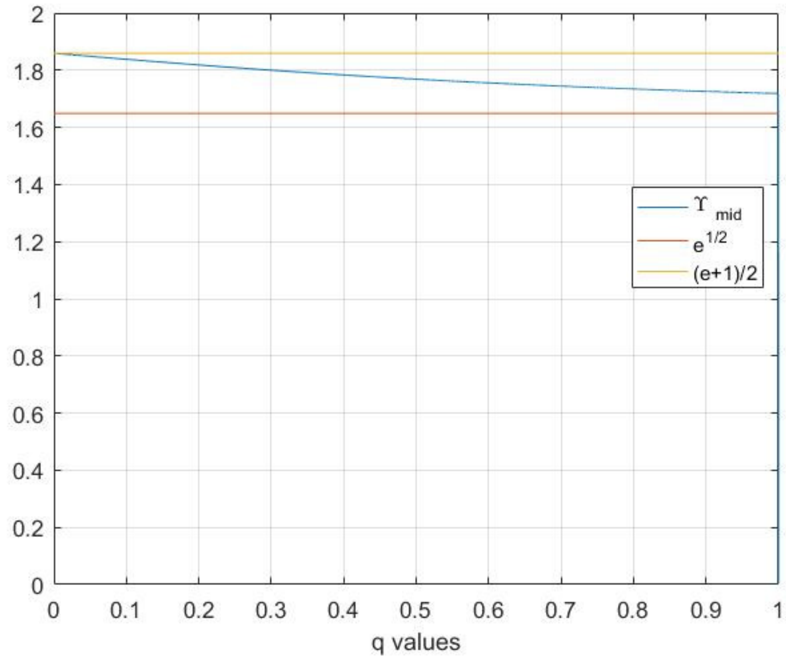

FIGURE 1. Example 1

We also have

$$\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right) = \varphi\left(\frac{1}{2}\right) = e^{1/2}$$

and

$$\frac{\varphi(\alpha_1)+\varphi(\alpha_2)}{2}=\frac{\varphi(0)+\varphi(1)}{2}=\frac{e+1}{2}.$$

It is obvious from Figure 1 that we have the inequality

$$e^{1/2}\leq \Upsilon_{mid}\leq \frac{e+1}{2}$$

for $q\in(0,1)$. On the other hand Figure 1 shows that $\Upsilon_{mid}\to\int_0^1 e^x dx=e-1$ for $q\to 1$.

4. Midpoint inequalities

In this section, we prove some left-estimates of the newly proved Hermite-Hadamard inequality (3.1) using the q-differentiability of the function.

Let us start with the following lemma.

Lemma 3. *Let $\varphi:[\alpha_1,\alpha_2]\subset\mathbb{R}\to\mathbb{R}$ be a function. If the functions ${}_{\alpha_1}D_q\varphi$ and ${}^{\alpha_2}D_q\varphi$ are continuous and integrable over $[\alpha_1,\alpha_2]$, then we have the following new equality:*

$$\frac{1}{\alpha_2-\alpha_1}\left[\begin{array}{c}\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx\\ \\ +\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\end{array}\right]-\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right) \tag{4.1}$$
$$=\frac{(\alpha_2-\alpha_1)}{4}\left[\int_0^1(1-qt)\ {}_{\alpha_1}D_q\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt\right.$$
$$\left.-\int_0^1(1-qt)\ {}^{\alpha_2}D_q\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_1-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt\right].$$

Proof. Let

$$\frac{(\alpha_2-\alpha_1)}{4}\left[\int_0^1(1-qt)\ {}_{\alpha_1}D_q\varphi\left(t\alpha_2+(1-t)\frac{\alpha_1+\alpha_2}{2}\right)d_qt\right. \tag{4.2}$$
$$\left.-\int_0^1(1-qt)\ {}^{\alpha_2}D_q\varphi\left(t\alpha_1+(1-t)\frac{\alpha_1+\alpha_2}{2}\right)d_qt\right]$$
$$=\frac{(\alpha_2-\alpha_1)}{4}[I_1-I_2].$$

By applying Lemma 1 to calculate the integral I_1, we have

$$I_1=\int_0^1(1-qt)_{\alpha_1}D_q\,\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt \tag{4.3}$$
$$=\int_0^1\ {}_{\alpha_1}D_q\,\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt$$

$$
\begin{aligned}
& -\int_0^1 qt_{\alpha_1}D_q\,\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt \\
= & \left.\frac{\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)}{\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)}\right|_0^1-\left.\frac{qt\,\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)}{\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)}\right|_0^1 \\
& +\int_0^1 \frac{q\,\varphi\left(\frac{\alpha_1+\alpha_2}{2}+tq\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)}{\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)}d_qt \\
= & \frac{\varphi\left(\frac{\alpha_1+\alpha_2}{2}+\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)}{\left(\frac{\alpha_2-\alpha_1}{2}\right)}-\frac{\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)}{\left(\frac{\alpha_2-\alpha_1}{2}\right)}-\frac{q\,\varphi\left(\frac{\alpha_1+\alpha_2}{2}+\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)}{\left(\frac{\alpha_2-\alpha_1}{2}\right)} \\
& +\frac{2q}{b-a}(1-q)(1-0)\sum_{n=0}^{\infty}q^n\varphi\left(\frac{\alpha_1+\alpha_2}{2}+q^{n+1}\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right) \\
= & \frac{2\varphi(\alpha_2)}{(\alpha_2-\alpha_1)}-\frac{2\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)}{(\alpha_2-\alpha_1)}-\frac{2q\,\varphi(\alpha_2)}{(\alpha_2-\alpha_1)} \\
& +\frac{2q}{b-a}(1-q)\sum_{n=0}^{\infty}q^n\varphi\left(\frac{\alpha_1+\alpha_2}{2}+q^{n+1}\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right) \\
= & \frac{2(1-q)\varphi(\alpha_2)}{(\alpha_2-\alpha_1)}-\frac{2\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)}{(\alpha_2-\alpha_1)} \\
& +\frac{2^2}{(\alpha_2-\alpha_1)^2}(1-q)\left(\frac{\alpha_2-\alpha_1}{2}\right)\sum_{n=0}^{\infty}q^{n+1}\varphi\left(\frac{\alpha_1+\alpha_2}{2}+q^{n+1}\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right) \\
= & \frac{2(1-q)\varphi(\alpha_2)}{(\alpha_2-\alpha_1)}-\frac{2\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)}{(\alpha_2-\alpha_1)} \\
& +\frac{4}{(\alpha_2-\alpha_1)^2}(1-q)\left(\frac{\alpha_2-\alpha_1}{2}\right)\sum_{n=1}^{\infty}q^{n}\varphi\left(\frac{\alpha_1+\alpha_2}{2}+q^{n}\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right) \\
= & \frac{2(1-q)\varphi(\alpha_2)}{(\alpha_2-\alpha_1)}-\frac{2\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)}{(\alpha_2-\alpha_1)} \\
& +\frac{4}{(\alpha_2-\alpha_1)^2}(1-q)\left(\frac{\alpha_2-\alpha_1}{2}\right)\sum_{n=0}^{\infty}q^{n}\varphi\left(\frac{\alpha_1+\alpha_2}{2}+q^{n}\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right) \\
& +\frac{4}{(\alpha_2-\alpha_1)^2}(1-q)\left(\frac{\alpha_2-\alpha_1}{2}\right)\varphi(\alpha_2) \\
= & -\frac{2}{(\alpha_2-\alpha_1)}\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)+\frac{4}{(\alpha_2-\alpha_1)^2}\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx.
\end{aligned}
$$

Similarly, by using Lemma 2 to calculate the integral I_2, we obtain

$$
\begin{aligned}
I_2 &= \int_0^1 (1-qt)\ {}^{\alpha_2}D_q\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_1-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt \qquad (4.4)\\
&= \frac{2}{(\alpha_2-\alpha_1)}\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)-\frac{4}{(\alpha_2-\alpha_1)^2}\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx.
\end{aligned}
$$

Thus, we establish the desired equality by putting (4.3) and (4.4) in (4.2). The proof is completed. ☐

Theorem 2. *We assume that the conditions Lemma 3 hold. If the functions $\left|{}_{\alpha_1}D_q\varphi\right|$ and $\left|{}^{\alpha_2}D_q\varphi\right|$ are convex, then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{1}{\alpha_2-\alpha_1}\left[\begin{array}{c}\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx\\ +\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\end{array}\right]-\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right| \qquad (4.5)\\
&\leq \frac{(\alpha_2-\alpha_1)}{8[2]_q[3]_q}\left[q[2]_q\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right|+\left(1+[3]_q\right)\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right|\right.\\
&\quad\left.+\left(1+[3]_q\right)\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right|+q[2]_q\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|\right].
\end{aligned}
$$

Proof. By taking modulus in (4.1), we have

$$
\begin{aligned}
&\left|\frac{1}{\alpha_2-\alpha_1}\left[\begin{array}{c}\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx\\ +\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\end{array}\right]-\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right| \qquad (4.6)\\
&\leq \frac{(\alpha_2-\alpha_1)}{4}\left[\int_0^1(1-qt)\left|{}_{\alpha_1}D_q\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)\right|d_qt\right.\\
&\quad\left.+\int_0^1(1-qt)\left|{}^{\alpha_2}D_q\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_1-\frac{\alpha_1+\alpha_2}{2}\right)\right)\right|d_qt\right].
\end{aligned}
$$

Since the functions $\left|{}_{\alpha_1}D_q\varphi\right|$ and $\left|{}^{\alpha_2}D_q\varphi\right|$ are convex, we have

$$
\begin{aligned}
&\left|\frac{1}{\alpha_2-\alpha_1}\left[\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx+\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\right]-\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|\\
&\leq \frac{(\alpha_2-\alpha_1)}{4}\left[\int_0^1(1-qt)\left(\left(\frac{1-t}{2}\right)\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right|+\left(\frac{1+t}{2}\right)\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right|\right)d_qt\right.\\
&\quad\left.+\int_0^1(1-qt)\left(\left(\frac{1+t}{2}\right)\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right|+\left(\frac{1-t}{2}\right)\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|\right)d_qt\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(\alpha_2-\alpha_1)}{4}\left[\frac{q}{2[3]_q}\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right| + \frac{1+[3]_q}{2[2]_q[3]_q}\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right|\right. \\
&\quad \left. + \frac{1+[3]_q}{2[2]_q[3]_q}\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right| + \frac{q}{2[3]_q}\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|\right].
\end{aligned}
$$

Thus, the proof is completed. □

Remark 2. In Theorem 2, if we set the limit as $q \to 1^-$, then we obtain [11, Theorem 2.2].

Theorem 3. *We assume that the conditions Lemma 3 hold. If the functions* $\left|{}_{\alpha_1}D_q\varphi\right|^s$ *and* $\left|{}^{\alpha_2}D_q\varphi\right|^s$, $s \geq 1$ *are convex, then the following inequality holds:*

$$
\left|\frac{1}{\alpha_2-\alpha_1}\left[\begin{array}{c}\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx \\ +\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\end{array}\right] - \varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right| \tag{4.7}
$$

$$
\begin{aligned}
&\leq \frac{(\alpha_2-\alpha_1)}{4}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{s}}\left[\left(\frac{q[2]_q\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right|^s + \left(1+[3]_q\right)\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right|^s}{2[2]_q[3]_q}\right)^{\frac{1}{s}}\right. \\
&\quad \left. + \left(\frac{\left(1+[3]_q\right)\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right|^s + q[2]_q\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|^s}{2[2]_q[3]_q}\right)^{\frac{1}{s}}\right].
\end{aligned}
$$

Proof. By using the power mean inequality in (4.6), we have

$$
\begin{aligned}
&\left|\frac{1}{\alpha_2-\alpha_1}\left[\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx + \int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\right] - \varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right| \\
&\leq \frac{(\alpha_2-\alpha_1)}{4}\left(\int_0^1(1-qt)\,d_qt\right)^{1-\frac{1}{s}} \\
&\quad \times\left[\left(\int_0^1(1-qt)\left|{}_{\alpha_1}D_q\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)\right|^s d_qt\right)^{\frac{1}{s}}\right. \\
&\quad \left. + \left(\int_0^1(1-qt)\left|{}^{\alpha_2}D_q\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_1-\frac{\alpha_1+\alpha_2}{2}\right)\right)\right|^s d_qt\right)^{\frac{1}{s}}\right].
\end{aligned}
$$

By using the convexity of the functions $\left|{}_{\alpha_1}D_q\varphi\right|^s$ and $\left|{}^{\alpha_2}D_q\varphi\right|^s$, we have

$$
\left|\frac{1}{\alpha_2-\alpha_1}\left[\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx + \int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\right] - \varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|
$$

$$\leq \frac{(\alpha_2-\alpha_1)}{4}\left(\int_0^1 (1-qt)\,d_qt\right)^{1-\frac{1}{s}}$$
$$\times\left[\left(\int_0^1 (1-qt)\left(\frac{1-t}{2}\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right|^s+\frac{1+t}{2}\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right|^s\right)d_qt\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_0^1 (1-qt)\left(\frac{1+t}{2}\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right|^s+\frac{1-t}{2}\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|^s\right)d_qt\right)^{\frac{1}{s}}\right]$$
$$=\frac{(\alpha_2-\alpha_1)}{4}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{s}}\left[\left(\frac{q}{2[3]_q}\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right|^s+\frac{1+[3]_q}{2[2]_q[3]_q}\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right|^s\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\frac{1+[3]_q}{2[2]_q[3]_q}\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right|^s+\frac{q}{2[3]_q}\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|^s\right)^{\frac{1}{s}}\right].$$

Thus, the proof is completed. □

Theorem 4. *We assume that the conditions Lemma 3 hold. If the functions* $\left|{}_{\alpha_1}D_q\varphi\right|^s$ *and* $\left|{}^{\alpha_2}D_q\varphi\right|^s$, $s>1$ *are convex, then the following inequality holds:*

$$\left|\frac{1}{\alpha_2-\alpha_1}\left[\begin{array}{c}\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\;{}_{\frac{\alpha_1+\alpha_2}{2}}d_qx\\ +\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\;{}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\end{array}\right]-\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right| \tag{4.8}$$
$$\leq\frac{(\alpha_2-\alpha_1)}{4}\left(\frac{1-(1-q)^{r+1}}{q[r+1]_q}\right)^{\frac{1}{r}}\left[\left(\frac{q\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right|^s+\left([2]_q+1\right)\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\frac{\left([2]_q+1\right)\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right|^s+q\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right],$$

where $s^{-1}+r^{-1}=1$.

Proof. By applying Hölder's inequality in (4.6), we have

$$\left|\frac{1}{\alpha_2-\alpha_1}\left[\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\;{}_{\frac{\alpha_1+\alpha_2}{2}}d_qx+\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\;{}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\right]-\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|$$
$$\leq\frac{(\alpha_2-\alpha_1)}{4}\left(\int_0^1(1-qt)^r\,d_qt\right)^{\frac{1}{r}}$$

$$\times\left[\left(\int_0^1\left|{}_{\alpha_1}D_q\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)\right|^s d_qt\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_0^1\left|{}^{\alpha_2}D_q\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_1-\frac{\alpha_1+\alpha_2}{2}\right)\right)\right|^s d_qt\right)^{\frac{1}{s}}\right].$$

Since the functions $\left|{}_{\alpha_1}D_q\varphi\right|^s$ and $\left|{}^{\alpha_2}D_q\varphi\right|^s$ are convex, we have

$$\left|\frac{1}{\alpha_2-\alpha_1}\left[\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx+\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\right]-\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|$$
$$\leq\frac{(\alpha_2-\alpha_1)}{4}\left(\int_0^1(1-qt)^r d_qt\right)^{\frac{1}{r}}$$
$$\times\left[\left(\int_0^1\left(\frac{1-t}{2}\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right|^s+\frac{1+t}{2}\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right|^s\right)d_qt\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_0^1\left(\frac{1+t}{2}\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right|^s+\frac{1-t}{2}\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|^s\right)d_qt\right)^{\frac{1}{s}}\right]$$
$$=\frac{(\alpha_2-\alpha_1)}{4}\left(\frac{1-(1-q)^{r+1}}{q[r+1]_q}\right)^{\frac{1}{r}}\left[\left(\frac{q}{2[2]_q}\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right|^s+\frac{[2]_q+1}{2[2]_q}\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right|^s\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\frac{[2]_q+1}{2[2]_q}\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right|^s+\frac{q}{2[2]_q}\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|^s\right)^{\frac{1}{s}}\right].$$

Hence, the proof is completed. □

5. Example

In this section, we give an example to support one of the newly established inequalities.

Example 2. For a convex functions $\varphi:[1,3]\to\mathbb{R}$ is defined by $\varphi(x)=x+2$. From Theorem 2 with $q=\frac{1}{4}$, the left hand side of inequality (4.5) becomes

$$\left|\frac{1}{\alpha_2-\alpha_1}\left[\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx+\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\right]-\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|$$
$$=\left|\frac{1}{2}\left[\int_2^3(x+2)\ {}_2d_{\frac{1}{4}}x+\int_1^2(x+2)\ {}^2d_{\frac{1}{4}}x\right]-4\right|=\left|\frac{1}{2}(4+4)-4\right|=0$$

and the right side becomes

$$\frac{(\alpha_2-\alpha_1)}{8\,[2]_q\,[3]_q}\Big[q\,[2]_q\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right|+\Big(1+[3]_q\Big)\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right|$$
$$+\Big(1+[3]_q\Big)\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right|+q\,[2]_q\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|\Big]$$
$$=\frac{2}{8\,(1+q)\,(q^2+q+1)}\cdot\left[\left(2+q^2+q\right)+q\,(q+1)+q\,(q+1)+\left(2+q^2+q\right)\right]$$
$$=\frac{1}{4\,(1+q)\,(q^2+q+1)}\cdot 4\left(1+q^2+q\right)=\frac{1}{q+1}=\frac{4}{5}=0.8.$$

It is clear that

$$0<0.8$$

which demonstrates inequality (4.5).

6. Concluding remarks

In this paper, we have established some new Hermite-Hadamard inequality for convex functions in the context of quantum calculus. Moreover, we have proved some quantum estimates for midpoint-type inequalities for q -differentiable convex functions. It is also has proven by some mathematical examples that the newly established inequalities are valid for any convex function. It is an interesting and new problem that the upcoming researchers can obtain similar inequalities for coordinated convex functions.

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Authors' addresses

Thanin Sitthiwirattham
Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok, Thailand
E-mail address: thanin_sit@dusit.ac.th

Muhammad Aamir Ali
(**Corresponding author**) Jiangsu Key Laboratory of NSLSCS, School of Mathematical Sciences, Nanjing Normal University, 210023, China
E-mail address: mahr.muhammad.aamir@gmail.com

Asghar Ali
Department of Mathematics, Government Postgraduate College Sahiwal, Pakistan
E-mail address: leoasgharali@gmail.com

Hüseyin Budak
Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey
E-mail address: hsyn.budak@gmail.com