A NEW $q$-HERMITE-HADAMARD’S INEQUALITY AND ESTIMATES FOR MIDPOINT TYPE INEQUALITIES FOR CONVEX FUNCTIONS

THANIN SITTHIWIRATTHAM, MUHAMMAD AAMIR ALI, ASGHAR ALI, AND HÜSEYN BUDAK

Received 11 April, 2022

Abstract. This paper proves a new $q$-Hermite-Hadamard inequality for convex functions using quantum integrals. We also prove some new midpoint-type inequalities for $q$-differentiable convex functions. Moreover, we present some examples to illustrate our established results, supplemented with graphs.

2010 Mathematics Subject Classification: 26D10; 26D15; 26A51

Keywords: Hermite-Hadamard inequality, midpoint inequalities, convex functions

1. INTRODUCTION

A function $\phi : I \to \mathbb{R}$, where $I$ is an interval in $\mathbb{R}$ is called convex if it satisfies the inequality

$$\phi(tx + (1 - t)y) \leq t\phi(x) + (1 - t)\phi(y)$$

where $x, y \in I$ and $t \in [0, 1]$.

It is also well known that $\phi$ is convex if and only if it satisfies the Hermite-Hadamard inequality, stated below (see, [6]):

$$\phi\left(\frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \phi(x)dx \leq \frac{\phi(\alpha_1) + \phi(\alpha_2)}{2},$$

where $\phi : I \to \mathbb{R}$ is a convex function and $\alpha_1, \alpha_2 \in I$ with $\alpha_1 < \alpha_2$. In [7, 11], authors proved some bounds for the right and left sides of the inequality (1.1) which are called trapezoid and midpoint inequalities, respectively.

This project is funded by National Research Council of Thailand (NRCT) and Suan Dusit University, Grant No. N42A650384. This work is partially supported by the National Natural Science Foundation of China, Grant No. 11971241.

© 2023 Miskolc University Press
On the other hand, in [2], Alp et al. proved the following version of quantum Hermite-Hadamard type for convex functions using the left quantum integrals:

\[
\phi \left( \frac{q\alpha_1 + \alpha_2}{2} \right) \leq \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \phi(x) \alpha_1 d_q x \leq \frac{q\phi(\alpha_1) + \phi(\alpha_2)}{2}. \tag{1.2}
\]

Recently, Bermudo et al. [3] used the right quantum integrals and proved the following variant of Hermite-Hadamard type inequality for convex functions:

\[
\phi \left( \frac{\alpha_1 + q\alpha_2}{2} \right) \leq \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \phi(x) \alpha_2 d_q x \leq \frac{\phi(\alpha_1) + q\phi(\alpha_2)}{2}. \tag{1.3}
\]

Several papers are devoted to finding bounds for the left and right sides of inequalities (1.2) and (1.3). In [12], Noor et al. proved several bounds for the right-hand side of the inequality of (1.2). Alp et al., by using convex functions, established some new results for the left side of the inequality (1.2) in [2]. In [1, 4], quantum Simpson’s and Newton’s type inequalities for convex and co-ordinated convex functions were proved. For more recent results, one can consult [5, 10].

Inspired by the ongoing studies, we prove a new version of quantum Hermite–Hadamard inequalities for convex functions. We also prove some new corresponding quantum midpoint inequalities for \(q\)-differentiable convex functions. We present many examples to illustrate our results, supplemented with graphs.

2. BASICS OF \(q\)-CALCULUS

In this section, we recall some basics of quantum calculus, and throughout this paper, let \(0 < q < 1\) be a constant.

The \(q\)-number or \(q\)-analogue of \(n \in \mathbb{N}\) is given by (see [9])

\[
[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}. \tag{2.1}
\]

The \(q\)-Jackson integral for the function \(\phi\) over \([0, \alpha_1]\) is defined as (see, [8]):

\[
\int_0^{\alpha_1} \phi(x) d_q x = (1 - q) \alpha_2 \sum_{n=0}^{\infty} q^n \phi(\alpha_2 q^n) \tag{2.2}
\]

and \(q\)-Jackson integral for a function \(\phi\) over \([\alpha_1, \alpha_2]\) is as follows (see, [8]):

\[
\int_{\alpha_1}^{\alpha_2} \phi(x) d_q x = \int_0^{\alpha_2} \phi(x) d_q x - \int_0^{\alpha_1} \phi(x) d_q x. \tag{2.3}
\]

**Definition 1** ([15]). Let \(\phi: [\alpha_1, \alpha_2] \to \mathbb{R}\) be a continuous function. Then the left \(q\)-derivative of function \(\phi\) at \(x \in [\alpha_1, \alpha_2]\) is defined by

\[
\alpha_i D_q \phi(x) = \begin{cases} 
\frac{\phi(x) - \phi(qx + (1 - q)\alpha_1)}{(1 - q)(x - \alpha_1)}, & \text{if } x \neq \alpha_1; \\
\lim_{x \to \alpha_1} \alpha_i D_q \phi(x), & \text{if } x = \alpha_1.
\end{cases} \tag{2.4}
\]
The function $\phi$ is said to be $q$-differentiable function on $[\alpha_1, \alpha_2]$ if $\alpha_1D_q\phi(x)$ exists for all $x \in [\alpha_1, \alpha_2]$.

**Definition 2 ([15])**. Let $\phi : [\alpha_1, \alpha_2] \to \mathbb{R}$ be a continuous function. Then the left $q$-integral of function $\phi$ at $z \in [\alpha_1, \alpha_2]$ is defined by

$$\int_{\alpha_1}^{c} \phi(x)\alpha_1d_qx = (1-q)(z - \alpha_1) \sum_{n=0}^{\infty} q^n \phi(q^n z + (1-q^n)\alpha_1). \quad (2.5)$$

The function $\phi$ is said to be $q$-integrable function on $[\alpha_1, \alpha_2]$ if $\int_{\alpha_1}^{c} \phi(x)\alpha_1d_qx$ exists for all $z \in [\alpha_1, \alpha_2]$.

On the other hand, Bermudo et al. defined new quantum derivatives and quantum integral which are called right $q$-derivative and right $q$-integral:

**Definition 3 ([3])**. The right $q$-derivative of mapping $\phi : [\alpha_1, \alpha_2] \to \mathbb{R}$ is defined as:

$$\alpha_q D_q \phi(x) = \frac{\phi(qx + (1-q)\alpha_2) - \phi(x)}{(1-q)(\alpha_2 - x)} , \quad x \neq \alpha_2.$$

If $x = \alpha_2$, we define $\alpha_q D_q \phi(\alpha_2) = \lim_{x \to \alpha_2} \alpha_q D_q \phi(x)$ if it exists and it is finite.

**Definition 4 ([3])**. The right $q$-definite integral of mapping $\phi : [\alpha_1, \alpha_2] \to \mathbb{R}$ at $z \in [\alpha_1, \alpha_2]$ is defined as:

$$\int_{\alpha_1}^{c} \phi(x)\alpha_2 d_qx = (1-q)(\alpha_2 - z) \sum_{k=0}^{\infty} q^k \phi(q^k z + (1-q^k)\alpha_2).$$

**Lemma 1 ([13])**. For continuous functions $\phi, h : [\alpha_1, \alpha_2] \to \mathbb{R}$, the following equality is true:

$$\int_{0}^{c} h(t)\alpha_2 D_q \phi(t\alpha_1 + (1-t)\alpha_2) d_qt \hspace{1cm} = \frac{1}{\alpha_2 - \alpha_1} \int_{0}^{c} D_q h(t) \phi(qt\alpha_1 + (1-qt)\alpha_2) d_qt - \frac{h(0)\phi(t\alpha_1 + (1-t)\alpha_2)}{\alpha_2 - \alpha_1} \bigg|_{0}^{c}. $$

**Lemma 2 ([14])**. For continuous functions $\phi, h : [\alpha_1, \alpha_2] \to \mathbb{R}$, the following equality is true:

$$\int_{0}^{c} h(t)\alpha_1 D_q \phi(t\alpha_2 + (1-t)\alpha_1) d_qt \hspace{1cm} = \frac{h(0)\phi(t\alpha_2 + (1-t)\alpha_1)}{\alpha_2 - \alpha_1} \bigg|_{0}^{c} - \frac{1}{\alpha_2 - \alpha_1} \int_{0}^{c} D_q h(t) \phi(qt\alpha_2 + (1-qt)\alpha_1) d_qt.$$
3. **q-HERMITE-HADAMARD INEQUALITY**

In this section, we prove the new quantum Hermite-Hadamard inequality and give an example to illustrate the obtained inequality.

**Theorem 1.** Let \( \varphi : [\alpha_1, \alpha_2] \to \mathbb{R} \) be a convex mapping, then we have the following inequality:

\[
\varphi\left(\frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{1}{\alpha_2 - \alpha_1} \left[ \int_{\alpha_1}^{\alpha_2} \varphi(x) \frac{1}{x} \, dq_x + \int_{\alpha_1}^{\frac{\alpha_1 + \alpha_2}{2}} \varphi(x) \, dq_x \right] \tag{3.1}
\]

\[
\leq \frac{\varphi(\alpha_1) + \varphi(\alpha_2)}{2}.
\]

**Proof.** Since \( \varphi \) is a convex function, we have

\[
\varphi\left(\frac{\alpha_1 + \alpha_2}{2}\right) = \frac{1}{2} \left[ \varphi\left(\frac{1-t}{2} \alpha_1 + \frac{1+t}{2} \alpha_2\right) + \varphi\left(\frac{1+t}{2} \alpha_1 + \frac{1-t}{2} \alpha_2\right) \right] \tag{3.2}
\]

\[
\leq \frac{1}{2} \left[ \varphi\left(\frac{1-t}{2} \alpha_1 + \frac{1+t}{2} \alpha_2\right) + \varphi\left(\frac{1+t}{2} \alpha_1 + \frac{1-t}{2} \alpha_2\right) \right]
\]

\[
\leq \frac{\varphi(\alpha_1) + \varphi(\alpha_2)}{2}.
\]

\(q\)-integrating the inequality (3.2) over \([0, 1]\), we have

\[
\varphi\left(\frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{1}{\alpha_2 - \alpha_1} \left[ \int_{\alpha_1}^{\alpha_2} \varphi(x) \frac{1}{x} \, dq_x + \int_{\alpha_1}^{\frac{\alpha_1 + \alpha_2}{2}} \varphi(x) \, dq_x \right] \tag{3.3}
\]

It is obvious from Definition 2 and Definition 4 that

\[
\int_0^1 \varphi\left(\frac{\alpha_1 + \alpha_2}{2} + t \left(\alpha_2 - \frac{\alpha_1 + \alpha_2}{2}\right)\right) \, dq_t = \frac{2}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(x) \frac{1}{x} \, dq_x
\]

and

\[
\int_0^1 \varphi\left(\frac{\alpha_1 + \alpha_2}{2} + t \left(\alpha_1 - \frac{\alpha_1 + \alpha_2}{2}\right)\right) \, dq_t = \frac{2}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\frac{\alpha_1 + \alpha_2}{2}} \varphi(x) \frac{1}{x} \, dq_x.
\]

This completes the proof. \( \square \)

**Remark 1.** In Theorem 1, if we set the limit as \( q \to 1^- \), then we obtain the classical Hermite-Hadamard inequality (1.1).
Example 1. The function $\phi(x) = e^x$ is convex on $\mathbb{R}$ so that it also convex on $[0, 1]$.

For this convex function, we have

$$\int_{\frac{\alpha_1}{2}}^{\alpha_2} \phi(x) \frac{\alpha_1 + \alpha_2}{2} \, dq \, x = \int_{\frac{1}{2}}^{1} e^{x^2} \frac{1}{2} \, dq \, x = \frac{(1 - q) e^{1/2}}{2} \sum_{n=0}^{\infty} q^n e^{\frac{q^n}{2}}$$

and

$$\int_{\alpha_1}^{\frac{\alpha_1 + \alpha_2}{2}} \phi(x) \frac{\alpha_1 + \alpha_2}{2} \, dq \, x = \int_{0}^{\frac{1}{2}} e^{x^2} \frac{1}{2} \, dq \, x = \frac{(1 - q) e^{1/2}}{2} \sum_{n=0}^{\infty} q^n e^{-\frac{q^n}{2}}.$$

Hence, we have

$$\frac{1}{\alpha_2 - \alpha_1} \left[ \int_{\frac{\alpha_1}{2}}^{\alpha_2} \phi(x) \frac{\alpha_1 + \alpha_2}{2} \, dq \, x + \int_{\alpha_1}^{\frac{\alpha_1 + \alpha_2}{2}} \phi(x) \frac{\alpha_1 + \alpha_2}{2} \, dq \, x \right]$$

$$= \left( \frac{1 - q}{2} \right)^{1/2} \left[ \sum_{n=0}^{\infty} q^n e^{\frac{q^n}{2}} + \sum_{n=0}^{\infty} q^n e^{-\frac{q^n}{2}} \right] := \Upsilon_{\text{mid}},$$

We also have

$$\phi\left( \frac{\alpha_1 + \alpha_2}{2} \right) = \phi\left( \frac{1}{2} \right) = e^{1/2}.$$

![Figure 1. Example 1](image-url)
and
\[ \frac{\varphi(\alpha_1) + \varphi(\alpha_2)}{2} = \frac{\varphi(0) + \varphi(1)}{2} = \frac{e+1}{2}. \]

It is obvious from Figure 1 that we have the inequality
\[ e^{1/2} \leq \Upsilon_{\text{mid}} \leq \frac{e+1}{2} \]
for \( q \in (0,1) \). On the other hand Figure 1 shows that \( \Upsilon_{\text{mid}} \rightarrow \int_0^1 e^t \, dt = e - 1 \) for \( q \rightarrow 1 \).

### 4. MIDPOINT INEQUALITIES

In this section, we prove some left-estimates of the newly proved Hermite-Hadamard inequality (3.1) using the \( q \)-differentiability of the function.

Let us start with the following lemma.

**Lemma 3.** Let \( \varphi : [\alpha_1, \alpha_2] \subset \mathbb{R} \rightarrow \mathbb{R} \) be a function. If the functions \( \alpha_1 D_q \varphi \) and \( \alpha_2 D_q \varphi \) are continuous and integrable over \([\alpha_1, \alpha_2]\), then we have the following new equality:

\[
\frac{1}{\alpha_2 - \alpha_1} \left[ \int_{\alpha_1}^{\alpha_2} \varphi(x) \frac{\alpha_1 + \alpha_2}{2} \, d_q x \right] - \varphi \left( \frac{\alpha_1 + \alpha_2}{2} \right) \]

\[
= \frac{(\alpha_2 - \alpha_1)}{4} \left[ \int_0^1 (1 - qt) \alpha_1 D_q \varphi \left( \frac{\alpha_1 + \alpha_2}{2} + t \left( \alpha_2 - \frac{\alpha_1 + \alpha_2}{2} \right) \right) \, dt \right] - \int_0^1 (1 - qt) \alpha_2 D_q \varphi \left( \frac{\alpha_1 + \alpha_2}{2} + t \left( \alpha_1 - \frac{\alpha_1 + \alpha_2}{2} \right) \right) \, dt. \]  

**Proof.** Let

\[
\frac{(\alpha_2 - \alpha_1)}{4} \left[ \int_0^1 (1 - qt) \alpha_1 D_q \varphi \left( t \alpha_2 + (1-t) \frac{\alpha_1 + \alpha_2}{2} \right) \, dt \right]
\]

\[
- \int_0^1 (1 - qt) \alpha_2 D_q \varphi \left( t \alpha_1 + (1-t) \frac{\alpha_1 + \alpha_2}{2} \right) \, dt \]  

\[
= \frac{(\alpha_2 - \alpha_1)}{4} \left[ I_1 - I_2 \right]. \]

By applying Lemma 1 to calculate the integral \( I_1 \), we have

\[
I_1 = \int_0^1 (1 - qt) \alpha_1 D_q \varphi \left( \frac{\alpha_1 + \alpha_2}{2} + t \left( \alpha_2 - \frac{\alpha_1 + \alpha_2}{2} \right) \right) \, dt \]

\[
= \int_0^1 \alpha_1 D_q \varphi \left( \frac{\alpha_1 + \alpha_2}{2} + t \left( \alpha_2 - \frac{\alpha_1 + \alpha_2}{2} \right) \right) \, dt. \]
\[-\int_0^1 q t \alpha_1 D_q \phi \left( \frac{\alpha_1 + \alpha_2}{2} + t \left( \alpha_2 - \frac{\alpha_1 + \alpha_2}{2} \right) \right) dt = \phi \left( \frac{\alpha_1 + \alpha_2}{2} + t \left( \alpha_2 - \frac{\alpha_1 + \alpha_2}{2} \right) \right) \bigg|_0^1 - \int_0^1 q t \phi \left( \frac{\alpha_1 + \alpha_2}{2} + t \left( \alpha_2 - \frac{\alpha_1 + \alpha_2}{2} \right) \right) dt \]

\[= \phi \left( \frac{\alpha_1 + \alpha_2}{2} + \alpha_2 - \frac{\alpha_1 + \alpha_2}{2} \right) - q \phi \left( \frac{\alpha_1 + \alpha_2}{2} + q \phi \left( \frac{\alpha_1 + \alpha_2}{2} + \alpha_2 - \frac{\alpha_1 + \alpha_2}{2} \right) \right) \]

\[+ \frac{2q}{b - a} (1 - q) (1 - 0) \sum_{n=0}^{\infty} g^n \phi \left( \frac{\alpha_1 + \alpha_2}{2} + q^{n+1} \left( \alpha_2 - \frac{\alpha_1 + \alpha_2}{2} \right) \right) \]

\[= 2 \frac{2^q}{(\alpha_2 - \alpha_1)} \left( \frac{\alpha_2 - \alpha_1}{2^q} \right) (1 - q) \left( \frac{\alpha_2 - \alpha_1}{2} \right) \sum_{n=0}^{\infty} q^n \phi \left( \frac{\alpha_1 + \alpha_2}{2} + q^{n+1} \left( \alpha_2 - \frac{\alpha_1 + \alpha_2}{2} \right) \right) \]

\[= 2 \frac{2q}{(\alpha_2 - \alpha_1)} \left( \frac{\alpha_2 - \alpha_1}{2^q} \right) \left( \frac{\alpha_2 - \alpha_1}{2} \right) \sum_{n=1}^{\infty} q^n \phi \left( \frac{\alpha_1 + \alpha_2}{2} + q^{n} \left( \alpha_2 - \frac{\alpha_1 + \alpha_2}{2} \right) \right) \]

\[= 2 \frac{2q}{(\alpha_2 - \alpha_1)} \left( \frac{\alpha_2 - \alpha_1}{2^q} \right) \left( \frac{\alpha_2 - \alpha_1}{2} \right) \phi \left( \frac{\alpha_1 + \alpha_2}{2} \right) \]

\[\sum_{n=0}^{\infty} q^n \phi \left( \frac{\alpha_1 + \alpha_2}{2} + q^{n+1} \left( \alpha_2 - \frac{\alpha_1 + \alpha_2}{2} \right) \right) \]

\[= -2 \left( \frac{\alpha_2 - \alpha_1}{2} \right) \phi \left( \frac{\alpha_1 + \alpha_2}{2} \right) + \frac{4}{(\alpha_2 - \alpha_1)^2} \frac{\alpha_3}{q^{\alpha_1+\alpha_2}} d_q x \]
Similarly, by using Lemma 2 to calculate the integral $I_2$, we obtain

$$I_2 = \int_{0}^{1} (1 - qt) \alpha_2 D_q \phi \left( \frac{\alpha_1 + \alpha_2}{2} + t \left( \alpha_1 - \frac{\alpha_1 + \alpha_2}{2} \right) \right) d_q t$$

(4.4)

$$= \frac{2}{(\alpha_2 - \alpha_1)} \phi \left( \frac{\alpha_1 + \alpha_2}{2} \right) - \frac{4}{(\alpha_2 - \alpha_1)^2} \int_{\alpha_1}^{\alpha_2} \phi \left( \frac{\alpha_1 + \alpha_2}{2} \right) d_q x.$$

Thus, we establish the desired equality by putting (4.3) and (4.4) in (4.2). The proof is completed.

**Theorem 2.** We assume that the conditions Lemma 3 hold. If the functions $|\alpha_1 D_q \phi|$ and $|\alpha_2 D_q \phi|$ are convex, then the following inequality holds:

$$\left| \frac{1}{\alpha_2 - \alpha_1} \left[ \int_{\alpha_1}^{\alpha_2} \frac{\alpha_1 + \alpha_2}{2} d_q x \right] \phi \left( \frac{\alpha_1 + \alpha_2}{2} \right) - \frac{\alpha_1 + \alpha_2}{2} \right| \leq \frac{(\alpha_2 - \alpha_1)}{8} \left[ \left( \begin{array}{c} 2 \\ q \end{array} \right) |\alpha_1 D_q \phi (\alpha_1)| + \left( \begin{array}{c} 3 \\ q \end{array} \right) |\alpha_2 D_q \phi (\alpha_2)| + \left( \begin{array}{c} 1 \\ q \end{array} \right) |\alpha_1 D_q \phi (\alpha_1)| + \left( \begin{array}{c} 2 \\ q \end{array} \right) |\alpha_2 D_q \phi (\alpha_2)| \right].$$

(4.5)

**Proof.** By taking modulus in (4.1), we have

$$\left| \frac{1}{\alpha_2 - \alpha_1} \left[ \int_{\alpha_1}^{\alpha_2} \phi \left( \frac{\alpha_1 + \alpha_2}{2} \right) d_q x \right] - \frac{\alpha_1 + \alpha_2}{2} \right| \leq \frac{(\alpha_2 - \alpha_1)}{4} \left[ \int_{0}^{1} (1 - qt) |\alpha_1 D_q \phi \left( \frac{\alpha_1 + \alpha_2}{2} + t \left( \alpha_1 - \frac{\alpha_1 + \alpha_2}{2} \right) \right) | d_q tight] + \int_{0}^{1} (1 - qt) |\alpha_2 D_q \phi \left( \frac{\alpha_1 + \alpha_2}{2} + t \left( \alpha_1 - \frac{\alpha_1 + \alpha_2}{2} \right) \right) | d_q t].$$

(4.6)

Since the functions $|\alpha_1 D_q \phi|$ and $|\alpha_2 D_q \phi|$ are convex, we have

$$\left| \frac{1}{\alpha_2 - \alpha_1} \left[ \int_{\alpha_1}^{\alpha_2} \phi \left( \frac{\alpha_1 + \alpha_2}{2} \right) d_q x \right] - \frac{\alpha_1 + \alpha_2}{2} \right| \leq \frac{(\alpha_2 - \alpha_1)}{4} \left[ \int_{0}^{1} (1 - qt) \left( \frac{1 - t}{2} \right) |\alpha_1 D_q \phi (\alpha_1)| + \left( \frac{1 + t}{2} \right) |\alpha_1 D_q \phi (\alpha_2)| \right] d_q t + \int_{0}^{1} (1 - qt) \left( \frac{1 + t}{2} \right) |\alpha_2 D_q \phi (\alpha_1)| + \left( \frac{1 - t}{2} \right) |\alpha_2 D_q \phi (\alpha_2)| \right] d_q t.$$
Thus, the proof is completed.

**Remark 2.** In Theorem 2, if we set the limit as \( q \to 1^- \), then we obtain [11, Theorem 2.2].

**Theorem 3.** We assume that the conditions Lemma 3 hold. If the functions \( |\alpha_1 D_q \phi|^s \) and \( |\alpha_2 D_q \phi|^s \), \( s \geq 1 \) are convex, then the following inequality holds:

\[
\frac{1}{\alpha_2 - \alpha_1} \left[ \int_{\alpha_1}^{\alpha_2} \phi \left( x, \frac{\alpha_1 + \alpha_2}{2} \right) d_q x \right] - \varphi \left( \frac{\alpha_1 + \alpha_2}{2} \right) \leq \left( \frac{\alpha_2 - \alpha_1}{4} \right) \left( \frac{1}{[2]_q} \right)^{1 - \frac{1}{s}} \left[ \frac{q [2]_q |\alpha_1 D_q \phi(\alpha_1)|^s + \left( 1 + [3]_q \right) |\alpha_2 D_q \phi(\alpha_2)|^s}{2 [2]_q [3]_q} \right]^{\frac{1}{s}} + \left( \frac{1 + [3]_q}{2 [2]_q [3]_q} \right) \left( \int_{\alpha_1}^{\alpha_2} \phi \left( x, \frac{\alpha_1 + \alpha_2}{2} \right) d_q x \right).
\]

**Proof.** By using the power mean inequality in (4.6), we have

\[
\frac{1}{\alpha_2 - \alpha_1} \left[ \int_{\alpha_1}^{\alpha_2} \phi \left( x, \frac{\alpha_1 + \alpha_2}{2} \right) d_q x \right] - \varphi \left( \frac{\alpha_1 + \alpha_2}{2} \right) \leq \left( \frac{\alpha_2 - \alpha_1}{4} \right) \left( \int_{0}^{1} (1 - qt) d_q t \right)^{1 - \frac{1}{s}} \times \left[ \left( \int_{0}^{1} (1 - qt) \left| \alpha_1 D_q \phi \left( \frac{\alpha_1 + \alpha_2}{2} + t \frac{\alpha_2 - \alpha_1 + \alpha_2}{2} \right) \right|^s d_q t \right)^\frac{1}{s} + \left( \int_{0}^{1} (1 - qt) \left| \alpha_2 D_q \phi \left( \frac{\alpha_1 + \alpha_2}{2} + t \frac{\alpha_2 - \alpha_1 + \alpha_2}{2} \right) \right|^s d_q t \right)^\frac{1}{s} \right].
\]

By using the convexity of the functions \( |\alpha_1 D_q \phi|^s \) and \( |\alpha_2 D_q \phi|^s \), we have

\[
\frac{1}{\alpha_2 - \alpha_1} \left[ \int_{\alpha_1}^{\alpha_2} \phi \left( x, \frac{\alpha_1 + \alpha_2}{2} \right) d_q x \right] - \varphi \left( \frac{\alpha_1 + \alpha_2}{2} \right) \leq \left( \frac{\alpha_2 - \alpha_1}{4} \right) \left( \int_{0}^{1} (1 - qt) d_q t \right)^{1 - \frac{1}{s}} \times \left[ \left( \int_{0}^{1} (1 - qt) \left| \alpha_1 D_q \phi \left( \frac{\alpha_1 + \alpha_2}{2} + t \frac{\alpha_2 - \alpha_1 + \alpha_2}{2} \right) \right|^s d_q t \right)^\frac{1}{s} + \left( \int_{0}^{1} (1 - qt) \left| \alpha_2 D_q \phi \left( \frac{\alpha_1 + \alpha_2}{2} + t \frac{\alpha_2 - \alpha_1 + \alpha_2}{2} \right) \right|^s d_q t \right)^\frac{1}{s} \right].
\]
Theorem 4. We assume that the conditions Lemma 3 hold. If the functions $|\alpha_1 D_q \varphi|^s$ and $|\alpha_2 D_q \varphi|^s$, $s > 1$ are convex, then the following inequality holds:

$$\left| \frac{1}{\alpha_2 - \alpha_1} \left[ \int_{\frac{\alpha_2 - \alpha_1}{2}}^{\alpha_2 - \alpha_1} \varphi(x) \frac{\alpha_1 + \alpha_2}{2} d_q x + \int_{\alpha_1}^{\alpha_2} \varphi(x) \frac{\alpha_1 + \alpha_2}{2} d_q x \right] - \varphi\left( \frac{\alpha_1 + \alpha_2}{2} \right) \right| \leq \frac{(\alpha_2 - \alpha_1)}{4} \left( 1 - \frac{(1 - q)^{r+1}}{q[r + 1]} \right)^{\frac{1}{r+1}} \left[ \left( q \left| \alpha_1 D_q \varphi(\alpha_1) \right|^s + \left| 2 \frac{[2]_q}{[3]_q} \alpha_1 D_q \varphi(\alpha_2) \right|^s \right)^{\frac{1}{s}} \right.$$ 

$$+ \left( \frac{[2]_q + 1}{2 [2]_q} \right)^{\frac{1}{r}} \left| \alpha_1 D_q \varphi(\alpha_1) \right|^s + q \left| \alpha_2 D_q \varphi(\alpha_2) \right|^s \right]^{\frac{1}{2}} \left( \frac{[2]_q}{[3]_q} \right).$$

Thus, the proof is completed. \( \square \)

Proof. By applying Hölder’s inequality in (4.6), we have

$$\left| \frac{1}{\alpha_2 - \alpha_1} \left[ \int_{\frac{\alpha_2 - \alpha_1}{2}}^{\alpha_2 - \alpha_1} \varphi(x) \frac{\alpha_1 + \alpha_2}{2} d_q x + \int_{\alpha_1}^{\alpha_2} \varphi(x) \frac{\alpha_1 + \alpha_2}{2} d_q x \right] - \varphi\left( \frac{\alpha_1 + \alpha_2}{2} \right) \right| \leq \frac{(\alpha_2 - \alpha_1)}{4} \left( \int_0^1 (1 - qt)^{r+1} d_q t \right)^{\frac{1}{r+1}}.$$
Since the functions $|\alpha_1 D_q \phi|_s$ and $|\alpha_2 D_q \phi|_s$ are convex, we have

\[
\left| \frac{1}{\alpha_2 - \alpha_1} \left[ \int_{a_1 + a_2}^{\alpha_2} \phi(x) \frac{a_1 + a_2}{s} d_q x + \int_{a_1}^{a_1 + a_2} \phi(x) \frac{a_1 + a_2}{s} d_q x \right] - \phi \left( \frac{\alpha_1 + \alpha_2}{2} \right) \right|
\leq \frac{(\alpha_2 - \alpha_1)}{4} \left( \int_{0}^{1} (1 - qt)^{r} d_q t \right)^{\frac{1}{2}}
\times \left[ \left( \int_{0}^{1} \left( \frac{1 - r}{2} \right)^{\frac{1}{2}} |\alpha_1 D_q \phi(\alpha_1)|^s + \frac{1}{2} |\alpha_2 D_q \phi(\alpha_2)|^s \right) d_q t \right]^{\frac{1}{2}}
+ \left( \int_{0}^{1} \left( \frac{1 + r}{2} \right)^{\frac{1}{2}} |\alpha_1 D_q \phi(\alpha_1)|^s + \frac{1}{2} |\alpha_2 D_q \phi(\alpha_2)|^s \right) d_q t \right]^{\frac{1}{2}}
\]

\[
= \frac{(\alpha_2 - \alpha_1)}{4} \left[ \left( \int_{0}^{1} \left( \frac{1 - q}{2} \right)^{r+1} d_q t \right)^{\frac{1}{2}} \left[ \left( \frac{q}{2} \right)^{\frac{1}{2}} |\alpha_1 D_q \phi(\alpha_1)|^s + \frac{2q + 1}{2} |\alpha_2 D_q \phi(\alpha_2)|^s \right]^{\frac{1}{2}}
\right.
\]
\[+ \left. \left( \int_{0}^{1} \left( \frac{2q + 1}{2} |\alpha_2 D_q \phi(\alpha_1)|^s + \frac{q}{2} |\alpha_2 D_q \phi(\alpha_2)|^s \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \right].
\]

Hence, the proof is completed. \(\square\)

5. Example

In this section, we give an example to support one of the newly established inequalities.

Example 2. For a convex functions $\phi : [1, 3] \rightarrow \mathbb{R}$ is defined by $\phi(x) = x + 2$. From Theorem 2 with $q = \frac{1}{4}$, the left hand side of inequality (4.5) becomes

\[
\left| \frac{1}{\alpha_2 - \alpha_1} \left[ \int_{a_1 + a_2}^{\alpha_2} \phi(x) \frac{a_1 + a_2}{s} d_q x + \int_{a_1}^{a_1 + a_2} \phi(x) \frac{a_1 + a_2}{s} d_q x \right] - \phi \left( \frac{\alpha_1 + \alpha_2}{2} \right) \right|
= \frac{1}{2} \left[ \int_{2}^{3} (x + 2) \frac{a_1 + a_2}{s} d_q x + \int_{1}^{2} (x + 2) \frac{a_1 + a_2}{s} d_q x \right] = \frac{1}{2} (4 + 4) - 4 = 0
\]
and the right side becomes
\[
\frac{(\alpha_2 - \alpha_1)}{8 [2]_q [3]_q} \left[ q[2]_q |\alpha_1 D_q\varphi(\alpha_1)| + \left(1 + [3]_q \right) |\alpha_1 D_q\varphi(\alpha_2)| \right]
\]
\[
+ \left(1 + [3]_q \right) |\alpha_2 D_q\varphi(\alpha_1)| + q[2]_q |\alpha_2 D_q\varphi(\alpha_2)|
\]
\[
= \frac{2}{8(1+q)(q^2+q+1)} \cdot \left[ (2+q^2+q) + q(q+1) + q(q+1) + (2+q^2+q) \right]
\]
\[
= \frac{1}{4(1+q)(q^2+q+1)} \cdot 4(1+q^2+q) = \frac{1}{q+1} = 4.8 = 0.8.
\]

It is clear that
\[
0 < 0.8
\]

which demonstrates inequality (4.5).

6. CONCLUDING REMARKS

In this paper, we have established some new Hermite-Hadamard inequality for convex functions in the context of quantum calculus. Moreover, we have proved some quantum estimates for midpoint-type inequalities for \(q\)-differentiable convex functions. It is also has proven by some mathematical examples that the newly established inequalities are valid for any convex function. It is an interesting and new problem that the upcoming researchers can obtain similar inequalities for coordinated convex functions.

REFERENCES


Authors’ addresses

**Thanin Sitthiwirattham**
Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok, Thailand

*E-mail address:* thanin_sit@dusit.ac.th

**Muhammad Aamir Ali**
(Corresponding author) Jiangsu Key Laboratory of NSLSCS, School of Mathematical Sciences, Nanjing Normal University, 210023, China

*E-mail address:* mahr.muhammad.aamir@gmail.com

**Asghar Ali**
Department of Mathematics, Government Postgraduate College Sahiwal, Pakistan

*E-mail address:* leoasgharali@gmail.com

**Hüseyin Budak**
Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey

*E-mail address:* hsyn.budak@gmail.com