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NEW OSCILLATION CRITERIA FOR SECOND-ORDER DELAY DYNAMIC EQUATIONS WITH A SUB-LINEAR NEGATIVE NEUTRAL TERM ON TIME SCALES

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Abstract. In this paper, some sufficient conditions for the oscillation of all solutions of second order dynamic equations with a negative sub-linear neutral term are established. The obtained results provide a unified platform that adequately covers both discrete and continuous equations. Furthermore, it covers a wide range of equations by utilizing different time scales. Illustrative examples are provided.

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1. Introduction

The main focus of this paper is to provide new oscillation criteria for the second-order half-linear dynamic equation of the form

$$\left[r(t)\left(z^{\Delta}(t)\right)^{\gamma}\right]^{\Delta}+q(t)f(x(\delta(t)))=0, \tag{1.1}$$

where $z(t):=x(t)-p(t)x^{\alpha}(\tau(t))$. Under the following assumptions

(H1) $\alpha,\gamma\in\mathbb{Q}^{+}_{\text{odd}}$, where $\mathbb{Q}^{+}_{\text{odd}}:=\{a/b: a,b\in\mathbb{Z}^{+}\text{are odd}\}$, $\alpha\in(0,1]$;

(H2) $r\in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, $p,q\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $0\le p(t)\le p_0<1$, $q(t)\ge 0$ and $q(t)$ is not identically zero for large t;

(H3) $\tau,\delta\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{T}),\delta^{\Delta}\ge 0$, $\tau(t)\le t$, $\delta(t)\le t$, $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\delta(t)=\infty$ and $h(t)=\tau^{-1}(\delta(t))$;

(H4) $f\in C(\mathbb{R},\mathbb{R})$, $uf(u)>0$ for all $u\neq 0$, and there exists a positive constant k such that $f(u)/u^{\beta}\ge k$, β is a ratio of odd positive integers where $\beta\le\gamma$.

Furthermore, for sufficiently large t_1, we assume

$$R(v,u)=\int_u^v\frac{1}{r^{1/\gamma}(s)}\Delta s,\qquad v\ge u\ge t_0.$$

and assume that

$$R(t_0,t)\to\infty\qquad \text{as } t\to\infty. \tag{1.2}$$

Also, we define

$$Q(t)=\int_t^\infty kq(s)\Delta s,\qquad t\geq t_0.$$

By a solution of (1.1), we mean a function $x\in C_{rd}[T_x,\infty)_{\mathbb{T}}$, $T_x\in[t_0,\infty)_{\mathbb{T}}$ which has the property $r(z^\Delta)^\alpha\in C^1_{rd}[T_x,\infty)_{\mathbb{T}}$ and satisfies (1.1) on $[T_x,\infty)_{\mathbb{T}}$. We consider only those solutions x of (1.1) which satisfy $\sup|x(t)|:t\in[T_x,\infty)_{\mathbb{T}}>0$ for all $T\in[T_x,\infty)_{\mathbb{T}}$. A solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is termed nonoscillatory.

Differential, difference equations, and dynamic equations on time scales have an enormous potential for applications in biology, engineering, economics, physics, neural networks, social sciences, etc. In particular, half-linear equations have numerous applications in the analyses of p-Laplace equations, Emden-Fowler equations, non-Newtonian fluid theory, porous medium problems, chemotaxis models, and so forth; see, e.g., [6, 11, 19, 25]. We also refer the reader to the papers [1, 2, 4–11, 13–18, 20–24, 28–34] for the oscillation and asymptotic behavior of different classes of half-linear equations.

In previous years, many papers studied the oscillatory behavior for different classes of dynamic equations on time scale. Many studies have been devoted to the oscillatory behavior of solutions to different classes of equations with nonnegative neutral coefficients; see, e.g., [2, 4, 31, 32] and the references cited therein. However, for equations with nonpositive neutral coefficients, there are relatively fewer results in the literature; see [5, 7, 17, 18, 28–30, 34]. for the oscillation and asymptotic behavior of different classes of half-linear equations.

For instance, Zhang *et al.*[33] investigated oscillatory behavior of solutions to a class of second-order nonlinear neutral delay dynamic equations with nonpositive neutral coefficients of the form

$$\left[r(t)\left(z^\Delta(t)\right)^\gamma\right]^\Delta+q(t)f(x(\delta(t)))=0,\qquad t\in[t_0,\infty)_{\mathbb{T}}$$

where $\gamma\geq 1$ is a ratio of odd integers and $z(t)=x(t)-p(t)x(\tau(t))$ with $\int_{t_0}^\infty r^{-\frac{1}{\alpha}}(s)\Delta s=\infty$, and presented new oscillation criteria for

$$\left[r(t)\left(z'(t)\right)^\alpha\right]'+q(t)f(x(\delta(t)))=0,\qquad t\geq t_0$$

under the assumption

$$\int_{t_0}^\infty r^{-\frac{1}{\alpha}}(s)\mathrm{d}s<\infty.$$

For $\mathbb{T}=\mathbb{Z}$, in [14] Grace and Graef presented some new oscillation criteria for second order nonlinear difference equations with a nonlinear nonpositive neutral term of the form

$$\Delta\left(a(t)\left(\Delta\left(x(t)-p(t)x^\alpha(t-k)\right)\right)^\gamma\right)+q(t)x^\beta(t+1-m)=0$$

where α,β and γ are ratios of positive odd integers with $\gamma\geq\beta$ and $0<\alpha\leq 1$.

In [26], Lin studied $\Delta\left(x_n - p_n x_{n-\tau}^{\alpha}\right) + q_n x_{n-\sigma}^{\beta} = 0,\ n \geq n_0$ where α and β are quotients of odd positive integers with $0 < \alpha < 1$

More precisely, to the best of our knowledge, no paper on the oscillation of second-order dynamic equations on a time scale appears on (1.1). Our aim is not only present some oscillation criteria for solutions of equation (1.1) but also present sufficient conditions which ensure that all solutions of (1.1) are oscillatory.

2. Auxiliary results

Lemma 1 ([9, Theorem 1.93]). *Assume that $v\colon \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\check{\mathbb{T}} := v(\mathbb{T})$ is a time scale. Let $y\colon \check{\mathbb{T}} \to \mathbb{R}$. If $v^{\Delta}(t)$ and $y^{\check{\Delta}}(v(t))$ exist for $t \in \mathbb{T}^k$, then*

$$(y(v(t)))^{\Delta} = y^{\check{\Delta}}(v(t))v^{\Delta}(t).$$

Lemma 2. *Let conditions (H1)-(H4) and (1.2) hold. Assume that $x(t)$ is a positive solution of (1.1). Then we have the following two cases:*

(I) $z(t) > 0,\ z^{\Delta}(t) > 0,\ (r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} \leq 0,$

(II) $z(t) < 0,\ z^{\Delta}(t) > 0,\ (r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} \leq 0,$

for $t \in [t_1, \infty)_{\mathbb{T}}$, where $t_1 \in [t_0, \infty)_{\mathbb{T}}$, is sufficiently large.

Proof. Suppose that there exists a $t_1 \geq t_0$ such that $x(t) > 0, x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$. From (1.1) it follows that

$$[r(t)(z^{\Delta}(t))^{\gamma}]^{\Delta} \leq -kq(t)x^{\beta}(\delta(t)) < 0.$$

Hence, $[r(t)(z^{\Delta}(t))^{\gamma}]$ is nonincreasing and of one sign. That is, there exists $t_2 \geq t_1$ such that $z^{\Delta}(t) > 0$ or $z^{\Delta}(t) < 0$ for $t \geq t_2$. We claim that $z^{\Delta}(t) > 0$ for $t \geq t_2$. For this, we assume that $z^{\Delta}(t) < 0$ for $t \geq t_2$. Then,

$$r(t)(z^{\Delta}(t))^{\gamma} \leq -C < 0, \qquad \text{for } t \geq t_2,$$

where $C = -r(t_2)(z^{\Delta}(t_2))^{\gamma}$. Thus, we conclude that

$$z(t) \leq z(t_2) - c^{1/\gamma} \int_{t_2}^{t} r^{-1/\gamma}(s)\Delta s.$$

By virtue of condition (1.2), $\lim_{t\to\infty} z(t) = -\infty$. Now, we consider two cases.

Case 1: If x is unbounded, then there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} x(t_k) = \infty$, where $x(t_k) = \max\{x(s); t_0 \leq s \leq t_k\}$, since

$$\lim_{t\to\infty} \tau(t) = \infty, \qquad \tau(t_k) > t_0$$

for all sufficiently large k. By $\tau(t) \leq t$

$$x(\tau(t_k)) = \max\{x(s); t_0 \leq s \leq \tau(t_k)\} \leq \max\{x(s); t_0 \leq s \leq t_k\} = x(t_k).$$

Therefore, for all large k,

$$z(t_k) = x(t_k) - p(t_k)x^{\alpha}(\tau(t_k)) \geq x(t_k) - p(t_k)x^{\alpha}(t_k)$$

$$\geq \left(1-\frac{p(t_k)}{x^{1-\alpha}(t_k)}\right)x(t_k)>0$$

which contradicts the fact that $\lim_{t\to\infty} z(t)=-\infty$.

Case 2: If $x(t)$ is bounded, then $z(t)$ is also bounded, which contradicts $\lim_{t\to\infty} z(t) = -\infty$. This completes the proof.

□

3. Main results

For simplicity, we consider

$$K(t)=\begin{cases}1, & \gamma=\beta;\\ MR^{\frac{\beta-\gamma}{\gamma}}(t,t_2); & \gamma>\beta,\end{cases}\qquad \text{for some } M>0.$$

$$Q(t)=\int_t^\infty q(s)\Delta s \ \text{ and } \psi(t)=r^{1/\gamma}(t)R(T,t), \qquad \text{for } T\geq t_0.$$

Theorem 1. *Assume that conditions (H1)-(H4) and (1.2) hold. If there exists a positive nondecreasing continuously differentiable function $\varphi(t)$, such that*

$$\limsup_{t\to\infty}\Big[\varphi(t)Q(t) \tag{3.1}$$
$$+\int_{t_2}^t\left(kq(s)\varphi(s)-\frac{\gamma^\gamma}{\beta^\gamma(\gamma+1)^{\gamma+1}}\frac{r(\delta(s))(\varphi^\Delta(s))^{\gamma+1}}{(\delta^\Delta(s))^\gamma\varphi^\gamma(s)K^\gamma(\delta(\sigma(s)))}\right)\Delta s\Big]=\infty,$$

$$\limsup_{t\to\infty}\left[k\int_{h(t)}^t q(s)R^{\beta/\alpha}(h(s),h(t))\Delta s\right]>1, \qquad \textit{for } \beta=\alpha\gamma, \tag{3.2}$$

and

$$\limsup_{t\to\infty}\int_{h(t)}^t q(s)R^{\beta/\alpha}(h(s),h(t))\Delta s=\infty, \qquad \textit{for } \beta<\alpha\gamma, \tag{3.3}$$

then every solution of (1.1) is oscillatory.

Proof. Assume that x is a nonoscillatory solution of (1.1) such that $x(t)>0$, $x(\tau(t))>0$ and $x(\delta(t))>0$, for $t\in[t_1,\infty)_{\mathbb{T}}$. From Lemma 2, $z(t)$ satisfies either (I) or (II) for $t\in[t_1,\infty)_{\mathbb{T}}$.

Case 1 : Suppose that $z(t)$ satisfies Lemma 2 (I). From the definition of $z(t)$,

$$x(t)=z(t)+p(t)x^\alpha(\tau(t))\geq z(t),$$

in view of (1.1), we get

$$[r(t)(z^\Delta(t))^\alpha]^\Delta\leq -kq(t)z^\beta(\delta(t))<0. \tag{3.4}$$

Integrating (3.4) from t to u, letting $u\to\infty$, and using the increasing fact of $z(t)$, we get

$$r(t)(z^\Delta(t))^\gamma\geq z^\beta(\delta(t))\int_t^\infty kq(s)\Delta s=:Q(t)z^\beta(\delta(t)). \tag{3.5}$$

Define

$$\omega(t) = \varphi(t)\frac{r(t)(z^{\Delta}(t))^{\gamma}}{z^{\beta}(\delta(t))}.$$

It is clear that $\omega(t) > 0$ and

$$\begin{aligned}\omega^{\Delta}(t) &= [r(t)(z^{\Delta}(t))^{\gamma}]^{\Delta}\frac{\varphi(t)}{z^{\beta}(\delta(t))} + r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}\left(\frac{\varphi(t)}{z^{\beta}(\delta(t))}\right)^{\Delta} \\ &\leq -kq(t)\varphi(t) + \varphi^{\Delta}(t)\frac{r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}}{z^{\beta}(\delta(\sigma(t)))} \\ &\quad - \beta\varphi(t)\frac{r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}z^{\Delta}(\delta(t))\delta^{\Delta}(t)}{z^{\beta+1}(\delta(t))}.\end{aligned}$$

Since $z^{\Delta}(t) > 0$, $\delta^{\Delta}(t) \geq 0$ and from the definition of $\omega(t)$, we obtain

$$\begin{aligned}\omega^{\Delta}(t) &\leq -kq(t)\varphi(t) + \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t)) \\ &\quad - \beta\delta^{\Delta}(t)\varphi(t)\frac{r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}}{z^{\beta}(\delta(\sigma(t)))}\frac{z^{\Delta}(\delta(t))}{z(\delta(\sigma(t)))} \\ &\leq -kq(t)\varphi(t) + \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t)) \\ &\quad - \beta\delta^{\Delta}(t)\frac{\varphi(t)z^{\Delta}(\delta(t))}{\varphi(\sigma(t))z(\delta(\sigma(t)))}\omega(\sigma(t)).\end{aligned} \tag{3.6}$$

From the definition of $\omega(t)$ and since $[r(t)(z^{\Delta})^{\gamma}]$ is nonincreasing, then we have

$$z^{\Delta}(\delta(t)) \geq \frac{z^{\frac{\beta}{\gamma}}(\delta(\sigma(t)))}{r^{\frac{1}{\gamma}}(\delta(t))\varphi^{\frac{1}{\gamma}}(\sigma(t))}\omega^{\frac{1}{\gamma}}(\sigma(t)).$$

This with (3.6) leads to

$$\begin{aligned}\omega^{\Delta}(t) &\leq -kq(t)\varphi(t) + \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t)) \\ &\quad - \beta\delta^{\Delta}(t)\frac{\varphi(t)}{\varphi^{\frac{\gamma+1}{\gamma}}(\sigma(t))r^{\frac{1}{\gamma}}(\delta(t))}\omega^{\frac{\gamma+1}{\gamma}}(\sigma(t))z^{\frac{\beta-\gamma}{\gamma}}(\delta(\sigma(t))).\end{aligned} \tag{3.7}$$

It is clear that $z^{\frac{\beta-\gamma}{\gamma}}(\delta(\sigma(t))) = 1$ for $\beta = \gamma$ and (3.7) takes the form

$$\begin{aligned}\omega^{\Delta}(t) &\leq -kq(t)\varphi(t) + \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t)) \\ &\quad - \beta\delta^{\Delta}(t)\frac{\varphi(t)}{\varphi^{\frac{\gamma+1}{\gamma}}(\sigma(t))r^{\frac{1}{\gamma}}(\delta(t))}\omega^{\frac{\gamma+1}{\gamma}}(\sigma(t)).\end{aligned} \tag{3.8}$$

On the other hand for $\beta < \gamma$. Since $[r(t)(z^{\Delta}(t))^{\gamma}]^{\Delta} \leq 0$, then there exists a constant $C > 0$ such that $r(t)(z^{\Delta}(t))^{\gamma} \leq r(t_2)(z^{\Delta}(t_2))^{\gamma} < C$ for some $t \geq t_2$, which leads to

$$z^{\Delta}(t) \leq C^{\frac{1}{\gamma}} r^{\frac{-1}{\gamma}}(t). \tag{3.9}$$

Integrating (3.9) from t to t_2, we get

$$\begin{aligned} z(t) &\leq z(t_2) + C^{\frac{1}{\gamma}} R(t_2, t) \\ &\leq C^{1/\gamma} R(t_2, t). \end{aligned}$$

This leads to,

$$z^{\frac{\beta-\gamma}{\gamma}}(\delta(\sigma(t))) > MR^{\frac{\beta-\gamma}{\gamma}}(\delta(\sigma(t)), t_2), \quad \beta < \gamma \quad \text{for some } M > 0. \tag{3.10}$$

Combining (3.8) and (3.10), we get

$$\begin{aligned} \omega^{\Delta}(t) \leq{}& -kq(t)\varphi(t) + \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t)) \\ &- \beta M \delta^{\Delta}(t)\varphi(t) \frac{R^{\frac{\beta-\gamma}{\gamma}}(\delta(\sigma(t)), t_2)}{\varphi^{\frac{\gamma+1}{\gamma}}(\sigma(t)) r^{\frac{1}{\gamma}}(\delta(t))} \omega^{\frac{\gamma+1}{\gamma}}(\sigma(t)) \\ \leq{}& -kq(t)\varphi(t) + \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t)) \\ &- \beta \delta^{\Delta}(t)\varphi(t) \frac{K(\delta(\sigma(t)))}{\varphi^{\frac{\gamma+1}{\gamma}}(\sigma(t)) r^{\frac{1}{\gamma}}(\delta(t))} \omega^{\frac{\gamma+1}{\gamma}}(\sigma(t)). \end{aligned} \tag{3.11}$$

Applying the inequality

$$B\omega - A\omega^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^{\gamma}},$$

with $B = \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}$, $A = \beta\delta^{\Delta}(t)\varphi(t) \dfrac{K(\delta(\sigma(t)))}{\varphi^{\frac{\gamma+1}{\gamma}}(\sigma(t)) r^{\frac{1}{\gamma}}(\delta(t))}$, we get

$$\omega^{\Delta}(t) \leq -kq(t)\varphi(t) + \frac{\gamma^{\gamma}\beta^{-\gamma}}{(\gamma+1)^{\gamma+1}} \frac{r(\delta(t))(\varphi^{\Delta}(t))^{\gamma+1}}{(\delta^{\Delta}(t))^{\gamma}\varphi^{\gamma}(t)K^{\gamma}(\delta(\sigma(t)))}. \tag{3.12}$$

Integrating (3.12) from t_2 to t

$$\begin{aligned} \omega(t) \leq \omega(t_2) - \int_{t_2}^{t} \Bigg(& kq(t)\varphi(s) \\ & - \frac{\gamma^{\gamma}}{\beta^{\alpha}(\gamma+1)^{\gamma+1}} \frac{r(\delta(s))(\varphi^{\Delta}(s))^{\gamma+1}}{(\delta^{\Delta}(s))^{\gamma}\varphi^{\gamma}(s)K^{\gamma}(\delta(\sigma(s)))} \Bigg) \Delta s. \end{aligned} \tag{3.13}$$

From the definition of $\omega(t)$ and (3.5) we have

$$\omega(t) \geq \varphi(t)\frac{Q(t)z^{\beta}(\delta(t))}{z^{\beta}(\delta(t))} \geq \varphi(t)Q(t). \tag{3.14}$$

From (3.13) and taking (3.14) into account

$$\varphi(t)Q(t) + \int_{t_2}^{t}\left(kq(s)\varphi(s) - \frac{\alpha^{\alpha}}{\beta^{\alpha}(\alpha+1)^{\alpha+1}}\frac{r(\delta(s))(\varphi^{\Delta}(s))^{\alpha+1}}{(\delta^{\Delta}(s))^{\alpha}\varphi^{\alpha}(s)K^{\alpha}(\delta(\sigma(s)))}\right)\Delta s \leq \omega(t_2),$$

which contradicts (3.1).

Case 2 : Suppose that $z(t)$ satisfies Lemma 2 (II). Putting $y = -z > 0$, then $y^{\Delta} < 0$, and (1.1) takes the form

$$[r(t)(y^{\Delta}(t))^{\gamma}]^{\Delta} \geq kq(t)x^{\beta}(\delta(t)) \geq 0. \tag{3.15}$$

Since

$$\begin{aligned} y(t) &= -z(t) = p(t)x^{\alpha}(\tau(t)) - x(t) \\ &\leq p(t)x^{\alpha}(\tau(t)), \end{aligned}$$

by virtue of $0 \leq p(t) \leq p_0 < 1$, we have

$$y^{1/\alpha}(h(t)) \leq x(\delta(t)). \tag{3.16}$$

Now, inequalities (3.15) and (3.16) lead to

$$[r(t)(y^{\Delta}(t))^{\gamma}]^{\Delta} \geq kq(t)y^{\beta/\alpha}(h(t)). \tag{3.17}$$

Also for $t_2 \leq u \leq v$, we can write

$$y(u) - y(v) = \int_{u}^{v}\frac{1}{r^{1/\gamma}(s)}(-r(s)(y^{\Delta}(s))^{\gamma})^{1/\gamma}\Delta s,$$

$$y(u) \geq R(v,u)(-r(v)(y^{\Delta}(v))^{\gamma})^{1/\gamma}.$$

Setting $u = h(t)$ and $v = h(s)$, we get

$$y(h(s)) \geq R(h(s),h(t))(-r(h(t))(y^{\Delta}(h(t)))^{\gamma})^{1/\gamma}. \tag{3.18}$$

Integrating (3.17) from $h(t)$ to t, in view of (3.18), we get

$$-r(h(t))(y^{\Delta}(h(t)))^{\gamma} \geq k[-r(h(t))(y^{\Delta}(h(t)))^{\gamma}]^{\beta/\alpha\gamma}\int_{h(t)}^{t} q(s)R^{\beta/\alpha}(h(s),h(t))\Delta s,$$

which leads to

$$[Y(t)]^{1-\frac{\beta}{\alpha\gamma}} \geq k\int_{h(t)}^{t} q(s)R^{\beta/\alpha}(h(s),h(t))\Delta s,$$

where $Y(t) = -r(h(t))(y^{\Delta}(h(t)))^{\gamma}$. Therefore, we have

$$1 \geq k\int_{h(t)}^{t} q(s)R^{\beta/\alpha}(h(s),h(t))\Delta s, \qquad \text{for } \beta = \alpha\gamma,$$

which contradicts (3.2). Also, for $\beta < \gamma$, by (3.15) together with the fact that $Y^{\Delta}(t) \leq 0$ and $Y(t)$ is bounded, we get a contradiction with (3.3). This completes the proof.

□

Remark 1. Note that Theorem 1 holds when $Q(t) < \infty$ and the addition term $\varphi(t)Q(t)$ in condition (3.1) may improve some of the well-known results in the literature.

Corollary 1. *Let conditions (H1)-(H4) be satisfied and* (1.2) *hold. If* $Q(t) < \infty$, *then condition* (3.1) *replaced by*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left(kq(s)\varphi(s) - \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}} \frac{r(\delta(s))(\varphi^{\Delta}(s))^{\gamma+1}}{(\delta^{\Delta}(s))^{\gamma}\varphi^{\gamma}(s)K^{\gamma}(\delta(\sigma(s)))} \right) \Delta s = \infty, \quad (3.19)$$

and conclusion of Theorem 1 remains intact.

Corollary 2. *Let conditions (H1)-(H4) be satisfied and* (1.2) *hold. With* $\varphi^{\Delta}(t) \leq 0$, *then condition* (3.1) *replaced by*

$$\limsup_{t\to\infty} \left[\varphi(t)Q(t) + \int_{t_2}^{t} kq(s)\varphi(s)\Delta s \right] = \infty,$$

and conclusion of Theorem 1 remains intact.

To add variety, we present a different approach of the condition (3.1) in the following.

Theorem 2. *Assume that conditions (H1)-(H4) and* (1.2) *hold. If*

$$\limsup_{t\to\infty} \left[\varphi(t)Q(t) + \int_{t_2}^{t} \left(kq(s)\varphi(s) - \frac{1}{4\beta} \frac{r^{1/\gamma}(\delta(s))(\varphi^{\Delta}(s))^2}{\delta^{\Delta}(s)Q^{\frac{1-\gamma}{\gamma}}(\sigma(s))\varphi(s)K(\delta(\sigma(s)))} \right) \Delta s \right] = \infty, \quad (3.20)$$

and (3.2) *or* (3.3)*, then every solution of* (1.1) *is oscillatory.*

Proof. Assume that $x(t)$ is a nonoscillatory solution of (1.1) such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\delta(t)) > 0$, for $t \in [t_1, \infty)_{\mathbb{T}}$. From Lemma 2, $z(t)$ satisfies either (I) or (II) for $t \in [t_1, \infty)_{\mathbb{T}}$.

Case 1: First, we suppose that $z(t)$ satisfies (I). Since $\omega(t) = \varphi(t)\dfrac{r(t)(z^{\Delta}(t))^{\gamma}}{z^{\beta}(\delta(t))}$, then by (3.14), we conclude that

$$\left(\frac{\omega(\sigma(t))}{\varphi(\sigma(t))} \right)^{\frac{1-\gamma}{\gamma}} \geq Q^{\frac{1-\gamma}{\gamma}}(\sigma(t)). \quad (3.21)$$

Now, inequalities (3.11) and (3.21) imply

$$\omega^{\Delta}(t) \leq -kq(t)\varphi(t) + \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t)) - \beta\delta^{\Delta}(t)Q^{\frac{1-\gamma}{\gamma}}(\sigma(t))\frac{\varphi(t)K(\delta(\sigma(t)))}{\varphi^2(\sigma(t))r^{\frac{1}{\gamma}}(\delta(t))}\omega^2(\sigma(t)).$$

Apply the inequality

$$B\omega - A\omega^{\frac{\gamma+1}{\alpha}} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^{\gamma}}, \qquad \text{where } \gamma = 1,$$

with $B = \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}$, $A = \beta\delta^{\Delta}(t)Q^{\frac{1-\gamma}{\gamma}}(\sigma(t))\frac{\varphi(t)K(\delta(\sigma(t)))}{\varphi^2(\sigma(t))r^{\frac{1}{\gamma}}(\delta(t))}$, we get

$$\omega^{\Delta}(t) \leq -kq(t)\varphi(t) + \frac{1}{4\beta}\frac{r^{1/\gamma}(\delta(t))(\varphi^{\Delta}(t))^2}{\delta^{\Delta}(t)Q^{\frac{1-\gamma}{\gamma}}(\sigma(t))\varphi(t)K(\delta(\sigma(t)))}. \tag{3.22}$$

Integrating (3.22) from t_2 to t

$$\omega(t) \leq \omega(t_2) - \int_{t_2}^{t}\left(kq(s)\varphi(s) - \frac{1}{4\beta}\frac{r^{1/\gamma}(\delta(s))(\varphi^{\Delta}(s))^2}{\delta^{\Delta}(s)Q^{\frac{1-\gamma}{\gamma}}(\sigma(s))\varphi(s)K(\delta(\sigma(s)))}\right)\Delta s, \tag{3.23}$$

In view of (3.14), inequality (3.23) takes the form

$$k\varphi(t)Q(t) + \int_{t_2}^{t}\left(kq(s)\varphi(s) - \frac{1}{4\beta}\frac{r^{1/\gamma}(\delta(s))(\varphi^{\Delta}(s))^2}{\delta^{\Delta}(s)Q^{\frac{1-\gamma}{\gamma}}(\sigma(s))\varphi(s)K(\delta(\sigma(s)))}\right)\Delta s \leq \omega(t_2),$$

which contradicts (3.20).

Case 2: Suppose that $z(t)$ satisfies Lemma 2 (II). The proof can be performed a similar manner as in the proof of Theorem1. This completes the proof.

□

4. Applications and Examples

This section introduces some special cases for Eq. (1.1). For the non-neutral equation, i.e., Eq. (1.1) with $p(t) \equiv 0$, and $q(t)$ is either positive or negative for all large t, Eq. (1.1) is reduced to the equation

$$\left[r(t)\left((x(t))^{\Delta}\right)^{\gamma}\right]^{\Delta} \pm q(t)f(x(\delta(t))) = 0. \tag{E$\pm$}$$

From Theorem1 we conclude the following results.

Corollary 3. *Assume that conditions (H1)-(H4) and (1.2) hold. If there exists a positive function $\phi(t)$ with $\phi^{\Delta}(t) \geq 0$ such that condition (3.1) holds, then Eq. (E+) is oscillatory.*

Proof. The proof is omitted because it is included in the proof of Theorem 1-Case (1). □

It's noted that, results in [3] and the references cited therein are related to Corollary 3 for $\mathbb{T}=\mathbb{R}$. Also the results can be extended to the difference equations when $\mathbb{T}=\mathbb{Z}$, see [12, 27].

Corollary 4. *Assume that conditions (H1)-(H4) and* (1.2) *hold. If there exists a positive function* $\phi(t)$ *with* $\phi^{\Delta}(t)\geq 0$ *such that condition* (3.2) *or* (3.3) *holds, then Eq. (E-) is oscillatory.*

Proof. The proof is omitted because it is included in the proof of Theorem 1-Case (2). □

In the following, we investigate another special case

Theorem 3. *Let* $\alpha=1$*, conditions (H1)-(H4), and* (1.2) *hold. Assume that condition* (3.2) *and*

$$\limsup_{t\to\infty} R^{\beta}(t_0,\delta(t))>1, \qquad \text{when } \beta=\gamma, \tag{4.1}$$

hold and condition (3.3) *and*

$$\limsup_{t\to\infty} R^{\beta}(t_0,\delta(t))>0, \qquad \text{when } \beta<\gamma, \tag{4.2}$$

hold. Then (1.1) *is oscillatory.*

Proof. Assume that (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we assume that there exists a $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$, $x(\tau(t))>0$, and $x(\delta(t))$ for $t\in\in[t_1,\infty)_{\mathbb{T}}$. According to Lemma 2, we have two possible cases for $z(t)$. Assume that $z(t)$ satisfies Lemma 2 (I). It follows that

$$\begin{aligned} z(t) &= z(t_2)+\int_{t_2}^{t}\frac{(r(s)(z^{\Delta}(s))^{\gamma})^{1/\gamma}}{a^{1/\gamma}(s)}\Delta s \\ &\geq (r(t)(z^{\Delta}(t))^{\gamma})^{1/\gamma}\int_{t_2}^{t}a^{-1/\gamma}(s)\Delta s \\ &=: \psi(t)z^{\Delta}(t). \end{aligned} \tag{4.3}$$

In view of (3.5) and (4.3), and using the decreasing fact of $r(t)z^{\Delta}(t)$, we find

$$\begin{aligned} w(t) &=: r(t)z^{\Delta}(t) \\ &\geq Q(t)\psi^{\beta}(\delta(t))(z^{\Delta}(\delta(t)))^{\beta} \\ &= Q(t)\psi^{\beta}(\delta(t))\left(r^{-\beta/\gamma}(\delta(t))\right)\left(r(\delta(t))(z^{\Delta}(\delta(t)))^{\gamma}\right)^{\beta/\gamma} \\ &\geq Q(t)\psi^{\beta}(\delta(t))\left(r^{-\beta/\gamma}(\delta(t))\right)\left(r(t)(z^{\Delta}(t))^{\gamma}\right)^{\beta/\gamma} \\ &= Q(t)\psi^{\beta}(\delta(t))\left(r^{-\beta/\gamma}(\delta(t))\right)w^{\beta/\gamma}(t), \end{aligned}$$

hence

$$\begin{aligned} w^{1-\beta/\gamma}(t) &\geq Q(t)\psi^{\beta}(\tau(t))\left(r^{-\beta/\gamma}(\delta(t))\right) \\ &= Q(t)\left(\int_{t_2}^{\delta(t)} r^{-1/\gamma}(s)\Delta s\right)^{\beta} \\ &= R^{\beta}(t_2,\delta(t))Q(t). \end{aligned} \tag{4.4}$$

Taking lim sup of both sides of (4.4) as $t \to \infty$. For $\beta = \gamma$, we get a contradiction to (4.1) and (4.2) when $\beta < \gamma$. If $z(t)$ satisfies Lemma 2 (II), then the proof is similar to that of Theorem 1. □

Remark 2. Note that the obtained results in Theorem 3 are an improvement of the results in [18, 34] which guarantee that every solution of (1.1) is oscillatory.

Example 1. Let $\mathbb{T} = \mathbb{R}$. Consider the second order differential equation

$$\left(t^2\left(\left(x(t)-\frac{1}{2}x(t/3)\right)'\right)^3\right)+\frac{\lambda}{t^2}x^3(t/2)=0, \qquad t\geq 1. \tag{4.5}$$

Here $\alpha = 1$, $\beta = 3$, $\gamma = 3$, $\lambda > 0$ is a constant $k = \lambda$, $r(t) = t^2$, $p(t) = 1/2$, $q(t) = 1/t^2$, $\tau(t) = t/3$, $\delta(t) = t/2$ and $h(t) = 3t/2$. It is clear that

$$\begin{aligned} R(t_0,t) &= \int_1^t s^{-2/3} ds \\ &= 3(\sqrt[3]{t}-1), \qquad R(t_0,t) \to \infty \text{ as } t \to \infty. \end{aligned}$$

Applying Theorem 3, we have

$$\limsup_{t\to\infty}\left[k\int_{h(t)}^t q(s)R^{\beta}(h(s),h(t))\Delta s\right] = \limsup_{t\to\infty}\left[\lambda\int_{3t/2}^t \frac{81(\sqrt[3]{s}-\sqrt[3]{t})^3}{2s^2}ds\right] > 1$$

and

$$\limsup_{t\to\infty} R^{\beta}(t_0,\delta(t)) = \limsup_{t\to\infty}\left(3(\sqrt[3]{3t/2}-1)\right) > 1.$$

For suitable λ and large t, every solution of Eq. (4.5) is oscillatory.

Remark 3. Theorem 3.1 of [18] can be applied to (4.5) which yields that every solution of equation (4.5) is oscillatory when $\lambda > \frac{2}{27}$ or $\lim_{t\to\infty} x(t) = 0$.

Example 2. Let $\mathbb{T} = \mathbb{R}$. Consider the second order differential equation

$$\left(\left(x(t)-\frac{1}{2}x(\sqrt{t})\right)^3\right)'' + \frac{m}{t^{5/4}}x(t/2) = 0, \qquad t \geq 1. \tag{4.6}$$

Here $\alpha = 1$, $\beta = 1$, $\gamma = 3$, $k = m > 0$ where m is a constant, $r(t) = 1$, $p(t) = 1/2$, $q(t) = 1/t^{5/4}$, $\tau(t) = \sqrt{t}$, $\delta(t) = t^{5/4}$ and $h(t) = \sqrt{t}$. Taking $\varphi(t) = t$, by using Theorem 1, we have

$$\limsup_{t\to\infty}\left[4mt^{3/4} + \int_1^t \left(\frac{m}{\sqrt[4]{s}} - \frac{9}{16\left(\frac{1}{4}s^{1/4}\right)^3 \left(\sqrt[4]{s}-1\right)^{-2}}\right)ds\right]$$
$$= \limsup_{t\to\infty}\left[4mt^{3/4} + \frac{4}{3}\left(m\left(t^{3/4}-1\right) - 36\left(\sqrt[4]{t}-1\right)^3\right)\right].$$

Hence, (4.6) oscillates for $m > 9$.
According to (3.2), we have

$$\limsup_{t\to\infty}\left[k\int_{h(t)}^t q(s)R^{\beta}(h(s),h(t))ds\right] = \limsup_{t\to\infty}\left[2m\int_{\sqrt{t}}^t \frac{\sqrt{t}-\sqrt{s}}{s^{5/4}}ds\right] \to \infty.$$

Therefore, every solution of (4.5) is oscillatory when $m > 9$.

Remark 4. It should be noted that, for $\mathbb{T} = \mathbb{R}$ and $\alpha = 1$, Theorem 1 improves the conditions of Theorem 2.1 of [13] and guarantees that every solution of (1.1) is oscillatory unlike in [13, 33, 34] .

Example 3. Let $\mathbb{T} = \mathbb{Z}$. Consider the second order difference equation

$$\Delta\left(\Delta\left(x(t) - \frac{1}{2}x^{1/3}(t-3)\right)^3\right) + 8(x-7) = 0, \qquad t \geq 1. \tag{4.7}$$

Here $\alpha = 1/3$, $\beta = 1$, $\gamma = 3$, $k = 1$, $r(t) = 1$, $p(t) = 1/2$, $q(t) = 8$, $\tau(t) = t-3$, $\delta(t) = t-7$ and $h(t) = t-4$. Taking $\varphi(t) = t$, by using Corollary 1, we have

$$\limsup_{t\to\infty}\left[\sum_{s=1}^t \left(kq(s)\varphi(s) - \frac{\gamma^\gamma}{\beta^\gamma(\gamma+1)^{\gamma+1}}\frac{r(\delta(s))(\Delta\varphi(s))^{\gamma+1}}{(\Delta\delta(s))^\gamma \varphi^\gamma(s)K^\gamma(\delta(\sigma(s)))}\right)\right]$$
$$= \limsup_{t\to\infty}\left[\sum_{s=1}^t \left(8s - \frac{9}{16s^3(s-6)^{-2}}\right)\right] \to \infty.$$

Also, condition (3.2) implies that

$$\limsup_{t\to\infty}\left[k\sum_{s=h(t)}^t q(s)R^{\beta/\alpha}(h(t),h(s))ds\right] \to \infty,$$

which yields that (4.7) satisfies conditions (3.19) and (3.2). Therefore, every solution of (4.7) is oscillatory.

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